PROCEEDINGS

OF THE

PHYSICAL SOCIETY OF LONDON.

From December 1905 *to December* 1907.

VOL. XX.

LONDON:
TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.
MDCCCCVII.

ALERE FLAMMAM.

PRINTED BY TAYLOR AND FRANCIS,
RED LION COURT, FLEET STREET.

CONTENTS.

VOL. XX.

PLATES.

OBITUARIES.

Ludwig Boltzmann was born at Vienna on 20th February, 1844. His father was Ludwig Boltzmann, District Financial Commissioner. He was educated at the Gymnasium at Linz and at the University of Vienna, where he took the degree of Ph.D. He also studied for short periods at Heidelberg and Berlin. In 1866 he published his first paper (on the second law of thermodynamics), and in 1867 he became Privat-Dozent in the University of Vienna and assistant in the Physical Institute. In 1869 he was appointed Professor of Theoretical Physics at Graz, and in 1873 he returned to Vienna as Professor of Mathematics. A few years later we find him again at Graz, where he had succeeded Toepler as Professor of Experimental Physics. In 1889 he accepted a call to the chair of Theoretical Physics at Munich. From this time onward his official subject continued to be Theoretical Physics, with changes of University in 1894 to Vienna, in 1900 to Leipzig, and in 1902 again to Vienna where he remained to the end. On the retirement of Mach a Lectureship on the History and Theory of the Exact Sciences was added to his duties in the University of Vienna. He became a Corresponding Member of the Vienna Academy in 1875 and an Ordinary Member in 1885. In 1883 he was elected an Honorary Fellow of the Physical Society of London, and in 1899 a Corresponding Fellow of the Royal Society. The Austrian Government conferred on him the title of Hofrath. In 1904 Physicists from many lands contributed memoirs to a Festschrift in honour of his sixtieth birthday.

He married Henrietta von Aigentier in 1876 and leaves four children.

In the evening of the fifth of September, 1906, he committed suicide while away on his holidays. It appears that he was subject to fits of severe depression and melancholy, which may also explain his numerous changes of habitat.

Boltzmann's published work includes experiments on the specific inductive capacity of gases and sulphur, and theoretical papers of great importance on the Kinetic Theory of Gases, the Electromagnetic Theory of Light and Radiation. He also published treatises on Maxwell's Electromagnetic Theory, the Kinetic Theory of Gases, and Theoretical Dynamics.

Samuel Pierpont Langley was born of "middle class parents" at Roxbury, Massachusetts, on 22nd August, 1834. He studied civil engineering and architecture and worked for seven years as an architect in Chicago and St Louis. He then abandoned his profession, visited Europe, and on his return at the age of 30 began his work in Astronomy. After some time spent in the observatories of Cambridge and Annapolis, he was appointed in 1867 Director of the Alleghany Observatory at Pittsburg. Finding it inadequately equipped and almost destitute of endowment, he showed his practical capacity in securing an income for the observatory by instituting a system of time signals worked from the observatory clock to the telegraph stations on 8000 miles of railway. After a few years he was able to enter upon the study of the sun and the corona, the mapping of the spectrum with bolometer and thermopile, and the investigation of atmospheric absorption, which established his reputation and occupied the best years of his life.

In 1887 he was appointed Secretary of the Smithsonian Institution, where for nearly twenty years his scientific and administrative capacity proved of great value. During this period also he devoted his attention to the problem of aerial flight. After a theoretical study of the subject he succeeded in producing small models, of which one at least flew three quarters of a mile. Subsequently, under the auspices of the Board of Ordnance and Fortification of the United States Army, he constructed an aerodrome large enough for war purposes. Unfortunately, however, some defect in detail led to a mishap at the first trial, and the funds necessary for a second attempt were withheld. It is alleged that disappointment and annoyance at ignorant misrepresentation in the press "shortened his useful life and brought on the attack of paralysis that ended his days." He died in March 1906.

In addition to his more purely scientific papers Langley published some semipopular works on astronomy. His writings are distinguished by lucidity and charm of style, and his fertility is shown by the fact that the list of his papers and other writings number two hundred and eighty-five.

He has been an Honorary Fellow of the Physical Society of London since 1902.

John Purser was born in Dublin on 24th August, 1835. His father J. T. Purser was engaged in the brewery of Messrs. A. Guinness, Son & Co. Ltd., where he subsequently became chief brewer and manager.

John Purser was educated at a private boarding school kept by his uncle, Dr R. W. Biggs, at Devizes, Wiltshire, and at Trinity College, Dublin, where in 1855 he gained the Lloyd Exhibition and was classed as Senior Moderator with the first gold medal in Mathematics. He graduated B.A. in 1856, gained the Moderation scholarship, the Bishop Laws mathematical premium, and the McCullagh prize, and became M.A. in 1859. The religious test system then existing, which was subsequently abolished by Fawcett's Act, prevented him as a member of the Moravian Church from competing for a Fellowship in Trinity College.

In 1863 he was elected Professor of Mathematics in Queen's College, Belfast, in succession to G. M. Slesser. This office he held until his resignation in 1901, and on 18th Oct. 1903 he died at his residence, Rathmines Castle, Dublin.

He was an enthusiastic and inspiring teacher and the testimony of his students shows that he produced a profound and lasting impression upon them. Three of his students became Senior Wranglers, viz., the Rev. A. J. C. Allen in 1878, Prof. Joseph Larmor in 1880, and Prof. W. McFadden Orr in 1888.

His most important paper was read at the Belfast Meeting of the British Association in 1874. In it he showed that the waste of energy involved in tidal friction was supplied at the expense of the energy of the rotation of the earth and of the orbital motion of the moon, and consequently tended to increase the mean distance of the moon from the earth. This solution of a problem which had baffled Sir George Airy formed a most important contribution to the theory of the earth-moon system which was afterwards so splendidly developed by Sir George Howard Darwin. He also published papers upon hydrodynamical subjects in the Report of the Glasgow Meeting of the British Association in 1876 and in the Philosophical Magazine for Nov. 1878. At the British Association Meeting at Belfast in 1902 he was President of Section A, and gave a historical address on the work of Sir William Rowan Hamilton and other mathematicians of the Dublin School. In collaboration with his brother Frederick Purser, Fellow of Trinity College and Professor of Natural Philosophy in the University of Dublin, he wrote the article *Surface* in the supplementary volumes of the 'Encyclopædia Britannica.' He became a Fellow of the Physical Society of London in 1876.

PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

I. *The Theory of Phasemeters.*

By W. E. SUMPNER, *D.Sc.**

PHASEMETERS are instruments of the dynamometer type for indicating the phase relations of the currents and potentials in alternating-current circuits. With few exceptions they are made for use on multiphase circuits. Single phase instruments have been constructed, but they are not very satisfactory, as their calibration alters with both frequency and wave-form.

Such instruments consist essentially of two sets of coils, of which one set is fixed and the other forms a single moving system which is not provided with any form of control. The currents in one set of coils are determined by the voltages of the main circuits; while those in the other set are produced by the circuit currents. The total number of coils used in the two systems must be at least three, and as a rule actual instruments only contain this minimum number of coils: the fixed system usually consists of a single coil conveying one of the line currents, while the moving system consists of two independent but relatively fixed coils, traversed by currents produced, through suitable non-inductive resistances, by two of the voltages of the multiphase circuit. There is one instrument made commercially for three-phase circuits which is of a more elaborate construction, and

* Read October 27, 1905.

contains three fixed coils for the currents, and a moving system—also of three coils—for the voltages. Such an instrument is more complicated to make and to connect to the circuit, but it possesses certain advantages as will be shown later. With rare exceptions the magnetic circuits of phasemeters are air circuits, containing no iron, so that the magnetic fields associated with them are weak, and the instruments in consequence are of somewhat delicate construction. There, however, appears to be no reason why the use of iron should be avoided in these instruments, and in what follows the writer proposes to show, that the theory of these instruments is the same whether they contain iron or not, and however the coils may be arranged; that they can be calibrated by direct current methods although for use on alternating current circuits; and that a new type of instrument, containing iron, conforms to the theory given. An investigation will also be made of their action when the circuit currents are of unequal magnitude.

The study of these instruments really resolves itself into an examination of the behaviour of multiple magnetic circuits when actuated by independent currents.

Theory of Four-Circuit Phasemeters.

We shall first consider the case of a three-phase instrument having three fixed coils for the circuit currents, and one movable coil, in series with a non-inductive resistance, traversed by a current produced by one of the circuit voltages.

In all cases, it will be assumed (i.) that the induction density at any point due to the current $\mathbf{A}$ in a fixed coil is represented by $\mathbf{A}$F, where $\mathbf{A}$ is the instantaneous value of the current A, and F is a quantity dependent merely on the coil and the position of the point considered; and (ii.) that the principle of superposition holds, viz.:

$$\mathbf{B} = \mathbf{A}_1 F_1 + \mathbf{A}_2 F_2 + \mathbf{A}_3 F_3,$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are the currents in the three fixed coils and the quantities F are functions of the position of the point corresponding to these fixed coils. The addition must be

made vectorially if the subsidiary fluxes are differently directed.

There is of course no doubt about either of these assumptions if the magnetic circuits associated with the fixed coils pass wholly through non-magnetic and non-conducting media such as air. When the path of the lines of force lies partly through (suitably laminated) iron it is necessary, however, to consider the effects of varying permeability and hysteresis. But any magnetic circuit with which the moving coil is concerned must necessarily contain an air-gap to permit the movement of this coil, and for all dimensions suitable for instruments this air-gap must be such that the reluctance of the iron path is negligibly small compared with that of the gap. The flux densities in the iron under these circumstances can only reach moderate values for which the permeability is fairly constant, while even a considerable change in the permeability will not appreciably affect the magnetic reluctance of the circuit. Similar considerations show that hysteresis cannot produce any sensible effect in magnetic circuits the reluctance of which is almost entirely that of an air path.

We may therefore assume that, however unsymmetrical the windings of the three fixed coils may be, the induction density **B** at any point on the conductors * of the moving coil for any deflexion x follows the law

$$\mathbf{B} = \mathbf{A}_1 F_1 + \mathbf{A}_2 F_2 + \mathbf{A}_3 F_3, \quad . \quad . \quad . \quad . \quad (1)$$

where F_1, F_2, F_3 are three quantities dependent only on x and quite independent of time, while $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are the instantaneous values of the currents in the three coils.

If the current in the moving coil is denoted by $\mathbf{V}$, the product $\mathbf{BV}$ represents the momentary value of the force per unit length acting on the portion of the coil considered. It will also be seen that we can regard the quantities F as line averages for the whole moving coil (for the deflexion x) and that the turning moment acting on the latter may be written

$$\overline{\mathbf{BV}} = F_1\,\overline{\mathbf{A}_1\mathbf{V}} + F_2\,\overline{\mathbf{A}_2\mathbf{V}} + F_3\,\overline{\mathbf{A}_3\mathbf{V}}, \quad . \quad . \quad . \quad (2)$$

where, if the currents are constant $\overline{\mathbf{AV}}$ denotes the product

* Strictly speaking **B** is the component of the flux density perpendicular to both the conductor and the direction of motion.

of the corresponding quantities after taking account of their sign, while if the currents are periodic the expression represents the average value of such product.

Now as the moving coil is not provided with control, if the above quantity is not equal to zero the coil will turn until this is the case. Such a position must exist, since if the coil be turned through 180 degrees in either direction, the torque acting on the coil will be reversed in sign, and thus, as the coil turns, must pass through a zero value in either case. One of these corresponds with a stable, and the other with an unstable, position of equilibrium.

If the second member of (2) be equated to zero, we get an equation for the deflexion x, the solution of which must be independent of the absolute, though not of the relative, magnitudes of the currents **A**, and which must also be unaffected by the magnitude of **V**. For alternating currents the position of balance can only depend on the phase differences of the currents **A** with regard to the voltage **V**, and on the relative magnitudes of these load currents. For direct currents the equilibrium value of x can only depend on the ratios between A_1, A_2, and A_3.

Now in the theory of phasemeters as hitherto given, the following assumptions are made :—

(i) That the currents and voltages vary with time according to the sine law.
(ii) That the functions F are of equal magnitude and vary with the deflexion x according to the sine law.
(iii) That the load currents are balanced between the phases, that is to say, the magnitudes of the currents A_1, A_2, and A_3 are equal.

It is well known that assumptions (i) and (iii) are only true in practice under most exceptional circumstances, while assumption (ii) is only true when the moving coil is of very small dimensions compared with those of the fixed coils and is placed at their common centre. This cannot be the case in an actual instrument. In what follows, we shall not find it necessary to make any of these assumptions, but we shall confine ourselves at present to the case of balanced loads for which the magnitudes of A_1, A_2, and A_3 are equal.

By the *magnitude* A, of any cyclic function $\mathbf{A}$, of a variable t and of period P, we understand that quantity whose square is equal to

$$A^2 = \frac{1}{P}\int_0^P \mathbf{A}^2\, dt.$$

It follows from the foregoing that with balanced loads the condition of equilibrium at the deflexion x is

$$F_1\, \overline{\mathbf{A}_1\mathbf{V}} + F_2\, \overline{\mathbf{A}_2\mathbf{V}} + F_3\, \overline{\mathbf{A}_3\mathbf{V}} = 0, \quad . \quad . \quad . \quad . \quad (3)$$

and is such that the coefficients of F_1, F_2, F_3 are proportional to quantities which merely depend on the phase differences between the currents and the voltage $\mathbf{V}$, and also that if we pass steady currents through all the coils and adjust those through the fixed coils till they are proportional respectively to $\overline{\mathbf{A}_1\mathbf{V}}$, $\overline{\mathbf{A}_2\mathbf{V}}$, $\overline{\mathbf{A}_3\mathbf{V}}$, we shall get the same deflexion x. A given deflexion thus always corresponds with a particular power factor of the load; and in order to calibrate the instrument by means of direct-current tests, it remains to show how to calculate the power factor from the ratios of the direct currents used in order to produce the corresponding deflexion of the instrument.

Now however the instrument may be calibrated, its readings will only be correct when its coils are all connected to the circuit in the particular manner corresponding with the calibration. We must thus adopt for each fixed coil one direction as positive, and, if we distinguish the ends of the coil as positive and negative respectively, we shall understand by a positive current through the coil a current flowing from its positive to its negative end. Moreover, if we consider a current in the mains as positive when flowing in a particular direction along the mains, we have the following equation true at every instant, whatever the law of variation of the alternating currents may be:

$$\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From this it follows that the mean products forming the coefficients in equation (3) must be connected by the relation

$$\overline{\mathbf{A}_1\mathbf{V}} + \overline{\mathbf{A}_2\mathbf{V}} + \overline{\mathbf{A}_3\mathbf{V}} = 0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

That is to say, in the calibration by means of direct currents

we must use three steady currents through the fixed coils such that their algebraic sum is always zero. This can easily be arranged by connecting together all the negative ends of the coils, and afterwards putting two of the coils in parallel through variable resistances, and in series with the third. The two selected for parallel connexion may of course be varied if necessary to calibrate the instrument throughout the scale.

But there is another consequence of (4) the truth of which is also quite independent of any assumption in regard to the variation of the currents with time. Equation (4) can be regarded as a vector equation such that if three vectors are drawn forming a triangle the sides of which are respectively proportional to the magnitudes of the three currents, the angles of this triangle will perfectly represent the phase relations of these three currents. It is also possible to find another vector representing the voltage **V** in both magnitude and phase so that the mean product of any two of the quantities considered is accurately equal to the scalar product of the corresponding vectors *. In all ordinary cases the vector **V** can be regarded as in the same plane as the current vectors, even if the currents vary in a manner widely departing from the sine law; but, in any case, if $\mathbf{V}_p$ is the perpendicular projection of the vector **V** on the plane of the current vectors, we have from elementary vector considerations the following relations between scalar products:

$$\overline{\mathbf{VA}_1}=\overline{\mathbf{V}_p\mathbf{A}_1},\quad \overline{\mathbf{VA}_2}=\overline{\mathbf{V}_p\mathbf{A}_2},\quad \overline{\mathbf{VA}_3}=\overline{\mathbf{V}_p\mathbf{A}_3}.$$

We also have

$$\cos\phi=\cos\alpha\cos\phi_p,$$

where ϕ is the phase-angle between **V** and any current **A**,
ϕ_p is the phase-angle between $\mathbf{V}_p$ and the same current **A**,
and α is the angle between the vectors **V** and $\mathbf{V}_p$.

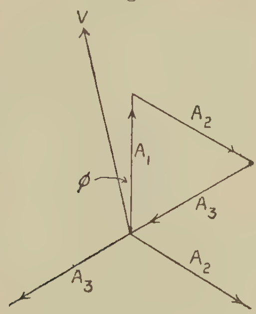
Hence, it will be apparent that in any such equation as (3) or (5) we can regard the vectors **V** and $\mathbf{V}_p$ as interchangeable, and also that if we may neglect the square of the small angle α, we can regard $\cos\phi$ and $\cos\phi_p$ as identical.

* See "The Vector Properties of Alternating Currents and other Periodic Quantities," Proceedings Royal Society, 1897, vol. lxi. p. 455.

Now the vector figure will be as represented in fig. 1.

In the case considered the load currents are equal, hence the vectors **A** will be of equal magnitude, and it follows that any two will differ in phase by 120°. If therefore ϕ is the angle between the vectors **V** and $\mathbf{A}_1$, and if we denote by

Fig. 1.



V and A the magnitudes of the voltage and current vectors, we have

$$\overline{\mathbf{V}_1\mathbf{A}_1} = \mathrm{VA}\cos\phi,$$

$$\overline{\mathbf{VA}_2} = \mathrm{VA}\cos(120+\phi) = -\mathrm{VA}\cos(60-\phi),$$

$$\overline{\mathbf{VA}_3} = \mathrm{VA}\cos(240+\phi) = -\mathrm{VA}\cos(60+\phi).$$

Substituting in (3) we have for balance the condition :

$$\mathrm{F}_1\cos\phi = \mathrm{F}_2\cos(60-\phi) + \mathrm{F}_3\cos(60+\phi). \quad . \quad . \quad (6)$$

If now in the direct-current test we make the steady currents such that

$$-\mathrm{A}_1 = \mathrm{A}_2 + \mathrm{A}_3,$$

and

$$\frac{\mathrm{A}_2}{\mathrm{A}_3} = \frac{\cos(60-\phi)}{\cos(60+\phi)} = \frac{1+3^{\frac{1}{2}}\tan\phi}{1-3^{\frac{1}{2}}\tan\phi}, \quad . \quad . \quad . \quad (7)$$

the position of balance will be the same as in the corresponding alternating current test, when the phase of the current through the moving coil differs from that of the current $\mathbf{A}_1$ by ϕ.

From (7) we have

$$\tan\phi = \frac{1}{3^{\frac{1}{2}}}\,\frac{\mathrm{A}_2-\mathrm{A}_3}{\mathrm{A}_2+\mathrm{A}_3}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

from which ϕ and $\cos\phi$ can be found. If $\cos\phi$ so found is to represent the power-factor of the alternate current load, all

that is necessary is to select for the voltage **V** that between the main conductor carrying the current $\mathbf{A}_1$ and the neutral point of the system, and to apply it through a non-inductive resistance to the moving coil. If, however, instead of this resistance a condenser be used, the voltage selected should be that between the lines carrying $\mathbf{A}_2$ and $\mathbf{A}_3$, since the effect of the condenser will be to cause the moving coil current to be in quadrature with the voltage producing it. The shape of the curve connecting the deflexion x with ϕ or $\cos\phi$ will depend on the nature of the functions F in (6), and these are determined by the structure of the instrument. With actual phasemeters the coils are arranged with the object of making the quantities F of equal magnitude, and as nearly as possible sinuous functions of the deflexion x; and the angle between the planes of any two coils is made equal to the angle representing the phase difference of the currents passing through the coils. Such assumptions are equivalent to putting:

$$F_1 = f\sin x;\quad F_2 = f\sin(x-120);\quad F_3 = f\sin(x+120);$$

and if we substitute these expressions in (6) and reduce, we easily find

$$\sin(\phi + x) = 0,$$

or the deflexion x in degrees is equal to ϕ when measured from an appropriate zero.

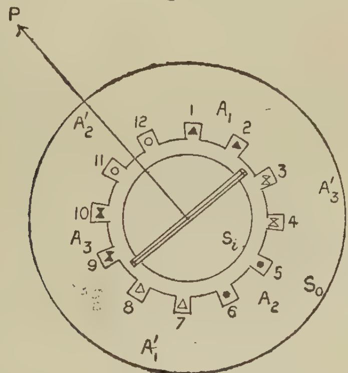
But the accuracy of the instrument as a phasemeter in no way depends upon the fulfilment of these conditions of construction. The introduction of iron into the magnetic circuits, and the use of unsymmetrical winding in the coils, will affect the shape of the curve connecting x and ϕ, but will not prevent x from being an accurate indication of ϕ. Nor is it even desirable that x should be proportional to ϕ. It would be much more useful to arrange to make x fairly proportional to $\cos\phi$. Phasemeters are wanted to measure not ϕ, but $\cos\phi$, or the power-factor, and this for load currents which in practice are always lagging. By making x proportional to ϕ the scale-readings for different power-factors are widely separated where least wanted, that is between power-factors 0·9 and 1·0, and are close together for the values most wanted, for power-factors between 0·6

and 0·8, the values of ϕ for which only differ by 16 degrees. The result is to produce an instrument having a scale the greater portion of which is hardly ever used, while the portion which is most used is small, and the readings undesirably crowded. Assuming bilateral symmetry in the moving coils, a change in x of 180 degrees must necessarily correspond with a reversal of the currents, or with a change in ϕ of 180 degrees, but there is no need for the range of x for leading currents to be as great as that for lagging currents, and it is advantageous to get a large range of x for the values of $\cos\phi$ most needed.

Tests on Phasemeters having Iron Cores.

The foregoing theory has been fully tested by the writer in reference to three new instruments each provided with

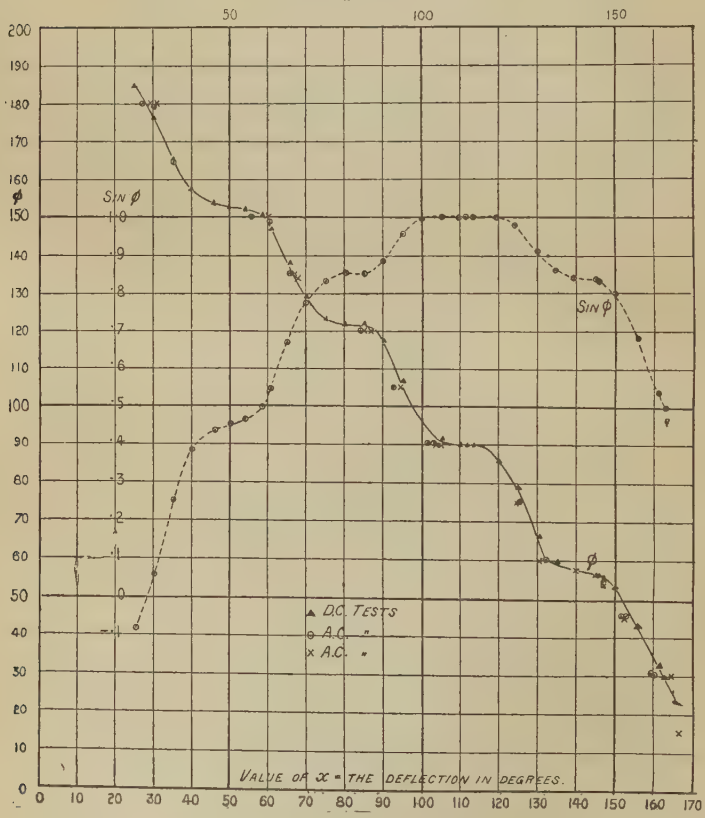
Fig. 2.



iron-cored magnetic circuits. The structure of one of these is sufficiently indicated in fig. 2. The iron parts are built up of thin circular stampings arranged in two fixed cylindrical coaxial blocks S_0 and S_i. The outer ones S_0 are provided with twelve slots on their inner periphery arranged symmetrically at angular intervals of 30°. In each pair of diametrically opposite slots a coil of 8 turns is wound, and adjoining coils are series connected in pairs to form the three fixed coils A_1, A_2, A_3. When positive currents are sent through the coils the currents flow downwards through the

slots 1, 2, 5, 6, 9, and 10, and upwards through the remaining slots. This is indicated in the figure by the use of full signs and corresponding outline signs. The moving coil is rectangular in shape and turns with its sides in the circular air-gap about the common axis of the system. To this moving coil a pointer P is attached, which reads on a fixed scale of degrees. For any given values of the currents in the fixed coils, the space distribution of induction-density in the gap tends to vary rapidly at the slots and to be nearly

Fig. 3.



uniform in the space between one slot and the next. To overcome this effect, which was foreseen, successive stampings had been slightly sheared circumferentially, so that the line

of centres for any slot, instead of being quite parallel to the axis, was slightly spiral in reference to it. The stepped nature of the curve connecting x and ϕ shows that the effect in question was only partially neutralized. But this tends to make the correspondence of the direct-current and alternating-current tests the more convincing. It will be noted that the angular period of the steps in the curve is 30°, the same as the angular interval between successive slots.

The results are all set forth in fig. 3. The direct current tests were 33 in number, and are indicated on the curve by delta signs. About 40 tests were taken with alternating currents, half of them with a non-inductive resistance in series with a moving coil, and the rest with a condenser substituted for this resistance. These tests are separately marked on the curve with rings and crosses. The observations are not all plotted, as several of the points were too close to be distinguishable, but a fair selection is given.

A few typical observations with direct currents are given in Table I. and will be sufficient to show the conditions of the test. The currents A_2, A_3 are in amperes, x is the observed deflexion of the pointer in degrees, and ϕ is calculated from formula (8).

TABLE I.

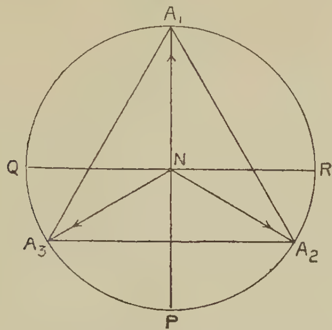
A_2 ...	+2·2	+1·1	+·42	−·5	−2·8	−5·2	−6·2	−8·9	−9·3	−9·5
A_3 ...	2·7	7·0	7·2	7·45	7·6	10·9	7·2	8·2	4·3	0·5
x ...	30	40	50	60·5	70	80	100	119·7	139	161
ϕ ...	176·4	157·3	153·0	146·7	128·9	121·7	97·4	85·9	57·3	32·6

The value of ϕ so calculated corresponds in the alternating tests with the phase difference between the current A_1 through the first fixed coil, and the moving coil current. For both the direct and alternating current tests, the deflexion x of the instrument could be read to, and appeared reliable to, one-tenth of a degree. The value of ϕ was determined with almost equal accuracy for the direct current tests, the conditions for which were simple and could easily be kept constant, while the value of ϕ depended merely on the ratio

of the readings of two excellent instruments of the permanent magnet type. But ϕ could not be determined with such accuracy in the alternating-current tests, for several reasons. The testing conditions with running machinery involved cannot easily be kept very constant; alternating-current instruments are not so reliable as those for direct currents; while ϕ has to be determined from the readings of three instruments instead of from those of two. The alternator used was provided with six terminals arranged for the supply of two-phase and three-phase current. The three-phase terminals were connected to the positive ends of the fixed coils through ammeters and banks of lamps, the negative ends being connected together to form the neutral point. The currents were adjusted to approximate equality, but only one good hot-wire ammeter of the range required was available, and this was put in the A_1 circuit. At first the wattmeter method of testing ϕ was used, a Mather-Duddell instrument being utilized, with its current-coil placed in the A_1 circuit. The pressure circuit of this instrument was put in parallel with the moving-coil circuit of the phasemeter, and with a Weston dynamometer voltmeter. The current circuits were not altered during the tests. The six terminals of the alternator, together with the neutral point, formed seven points, any two of which could be selected for application to the pressure circuits. In this way many different values of ϕ could be obtained, and a new set of tests could be made with a condenser substituted for the non-inductive resistance in series with the moving coil of the phasemeter. Of course, when interpreting the results in the latter case, the value of ϕ as deduced from the wattmeter readings had to be changed by 90 degrees to get the corresponding phase-difference between the moving coil current and the current A_1. The wattmeter tests were made on the instrument shown in fig. 3, but not with the particular arrangement of coils there indicated; and the results found were in close agreement with the curve between x and ϕ as determined by the direct-current method. The wattmeter method proved laborious when a large number of tests were needed; and it was found sufficiently accurate, and much quicker and simpler, to assume that the six terminals of the alternator

gave voltages in the precise phase relation indicated by the geometrical properties of fig. 4, in which the points A_1, A_2,

Fig. 4.



A_3 are 120° apart and correspond with the current circuits, while the points A_1P and QR are the extremities of two perpendicular diameters. The centre of the circle, N denotes the neutral point. The current vectors are NA_1 NA_2, and NA_3. The vector NA_1 represents zero phase, while the angle between NA_1 and the vector joining the two points selected for the voltage applied to the moving coil circuit is ϕ_1 the phase of this voltage. When a non-inductive resistance was used with the moving coil, ϕ_1 was the same as ϕ the phase of the moving coil current ; but when a condenser was used, it was found that ϕ_1 had to be diminished by 90°, or increased by 270°, in order to get ϕ the value to

Table II.

With Resistance.			With Condenser.			
V.	$\phi_1=\phi$.	x.	V.	ϕ_1.	ϕ.	x.
A_1A_3	150	60	A_3A_2	− 90	180	27
RP	135	67	NA_2	−120	150	55·5
NA_3	120	87	A_1A_2	−150	120	84·2
RA_3	105	94·5	RA_2	165	75	125
A_2Q	75	124	A_2P	120	30	158·7

plot with x. A few of the observations taken are given in Table II. Particulars are given in the column V of

the voltage points chosen for the pressure circuits. ϕ_1 was not a measured quantity, but one calculated on certain assumptions from fig. 4. Any error in these assumptions may serve to explain the slight divergence of some of the points obtained with the alternating-current tests from the curve deduced from the direct-current measurements, but the general agreement observed between the two sets of tests can only be made closer by putting any such error right.

Another complete set of tests was made on this instrument with one of the coils, A_2, reversed, so as to make the instrument unsymmetrical. The end of the coil A_2, formerly considered positive, was thus made negative. The direct-current test yielded a curve between x and ϕ of a most extraordinary character, yet a number of tests made by the alternating-current wattmeter method yielded points which plotted perfectly on the curve so found. The other two instruments referred to were also thoroughly tested, and with equally satisfactory results. Their structure differed from that indicated in fig. 2 merely in the arrangement and number of the fixed coils. There were three such coils in each case. In one instrument these were wound through six holes, stamped in the laminations of the outer stator S_0, and arranged symmetrically at angular intervals of 60°. In the other instrument, the coils were wound in six slots stamped symmetrically in the periphery of the inner stator S_i. In either case, the curve found between x and ϕ was of the same general character as that shown in fig. 3, except that, as the windings were in these instruments 60 degrees apart, it was found that the flat parts of the curve covered a greater angular range, and that the angular period of the steps was 60° instead of 30°. In each instrument the direct and alternating current tests were found to be in close agreement, and the direct-current calibration of one of them was found to coincide with a large number of alternating-current tests taken on the same instrument more than eight months previously.

It therefore appears clear, both from theory and experiment, that the accuracy of these instruments on balanced loads in no way depends upon the structure of the coils, or upon the presence or absence of iron, or upon the mode of variation

of the currents. The theory given of the four-circuit instrument described will be seen to be equally applicable whichever of the two systems of coils is fixed, so that if the fixed system consists of a single coil and the movable system consists of three relatively fixed coils, the same theory applies.

Three-Circuit and Monophase Instruments.

If the three-coil system is reduced to a two-coil system, as in most actual phasemeters, only a slight modification of the theory is required, and the instrument can still be tested by direct-current methods. All that is needed is to put $F_1=0$ in equations 1, 2, 3, and 6. Equations 4, 5, 7, and 8 still hold good, and in the direct-current calibration there are only two currents to consider, and these may have any values. For a single-phase instrument, which merely differs from a multi-phase instrument in that its two-coil system is parallel connected, and the branch circuits made of different inductive properties, a similar theory applies. For a fixed frequency and wave form there will be a constant ratio between P and Q, the two alternating currents in the two-coil system, and a constant phase difference α between these currents. If, then, ϕ is the phase difference between the alternating current in the single-coil system and that through P, the corresponding direct currents through the coils P and Q which will produce the same deflexion will be

$$A_2=P\cos\phi,$$
$$A_3=Q\cos(\phi-\alpha).$$

If therefore α and the ratio $P:Q$ are known, it is possible to determine ϕ for any ratio $A_2:A_3$ of the currents producing the observed deflexion.

Power-factor for Unbalanced Loads.

It has been assumed throughout the foregoing theory that the load currents are balanced ; a most desirable condition in practice, but one rarely attained. So far as the writer is aware, the behaviour of phasemeters on unbalanced loads has not previously been investigated. There is a fairly general impression that they are inaccurate under these circumstances, and for this phasemeters have been blamed ; though the fault rather lies with those who assume that an

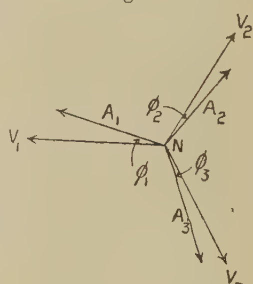
instrument, which is only affected by a few of the conditions of a circuit, can be reasonably expected to measure the average of such conditions.

Now, in the first place, there is not even any generally accepted definition of the power-factor of a three-phase circuit when the load currents are unbalanced. There are really three circuits, consisting of each line current and the voltage between this line and the neutral point. There are thus three power-factors, one for each circuit. Fortunately the voltages, being fixed by the generator, can always be regarded as equal and in symmetrical phase relation; otherwise the definition of the average power-factor of the load would be most complicated. But we shall take the following definition of $\cos\phi$ for unbalanced loads:—

$$\cos\phi = \frac{A_1\cos\phi_1 + A_2\cos\phi_2 + A_3\cos\phi_3}{A_1 + A_2 + A_3}; \quad . \qquad (9)$$

where A_1, A_2, A_3 are the magnitudes of the three load currents, and ϕ_1, ϕ_2, ϕ_3 are the phase angles between these currents and the corresponding voltages to the neutral point. This formula for $\cos\phi$ merely amounts to defining the volt-amperes of an unbalanced three-phase circuit as being the product of the voltage to the neutral point and the sum of the three-line currents. The relations of the different quantities are indicated in fig. 5, in which the voltage vectors are supposed to be of equal magnitude and symmetrically spaced. If this were not the case, we could still determine a neutral point such that the voltages between it and each line would be equal; and the definition (9) would still hold good provided the phase angles ϕ_1, ϕ_2, ϕ_3 were reckoned from the corresponding voltages to the neutral point so determined.

Fig. 5.



Symmetrical Six-Circuit Phasemeter.

Now the most likely form of phasemeter to be accurate on three-phase circuits with unbalanced loads, is naturally one in which each system of coils consists of three similar and

symmetrically arranged coils, having a common line of symmetry, and with their planes fixed at 120 degrees from each other. One of these sets of coils must be actuated by the load currents, and the other set by the line voltages. With such an instrument, we must have three equations like (1) for the induction densities at the three moving coils, or we have :—

$$\left.\begin{aligned}\mathbf{B}_1&=\mathbf{A}_1F_{11}+\mathbf{A}_2F_{21}+\mathbf{A}_3F_{31}\\ \mathbf{B}_2&=\mathbf{A}_1F_{12}+\mathbf{A}_2F_{22}+\mathbf{A}_3F_{32}\\ \mathbf{B}_3&=\mathbf{A}_1F_{13}+\mathbf{A}_2F_{23}+\mathbf{A}_3F_{33}\end{aligned}\right\}; \quad . \quad . \quad . \quad (10)$$

where in the quantities F the first suffix refers to the fixed coil, assumed to take one of the load currents, and the second suffix to the moving coil. Thus F_{23} means the line average effective induction density due to a unit current in the second fixed coil at the conductors of the third moving coil when the deflexion of the pointer is x. Now under the symmetrical conditions assumed we have :—

$$\left.\begin{aligned}F_{11}&=F_{22}=F_{33}=P\\ F_{23}&=F_{31}=F_{12}=Q\\ F_{32}&=F_{13}=F_{21}=R\end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (11)$$

but Q and R are not necessarily the same. Hence we have :

$$\left.\begin{aligned}\mathbf{B}_1&=\mathbf{A}_1P+\mathbf{A}_2R+\mathbf{A}_3Q\\ \mathbf{B}_2&=\mathbf{A}_1Q+\mathbf{A}_2P+\mathbf{A}_3R\\ \mathbf{B}_3&=\mathbf{A}_1R+\mathbf{A}_2Q+\mathbf{A}_3P\end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (12)$$

Now if the three moving coils are star connected, and their three free ends joined through three equal non-inductive resistances to the three mains, the currents through the three coils will be represented by $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, the voltages between the neutral point and the corresponding mains. The condition for balance constituting an equation for x is :

$$\overline{\mathbf{V}_1\mathbf{B}_1}+\overline{\mathbf{V}_2\mathbf{B}_2}+\overline{\mathbf{V}_3\mathbf{B}_3}=0,$$

or

$$P\Sigma\,\overline{\mathbf{V}_1\mathbf{A}_1}+Q\Sigma\,\overline{\mathbf{V}_1\mathbf{A}_3}+R\Sigma\,\overline{\mathbf{V}_1\mathbf{A}_2}=0.$$

Now referring to fig. 5, and remembering that the voltage vectors are of equal magnitude, and are symmetrically spaced,

we have for the different angles required :—

$$\widehat{V_1A_1}=\phi_1,\quad \widehat{V_1A_2}=\phi_2+120,\quad \widehat{V_1A_3}=\phi_3+240,\quad \&c.,\quad \&c.$$

If we substitute in the above equations, using such relations as

$$\overline{V_1A_3} = VA_3 \cos \widehat{V_1A_3},$$

and divide out by V we get,

$$\left.\begin{array}{l} P\Sigma A_1 \cos \phi_1 + Q\Sigma A_3 \cos (\phi_3+240) + R\Sigma A_2 \cos (\phi_2+120) = 0 \\ \text{or} \\ \quad P\Sigma A_1 \cos \phi_1 = Q\Sigma A_1 \cos (\phi_1+60) + R\Sigma A_1 \cos (60-\phi_1) \end{array}\right\} \quad (13)$$

Let us define two quantities C and S such that

$$\left.\begin{array}{l} C = A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 \\ S = A_1 \sin \phi_1 + A_2 \sin \phi_2 + A_3 \sin \phi_3 \end{array}\right\} \quad . \quad . \quad (14)$$

We then find on expanding (13) that

$$PC = Q\frac{C-3^{\frac{1}{2}}S}{2} + R\,\frac{C+3^{\frac{1}{2}}S}{2}; \quad . \quad . \quad . \quad (15)$$

but from (14) we get

$$C^2+S^2=A_1^2+A_2^2+A_3^2+2\Sigma A_2A_3 \cos (\phi_2-\phi_3) \quad . \quad (16)$$

Now we only propose to consider the case of a moderate want of balance for which the angles ϕ_1, ϕ_2, ϕ_3 differ from their mean value by only small amounts whose squares and products can be neglected compared with unity. Under these conditions, the cosine of the difference of any two of these angles can be considered unity, and (16) reduces to :—

$$C^2+S^2=(A_1+A_2+A_3)^2,$$

but from (9) and (14)

$$\therefore \qquad \left.\begin{array}{l} C=(A_1+A_2+A_3) \cos \phi \\ S=(A_1+A_2+A_3) \sin \phi \end{array}\right\} . \quad . \quad . \quad . \quad (17)$$

If now we substitute in (15) and simplify we obtain :

$$P \cos \phi = Q \cos (60+\phi) + R \cos (60-\phi), \quad . \quad (18)$$

and substituting from (11)

$$P = F_{11}, \quad Q = F_{31}, \quad R = F_{21},$$

we have

$$F_{11} \cos \phi = F_{21} \cos (60-\phi) + F_{31} \cos (60+\phi);$$

an equation exactly similar to (6) and showing that the deflexion is the same as on a balanced load of power-factor $\cos\phi$. Also it appears that for balanced loads only one of the moving coils need be used if it is connected to its appropriate voltage. The deflexion will be the same whichever moving coil is connected to the circuit. For unbalanced loads it is, however, necessary to have all the coils so connected if the instrument is to read $\cos\phi$ as defined in (9).

To calibrate such an instrument by the direct-current method, the process already described may be applied, the three coils of one system, but only one coil from the second, being used for the test.

To confirm this, tests were made on a commercial instrument of the symmetrical six-circuit type, manufactured by Messrs. Everett, Edgcumbe & Co. The moving system consisted of three star-connected coils designed for shunt connexion with the mains. A constant current was passed through one of the three fixed coils, various small measured currents were passed through the moving coils in accordance with the method above described, and the value of ϕ corresponding with each observed deflexion x was calculated by means of formula (8). The scale of the instrument was so constructed that with alternating currents the deflexion x was a direct measure of the phase angle. The observed value of x was found to agree with ϕ when a particular fixed coil was used for the current circuit. Two additional sets of tests were taken using the remaining fixed coils in succession. In one of these cases ϕ had to be increased by 120 degrees, and in the other diminished by the same amount, to make the converted value of ϕ agree with x. The tests made were comprehensive, each set consisting of about 35 determinations ranging over the whole scale. The three curves obtained by plotting the (converted) value of ϕ with x were nearly straight lines, and were nearly the same; but they were by no means absolutely so, and the differences amounted in some cases to as much as ten degrees. The arrangement of the coils was thus not symmetrical in the mathematical sense assumed in the above proof, and it is difficult to estimate to what extent the indications of the instrument are dependent upon the balance, or lack of balance, of the load currents.

To test such an instrument thoroughly with alternating currents on unbalanced loads would be so complicated and laborious that it is unlikely that such a test ever has, or will be, made. There would be needed no fewer than *nine* electrical instruments (three ammeters, three voltmeters, and three wattmeters) besides a number of troublesome adjustments and subsequent calculations. The labour of the nine or more observers would be of no avail unless all the readings for each test were taken accurately and simultaneously.

Symmetrical Five-Circuit Phasemeter.

Before leaving the consideration of symmetrical instruments, it may be well here to state the result of an examination of the case of one having five coils instead of six.

If the first coil of the moving system is omitted, making $F_{11} = F_{21} = F_{31} = 0$ in equations (10), (11), and (12), it will be found on making the changes which necessarily follow that equation (13) becomes

$$P[C - A_1 \cos \phi_1] = Q\left[\frac{C - 3^{\frac{1}{2}}S}{2} - A_3 \cos(\phi_3 + 60)\right] + R\left[\frac{C + 3^{\frac{1}{2}}S}{2} - A_2 \cos(\phi_2 - 60)\right];$$

now in the above equation the coefficient of P is equal to the sum of the coefficients of Q and R (see (21) and (22) below), and we can thus always find an angle χ such that

$$P \cos \chi = Q \cos(60 + \chi) + R \cos(\chi - 60),$$

and hence by (18) the reading of the phasemeter will be that due to a balanced load of power-factor $\cos \chi$.

With the help of (17) and relations proved below (21), (22), and (25), it will be found that

$$\frac{3A \cos \phi - A_0 \cos \theta}{\cos \chi} = \frac{3A \cos(\phi + 60) - A_0 \cos(\theta + 60)}{\cos(\chi + 60)} = \frac{3A \cos(\phi - 60) - A_0 \cos(\theta - 60)}{\cos(\chi - 60)},$$

and assuming that θ and χ exceed ϕ by small amounts θ_0 and

χ_0 respectively, it will be found that the above equations involve the relation

$$\chi_0 = \tfrac{1}{3}\theta_0, \qquad \ldots\ldots\ldots \quad (19)$$

or the phasemeter on an unbalanced load will read, instead of the true power-factor $\cos\phi$, the value

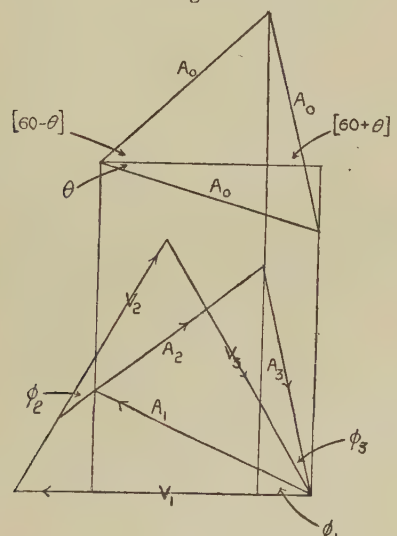
$$\cos(\phi + \tfrac{1}{3}\theta_0),$$

where θ_0 has the value given by (31) below.

Phase Error due to Unbalanced Loads.

If the instrument is unsymmetrical and of less complicated construction than those just considered, the next best form

Fig. 6.



will be one having a three-circuit system for the currents, and a single-circuit system for the volts, like the one first described. Referring to equation (3) and assuming that the load currents are no longer balanced, we see that instead of equation (6) we shall have as the condition for balance :

$$F_1A_1\cos\phi_1 = F_2A_2\cos(60-\phi_2) + F_3A_3\cos(60+\phi_3). \quad (20)$$

Now the vectors of fig. 5 are the same as those of fig. 6, in

which latter the current vectors, and also the voltage vectors, are drawn so as to form closed triangles. Bearing in mind that the voltage triangle is equilateral, it is readily seen that

$$A_1 \cos \phi_1 = A_2 \cos (60 - \phi_2) + A_3 \cos (60 + \phi_3) \; ; \quad (21)$$

and as we have, whatever θ may be,

$$\cos \theta = \cos (60 - \theta) + \cos (60 + \theta),$$

we see that we can always determine an angle θ from the consistent equations

$$A_0 = \frac{A_1 \cos \phi_1}{\cos \theta} = \frac{A_2 \cos (60 - \phi_2)}{\cos (60 - \theta)} = \frac{A_3 \cos (60 + \phi_3)}{\cos (60 + \theta)} \quad . \quad (22)$$

and that for such value of θ we have from (20)

$$F_1 \cos \theta = F_2 \cos (60 - \theta) + F_3 \cos (60 + \theta) \quad . \quad . \quad (23)$$

If we compare this equation with (6), we see that if the instrument has been correctly calibrated for power-factor on balanced loads, the reading of the instrument will, for the unbalanced load assumed, be $\cos \theta$ as determined from (22), and that this value is absolutely *independent of the quantities* F resulting from a particular structure of the instrument. In other words, if a number of such phasemeters be connected to the mains in precisely the same manner, and if these instruments have all been calibrated for balanced loads, each of the instruments will indicate the same reading whatever the nature of the load, and however different the various instruments may be as regards internal structure. But it does not follow that this common reading correctly gives the power-factor of the load.

There are certain relations between A_1, A_2, A_3, ϕ_1, ϕ_2, ϕ_3, resulting from the geometrical properties of fig. 6, but it cannot in general be true that the value of $\cos \theta$ determined from (22) is the same as that of $\cos \phi$ as defined in (9). This can readily be seen by putting $A_3 = 0$, in which case θ must be 30 degrees while ϕ may have any value. It may all the same be the fact for small divergences of the load currents from A their mean value, that $\cos \theta$ and $\cos \phi$ differ

but very slightly. We thus want to find θ_0 where

$$\theta = \phi + \theta_0,$$

since if θ_0 is small, the error, considered as a fraction of the true power-factor, made in taking $\cos\theta$ (the reading) to represent $\cos\phi$ (the true power-factor) will be:

$$\frac{\theta_0}{\cos\phi}\frac{d\cos\phi}{d\phi} = -\theta_0\tan\phi, \quad . \quad . \quad . \quad . \quad (24)$$

and θ_0 will correspond with the phase error of an ordinary wattmeter due to inductance in its pressure-coil. For equal phase errors the two instruments will read erroneously by precisely the same percentage on loads of the same power-factor. The question to determine is whether θ_0, for a moderately unbalanced load, is sufficiently great to render the phasemeter unsatisfactory.

While θ_0 is small the instrument remains a phasemeter, but when θ_0 becomes large the instrument tends to indicate the want of balance of the currents, rather than the average power-factor of the load.

If we put each of the ratios in (22) equal to A_0 and consider fig. 6, it will be noticed that A_0 is the length of each of the sides of an equilateral triangle drawn so that these sides make angles θ with the sides of the triangle representing the voltages, and so that the vertices lie on lines perpendicular to the voltage V_1, and passing through the angular points of the current triangle. For balanced loads the A_0 triangle must coincide with the current triangle, and for moderate variations from balance we may put:—

$$A_1 = A(1+\epsilon_1), \quad A_2 = A(1+\epsilon_2), \quad A_3 = A(1+\epsilon_3),$$
$$A_0 = A(1+\epsilon_0),$$

where A is the mean value of the load currents, so that

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0,$$

and where we may regard all the quantities ϵ as small fractions whose squares and products may be neglected compared with unity.

Since when the load currents are equal $\phi_1=\phi_2=\phi_3$, the above assumptions necessarily imply that if

$$\phi_1=\chi+\theta_1,\quad \phi_2=\chi+\theta_2,\quad \phi_3=\chi+\theta_3,$$

where χ is the mean value of the quantities ϕ, so that

$$\theta_1+\theta_2+\theta_2=0,$$

we may also neglect the squares and products of the quantities θ.

Now, if we substitute in (9) and simplify, neglecting squares and products of small quantities, we easily find

$$3\cos\phi=3\cos\chi+(\epsilon_1+\epsilon_2+\epsilon_3)\cos\chi-(\theta_1+\theta_2+\theta_3)\sin\chi,$$

or, using the above relations, we have

$$\cos\phi = \cos\chi,$$

or ϕ is the mean value of ϕ_1, ϕ_2, ϕ_3.

Collecting formulæ we thus have

$$\left.\begin{array}{lll} \phi_1=\phi+\theta_1, & \phi_2=\phi+\theta_2, & \phi_3=\phi+\theta_3, \\ A_1=A(1+\epsilon_1), & A_2=A(1+\epsilon_2), & A_3=A(1+\epsilon_3)\,; \\ \quad\theta=\phi+\theta_0, & \theta_1+\theta_2+\theta_3=0, & \\ \quad A_0=A(1+\epsilon_0), & \epsilon_1+\epsilon_2+\epsilon_3=0. & \end{array}\right\} \quad . \quad . \quad (25)$$

Now certain relations can be found between the quantities ϵ and θ for the current triangle shown in fig. 6, by equating the ratios of the sides to the sines of the opposite angles and reducing.

These relations prove to be

$$\left.\begin{array}{ll} 3^{\frac{1}{2}}\,\epsilon_1 = \theta_2-\theta_3, & -3^{\frac{1}{2}}\,\theta_1 = \epsilon_2-\epsilon_3, \\ 3^{\frac{1}{2}}\,\epsilon_2 = \theta_3-\theta_1, & -3^{\frac{1}{2}}\,\theta_2 = \epsilon_3-\epsilon_1, \\ 3^{\frac{1}{2}}\,\epsilon_3 = \theta_1-\theta_2, & -3^{\frac{1}{2}}\,\theta_3 = \epsilon_1-\epsilon_2. \end{array}\right\} \quad . \quad . \quad (26)$$

From these equations it can be shown, with the help of the last relations of (25) and equivalent relations such as

$$2\,\epsilon_2\epsilon_3=\epsilon_1^2-\epsilon_2^2-\epsilon_3^2,$$

that

$$\left.\begin{array}{c} \epsilon_1^2+\theta_1^2=\epsilon_2^2+\theta_2^2=\epsilon_3^2+\theta_3^2=2\,\epsilon^2, \\ 3\,\epsilon^2=\epsilon_1^2+\epsilon_2^2+\epsilon_3^2\,; \end{array}\right\} \quad . \quad . \quad . \quad (27)$$

where

or ϵ is the square root of mean square of the quantities ϵ_1, ϵ_2, ϵ_3, and may be said to measure the extent to which the load-currents are "out of balance." For small values, ϵ is essentially the same as the arithmetic mean of ϵ_1, ϵ_2, and ϵ_3. Since algebraically the sum of these quantities is zero, it follows that the greatest is equal numerically to the sum of the other two, and thus ϵ is two thirds of the greatest of the quantities ϵ_1, ϵ_2, ϵ_3.

Now, if we equate to A_0 each of the three ratios of (22) and multiply up and reduce, we obtain the three equations

$$\left.\begin{aligned} \epsilon_0 \cos\phi - \theta_0 \sin\phi &= \epsilon_1 \cos\phi - \theta_1 \sin\phi, \\ \epsilon_0 \cos(\phi-60) - \theta_0 \sin(\phi-60) &= \epsilon_2 \cos(\phi-60) - \theta_2 \sin(\phi-60), \\ \epsilon_0 \cos(\phi+60) - \theta_0 \sin(\phi+60) &= \epsilon_3 \cos(\phi+60) - \theta_3 \sin(\phi+60). \end{aligned}\right\} \quad (28)$$

If we subtract the third of these equations from the second and simplify with the aid of (25) and (26), we get

$$\epsilon_0 \sin\phi + \theta_0 \cos\phi = -[\epsilon_1 \sin\phi + \theta_1 \cos\phi]. \quad . \quad . \quad (29)$$

By squaring this equation and adding the square of the first of (28) we get

$$\epsilon_0^2 + \theta_0^2 = \epsilon_1^2 + \theta_1^2 = 2\,\epsilon^2 \text{ by } (27)\,; \quad . \quad . \quad . \quad (30)$$

or θ_0 is necessarily less than $\epsilon 2^{\frac{1}{2}}$.

If we choose two angles β_0 and β_1 such that

$$\epsilon_0 = \theta_0 \tan\beta_0, \qquad \epsilon_1 = \theta_1 \tan\beta_1,$$

we can use (30), the first equation of (28), and (29) to prove

$$\sin(\beta_0 - \phi) = \sin(\beta_1 - \phi),$$

$$\cos(\beta_0 - \phi) = -\cos(\beta_1 - \phi)\,;$$

these simultaneous equations involve the relation

$$\beta_0 - \phi + \beta_1 - \phi = \pi$$

or

$$\beta_0 = \pi + 2\,\phi - \beta_1.$$

But from the above we have

$$\left.\begin{aligned} &\theta_0 = \epsilon 2^{\frac{1}{2}} \cos\beta_0, \\ \text{or} \quad &\theta_0 = -\epsilon 2^{\frac{1}{2}} \cos(2\,\phi - \beta_1), \\ \text{or} \quad &\theta_0 = -[\theta_1 \cos 2\,\phi + \epsilon_1 \sin 2\phi], \end{aligned}\right\} \quad . \quad . \quad (31)$$

where

$$\tan\beta_1 = \frac{\epsilon_1}{\theta_1} = -3^{\frac{1}{2}}\frac{\epsilon_1}{\epsilon_2-\epsilon_3} = +3^{\frac{1}{2}}\frac{\epsilon_2+\epsilon_3}{\epsilon_2-\epsilon_3},$$

and the fractional error made in reading $\cos\phi$, which by (24) is $-\theta_0\tan\phi$, becomes

$$\frac{\Delta\cos\phi}{\cos\phi} = \epsilon 2^{\frac{1}{2}}\cos(2\phi-\beta_1)\tan\phi, \quad . \quad . \quad . \quad (32)$$

in which the quantities ϵ and β_1 are determined solely by the divergences of the load-currents A_1, A_2, A_3 from their arithmetical mean value.

The same formula can be obtained by equating any two of the ratios (22) without bringing the quantities A_0 and ϵ_0 into the equations, but the working is not any shorter and the information yielded is less.

It will be apparent, on inspection of the error formula (32), that the phasemeter may give very erroneous readings when the load-currents are badly out of balance. This can be seen best by considering a numerical case. Suppose the three load-currents are 21, 22, and 17 amperes. The mean current is 20, and the errors ϵ_1, ϵ_2, ϵ_3 from the mean are 5, 10, and 15 per cent. respectively. It follows that ϵ is 10·8 per cent. and $\epsilon 2^{\frac{1}{2}}$ is 15·3 per cent. Disregarding for the moment the factor $\cos(2\phi-\beta_1)$, the above error has to be multiplied by $\tan\phi$ to get the percentage error in reading $\cos\phi$. The error will thus be much reduced on circuits of high-power factor, but for power-factors below 0·71 the value of $\tan\phi$ becomes greater than unity, and the percentage error in reading $\cos\phi$ will be correspondingly increased, though it must be noted that if the scale is graduated to read $\cos\phi$ the absolute error of the reading in the case assumed is never greater than $0·153\sin\phi$.

Now, a circuit having three currents proportional to those assumed would be considered badly out of balance, and as a rule a much better state of things obtains, yet the conditions instanced are quite possible in practice. The influence of the factor $(\cos 2\phi-\beta_1)$ must also be considered. This factor, though it must always decrease the magnitude of the error,

can alter its value in a striking manner, and is the quantity which determines whether the instrument reads high or low. The quantity ϵ in (32) depends merely on the average square of the fractional divergences ϵ_1, ϵ_2, ϵ_3 of the load-currents A_1, A_2, A_3 from their mean value. The quantity β_1, on the other hand, does not depend upon ϵ but upon the ratios between ϵ_1, ϵ_2, ϵ_3. Whatever the value of ϵ, β_1 may have any value in accordance with these ratios, so that $\cos(2\phi-\beta_1)$ varies sinuously with β_1 and will be numerically equal to unity for a particular relation between β_1 and ϕ. Even for the same three currents in the mains, β_1 may have no less than six values, since we can select any of the three currents for the line A_1, and after this selection is made we may choose either of the two remaining currents for the line A_2. The six possible values of β_1 for the currents assumed in the above case can be shown to be $\pm 19°$, $\pm 41°$, and $\pm 79°$, and the percentage errors for these values of β_1 and for a value of ϵ equal to 10 per cent. are shown in Table III. for loads of different power-factor. Another column is added for the case $\beta_1=0$, and in the last column the maximum error $\epsilon 2^{\frac{1}{2}} \tan\phi$ is shown.

TABLE III.—Percentage Error of Phasemeters (for a load 10 per cent. out of balance).

Power-factor.	Values of β_1 in degrees.							Maximum.
	−79.	−41.	−19.	0.	+19.	+41.	+79.	
1·0 ...	0	0	0	0	0	0	0	0
·9 ...	6·1	6·7	5·7	4·2	2·2	− ·3	−4·4	6·9
·8 ...	10·6	8·9	6·1	3·0	− ·5	−4·5	−9·4	10·6
·7 ...	14·1	9·2	4·3	− ·3	−5·0	−9·6	−14·2	14·4
·6 ...	16·9	8·0	·8	−5·3	−10·9	−16·0	−18·9	18·9
·5 ...	18·4	4·6	−4·6	−12·2	−18·4	−23·1	−23·6	24·5

It will be seen that the error may be quite serious. Thus a 14 per cent. error on a load of power-factor 0·7 means that the instrument may read either 0·8 or 0·6 according as the

error increases or decreases the reading. Of course it must be remembered that the error is always proportional to ϵ, which may be called the "out-of-balance" of the load; so that if this out-of-balance is 2 per cent. instead of 10 per cent., all the above numbers must be divided by 5. In all cases the reading for a power-factor $\cos\phi$ fluctuates, according to the way the load is out of balance, between the values

$$\cos\phi \pm \epsilon 2^{\frac{1}{2}} \sin\phi,$$

so that for a load 2 per cent. out of balance the reading of the power-factor cannot vary from the true value by more than $\pm\cdot 028$ as extreme limits. The above formula applies to an instrument having three coils in one system and a single coil in the other and the error is independent of the structure of the instrument. The error is largely controlled by the value of $\tan\phi$, where ϕ is the angle which we have assumed to represent both the power-factor, and the phase-difference of the moving-coil current, in reference to the current in one of the fixed coils. As any voltage of the multi-phase system may be chosen for the moving coil, provided the instrument has been correspondingly calibrated, it might at first sight appear possible to select such a voltage for the moving system as to make the value of $\tan\phi$ small for the particular power-factors the instrument is most required to read, and thus render the error under practical conditions negligible on unbalanced loads. A careful examination will, however, show that this is not the case.

As already shown, an instrument having three current-coils and three voltage-coils, all symmetrically arranged, will read correctly whether the load-currents are balanced or not. But such perfect symmetry is easier to assume in a mathematical investigation than to ensure in the structure of so complicated an instrument. If one of the moving coils is dispensed with, as in the other symmetrical case considered, the instrument is simpler to construct and its symmetry easier to secure. Such a five-circuit instrument is not quite accurate on unbalanced loads; but (19) shows that the error is only one-third as great as in the four-circuit instrument to which Table III. applies; and it will be seen that for all

load-currents such as are likely to occur in practice, the error is small enough in this case to be neglected. The six-circuit instrument consists essentially of three single-phase instruments combined into one. The best solution of the phasemeter problem would no doubt consist of three single-phase instruments used one on each circuit. There are difficulties, not yet overcome, in constructing good instruments for single-phase circuits. But such instruments, even if they existed, would at present be used mostly on multi-phase circuits, and on such circuits no difficulty arises. All that is needed is for the instrument to have one of its systems of coils arranged multi-phase for two or more of the voltages, and to have its other system single-phase for the current whose power-factor is required. The fixed phase relation of the voltages will be unaffected by the variation of the load-currents, and thus the instrument will indicate the true power-factor of the particular current chosen. Indeed, most actual phasemeters are constructed and used in precisely this manner, and the error on unbalanced loads is given by a formula like (24) with θ_1 substituted for θ_0. Reference to (30) and (31) will show that the error made in assuming that the reading indicates the average power-factor of the load is essentially the same as in the case considered, and to which Table III. applies.

In conclusion, the main results of the foregoing investigation may be thus summarised :—

Summary.

1. Phasemeters for multi-phase circuits are all equally accurate on balanced loads provided they have been correctly calibrated and possess no faults due to purely mechanical causes. Their accuracy is not affected by variations in wave-form or in current-frequency. The calibration of the scale is affected by the number of coils used in the instrument, by the ratios of the ampere-turns used with these coils, by the distribution of the windings, and by the magnetic nature and properties of the magnetic circuits, especially if these contain iron ; but the accuracy of the indications is not dependent upon any of these considerations.

2. Phasemeters can be simply and accurately calibrated for balanced loads by means of a direct-current method of test.

3. The error of phasemeters on unbalanced circuits is generally serious for loads which are badly out of balance. The error, like that of a wattmeter, increases rapidly as the power-factor of the load diminishes. It can only be reduced at the expense of complication in the instrument, by increasing the number of coils used in the fixed and moving systems, and by arranging the coils and magnetic circuits to be symmetrical in regard to one another. If the true power-factor of the load is $\cos\phi$, the reading of the instrument is

$$\cos\phi + \theta \sin\phi,$$

where θ is the phase error due not to the instrument but to the unbalanced load, and is the product of two factors one of which is the maximum value of θ determined by the amount the load is out of balance, and the other may have any value between $+1$ and -1 and is the factor which determines whether the instrument reads high or low. The maximum value of θ is as follows :—

(i) For an instrument consisting of a single coil for one-coil system, and of either two or three coils for the other system,

$$\theta = \epsilon 2^{\frac{1}{2}}.$$

(ii) For a symmetrical instrument containing three coils in one system and two in the other,

$$\theta = \tfrac{1}{3}\epsilon 2^{\frac{1}{2}}.$$

(iii) For a mathematically symmetrical six-coil instrument

$$\theta = 0,$$

where ϵ is the square root of mean square of the fractions ϵ_1, ϵ_2, ϵ_3 by which the three load-currents differ from their mean value. It is also very approximately the arithmetical mean of the quantities ϵ_1, ϵ_2, ϵ_3 or two-thirds of the greatest of these three quantities, so that $\epsilon 2^{\frac{1}{2}}$ is essentially the same as the greatest of the quantities ϵ_1, ϵ_2, and ϵ_3.

Discussion.

The Secretary read a letter from Mr. A. Russell referring to one of the fundamental assumptions made by Dr. Sumpner. The magnetic force **B**, at any point on the conductor of the moving coil had been represented by $\mathbf{A}_1F_1+\mathbf{A}_2F_2+\mathbf{A}_3F_3$, where F_1, F_2, and F_3 were considered constant. The instantaneous value of the magnetic force at any point was the resultant of the three magnetic forces $k_1\mathbf{A}_1$, $k_2\mathbf{A}_2$, and $k_3\mathbf{A}_3$, acting in fixed directions at that point. At any instant therefore

$$\mathbf{B}=k_1\mathbf{A}_1\cos\phi_1+k_2\mathbf{A}_2\cos\phi_2+k_3\mathbf{A}_3\cos\phi_3,$$

where ϕ_1, ϕ_2, and ϕ_3 are the angles between **B** and $k_1\mathbf{A}_1$, $k_2\mathbf{A}_2$, and $k_3\mathbf{A}_3$. As $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are not in phase with one another, the direction of the resultant magnetic force is continually altering, and thus ϕ_1, ϕ_2, and ϕ_3 are functions of the time. Hence also the Author's factors F_1, F_2, and F_3 are not constants but functions of the time, and thus the conclusions arrived at in the paper must be modified. Dr. Sumpner's method of calibrating by means of direct currents was novel and valuable, but little confidence could be placed in the readings of phasemeters when the applied waves of potential difference did not follow the harmonic law.

Mr. A. Campbell, referring to the stepped nature of one of the calibration curves obtained by the Author, asked if the use of a smooth core instead of a channelled one would result in a smoother curve.

Mr. K. Edgcumbe expressed his interest in the paper, and said it was valuable because phasemeters were coming into general use and the literature on the subject was small. Dr. Sumpner had stated in his paper that previous theories had neglected the question of the currents being unbalanced, but he pointed out that Punga in 1902 had described power-factor indicators and showed how they could be used with unbalanced loads. It was important to define exactly what was meant by average power-factor. The Author had given a mathematical definition, but a simpler definition would be

that the average power-factor is that quantity which multiplied by the phase-voltage and the sum of the three currents gives the total power supplied to the line. A disadvantage of power-factor indicators was the difficulty of testing them, so that Dr. Sumpner's direct-current method was most valuable.

Mr. W. DUDDELL asked if the Author had assumed that the three voltages were in symmetrical phase relation when he stated that the six-coil instrument gave accurate results on unbalanced circuits, and pointed out that in practice differences of as much as 10° might occur.

Dr. SUMPNER, in reply, said that in the proof of the paper **B** was not the *maximum* flux but the effective part of it, that at right angles to the direction of motion of the conductor, and thus Mr. Russell's criticism did not apply. The effect of wave form and frequency had been carefully considered, and he was confident that the accuracy of phasemeters was independent of such effects. He was aware that it was possible to obtain smoother curves, and had partially done it by a suitable distribution of winding giving a gradually varying flux. In reply to Mr. Duddell, he said he had assumed equal voltages in symmetrical phase relation, but the mathematical symmetry of the problem in regard to currents and potentials implied that an instrument whose indications were independent of inequalities in the load currents, would also be unaffected by inequalities in the line voltages.

II. *The Question of Temperature and Efficiency of Thermal Radiation.* By JAMES SWINBURNE.*

ARGUMENT.

It is generally supposed that some surfaces can radiate light more economically than others. Twenty years ago the author inferred from Kirchhoff's and others' work that efficiency depended on temperature alone. This theory has been contradicted repeatedly. Evans's experiments do not support this theory. Explanation of his results. Professor Bottomley's results misleading owing to absence of photometric measurements. Lummer and Jahnke show little difference between platinum and a black body. Theories as to the thoria mantle. Explanation of phenomena.

It has long been known that at the same high temperatures some bodies, for instance those with black surfaces, radiate more energy per unit surface than white or polished surfaces; but it is now generally held that certain surfaces at a given temperature have a convenient property of selective radiation, in virtue of which they can give out radiation on the whole of a higher refrangibility than that given out by a black body at the same temperature.

As this question is of great importance in technology, I may be allowed to refer to incandescent lamps and mantles.

More than twenty years ago, when engaged on electric-lamp making, I worked on the theory that radiant light efficiency depends on the temperature only, and is independent of the nature of the surface; and that the colour of the light is therefore a test of the temperature. In those days, the work of Balfour-Stewart, Kirchhoff, Prévost, and Provostaye and Desains were available to the technologist. The perfectly black body had not been invented, and the perfectly black gas was not even a familiar idea. It appeared to me that, if a grey or white body at a given temperature could radiate energy corresponding to a black body at a higher temperature, the two bodies might be enclosed in a reflecting case, and the grey surface would be in equilibrium with a black surface at a higher temperature. If the temperature of the black body were reduced a little, so as still to be higher than that of the grey body, the grey body would radiate and the black absorb,

* Read November 10, 1905.

so that heat would move from a body to a hotter body. Experience agreed with this reasoning. Carbons that were made grey by "flashing" did not last any longer than black carbons of the same material and density, when run at the same efficiency ; in fact, the flashed carbons were less durable, probably because they were not homogeneous.

I was not aware at that time that the opposite opinion was generally held among scientific men, though it was then widely spread among specialists in incandescent lamps. The statement has been repeatedly contradicted during the last twenty years, but without meeting the arguments, or production of good evidence.

I am thus to some extent compelled to act as a controversialist on the defensive. To take recent examples, Prof. Bottomley * quotes me, and brings forward experiments he has made to disprove my assertions. Mr. M. Solomon † also quotes me from a discussion, and brings forward arguments to show that bodies at the same temperature can have different efficiencies.

As I have also been frequently represented as saying various curious things, such as, that all surfaces have the same emissivity, it will be well to state the alleged law clearly and definitely, and then to leave it for discussion, so that if wrong, it may be cleared away as soon as possible. It is, "In the case of pure temperature-radiation, the light-efficiency is a function of the surface temperature only." Light is taken to mean, not only what affects the eye, but what includes ultra-violet radiation also. If this is the case, the colour of the light of a thermally radiating body depends on its temperature only.

The prevalent opinion that high efficiencies can be obtained at given temperatures from special surfaces is largely based on a paper by Mr. M. Evans ‡. The curious thing is that there is nothing in this paper to support the proposition it was written to maintain.

Before dealing with this paper, let us first consider what the effect of "flashing" an incandescent lamp is. Imagine a black carbon thread of a certain superficial area, and a

* Phil. Mag. Nov. 1902. † 'Nature,' 1902.

‡ Proc. Roy. Soc. No. 243, 1886, p. 207.

certain resistance, the resistance varying with the temperature. At a certain temperature this filament gives, let us say, 20 candles measured in a given direction, and takes, say, 80 watts. Now imagine the filament coated with something not nearly so black as the original carbon ; for instance, let it be "flashed" so that it is coated with a light steely-grey carbon. It is then found that with the same expenditure of power the lamp gives much more light, while if run at the same candle-power as before, it takes less electrical power. From this it has been argued that the lamp is more efficient after flashing, and that it is a good thing to have a grey surface rather than a black, because at a given temperature and expenditure for a given power, it gives more light. The idea appears to be that the grey surface gives a higher efficiency at a given temperature, and is therefore more durable. To quote the paper just mentioned—"The same filament at 3 watts per candle, when black, had to give off 31 candles, equal to 270 candles per square inch of its surface, while in its polished state it required to give only 18 candles to equal 3 watts per candle, and its surface was strained only to the extent of 155 candles per square inch. It is certain, therefore, that its lasting power with its surface bright would be many times greater at the foregoing expenditure of energy than in its black condition." This is not very clear ; but unless it means that the carbon was at a higher temperature when giving 31 candles in the black state than when it was giving 18 in its grey, the whole paper has no point. Evans's paper is quoted as it is apparently the foundation of the theory that "flashing" improves the efficiency of a carbon. Professor Bottomley has made other experiments of his own.

It is generally admitted that a black body is a good absorbent when cold, and a good radiator when hot, and that a white body absorbs less and reflects or diffuses more when cold, and emits less when hot. Returning to the carbon filament, when black it is a good radiator. Let it be run at 0·3 candles per watt, and give 31 candles. It is then at such a temperature that, with its emissivity it can just radiate energy at the rate of 93 watts. If the power is kept at 93 watts, and the emissivity or power of radiation increased, the temperature will fall. The greater the emissivity the

greater the power radiated at a given temperature; or if the power is kept constant, the greater the emissivity the lower the temperature necessary to radiate at 93 watts. On the other hand, if the emissivity is reduced and the power kept constant, the temperature must rise until the increased temperature compensates for the decreased emissivity. If the filament is flashed, it will become grey or whitish, so that its emissivity may be expected to be less, not greater; as it reflects well and absorbs badly when cold. Assume its emissivity is less. Then if run at 93 watts it must get to a higher temperature before it can radiate at 93 watts. If, instead of running it up to 93 watts, it is run at 31 candles: as the emissivity is less the light at a given temperature is less, and in order to give the same light, 31 candles, as when black, the filament must be at a higher temperature. At the same candle-power the grey carbon is therefore hotter than the black, otherwise it cannot give the same light with reduced emissivity. Now run it at 0·3 candles per watt. It gives 18 candles. It is therefore colder than when it was giving 31 candles, and much colder than when it was taking 93 watts. There is no reason to suppose, however, that it is colder than when giving 31 candles with a black surface. At 18 candles the grey filament is at the same efficiency as the black at 31 candles, namely 0·3 w.p.c. There is no reason yet to suppose they are at the same temperature. If they are, when the grey is raised to 31 candles it will be at a higher temperature and higher efficiency, taking less than 93 watts. If raised to take 93 watts the temperature and efficiency are very much higher still. There is thus nothing in Evans's experiments to show that a grey carbon is at a lower temperature than a black when running at the same efficiency; the whole of the results are explained amply on the ground that a body that is black when cold has a higher emissivity (and a higher absorption) than a grey or white body.

Suppose, instead of flashing the black filament, it had been shortened by nearly half. It might then give 18 candles at 0·3 candles per watt, instead of 31; and if run at 31 candles it would take less than 93 watts, and at 93 watts would give much more than even 31 candles. From this it might have been argued on the same lines that a small carbon is much

more efficient than a large one. It is, if run at the same candle-power or same electrical power, but it is obviously at a higher temperature. It might be said that this is not a fair illustration, because in the case of the flashed and unflashed surface, the fact that the unflashed gives much more light per unit surface at a given efficiency shows it is at a higher temperature. But such reasoning is obviously fallacious. Imagine the flashed and unflashed filaments not heated electrically, in an enclosure kept at the temperature corresponding to 0·3 candles per watt for the black filament. Both are equally hot; but a black body, under such circumstances, radiates and absorbs much more heat than a grey or white one, which reflects a large proportion of the incident radiation, and therefore only absorbs the remainder, and radiates what it absorbs. That a black carbon at the same efficiency radiates more per unit surface than a grey or white, is not a proof that it is hotter: it is a necessary result of their being at the same temperature, and therefore no proof of their difference.

Though there is nothing in Evans's experiments to prove his own case, the greater emissivity of black compared with grey does not in itself show that bodies radiating at the same efficiency are at the same temperature.

Professor Bottomley's experiments consist of running two similar platinum wires under similar conditions in the same exhausted vessel. One is blackened with lampblack, the other is polished. The electrical measurements are carefully and no doubt accurately made, but no photometric measurements are made at all. The human eye is used to judge whether two wires of the same size, one of which is giving out more than four times the power of the other, are giving off an equal amount of light. Of course the eye is hopelessly inadequate for such a purpose.

Professor Bottomley seems to think that if bodies have different emissivities they must have different radiating efficiencies, and that proving one is proving the other. He says:—"In 1887, together with Mr. Mortimer Evans, I pointed out the marked difference in emissivity between a polished metallic-like surface and a dull sooted surface." But it was known long before this that black bodies had

higher emissivities than polished or white. I remember in my early schooldays reading why a polished silver tea-pot kept hot. That certainly was not the point of Mr. Evans's paper: it was, that a grey surface gave a given efficiency at a lower temperature than a black. Professor Bottomley goes on:—"I think it is now generally admitted that such a difference does really exist; but at the time, my conclusions were controverted." He then quotes a passage of mine in which I controvert, not the statement that black is more emissive than grey, but that it is less efficient. Professor Bottomley, in a footnote, adds that a speaker in a discussion "expressed the view that incandescent-lamp makers do not find any difference between flashed and unflashed filaments. I venture to think, however, that this is incorrect. . . . I believe that all first-class makers are now fully alive to a difference in economy between a brightly flashed and a dull unflashed filament." Of course all lamp-makers know that a flashed filament has less emissivity, and at a given efficiency makes a lamp of less candle-power; but that is quite different from its being more economical. It is possible that Prof. Bottomley uses the word emissivity in some other sense. I have heard people say, for instance, that the emissivity of a carbon is increased by flashing. Prof. Bottomley's experiments show that the emissivity of a black surface is nearly 8 times that of polished platinum at 444°, the difference decreasing until at 895° it is over 4 times. He makes no photometric measurements, but trusts to the judgment of a skilled assistant to say whether the strips give an equal amount of light. The experiments show what would be an extraordinary result, though there is no remark on it. The strips always give equal amounts of light at the same temperature; though the theory is that in each case the blackened wire gave light of lower average frequency. The eye is a physiological instrument, and is only useful photometrically by null methods. No one who has not tried it has any idea of the inaccuracy of the eye in looking at lamps of different sizes run at different efficiencies. Professor Bottomley says that bodies at the same temperatures with widely different emissivities give the same amount of light, though one radiates four times the power of the other. Mr. Evans, on

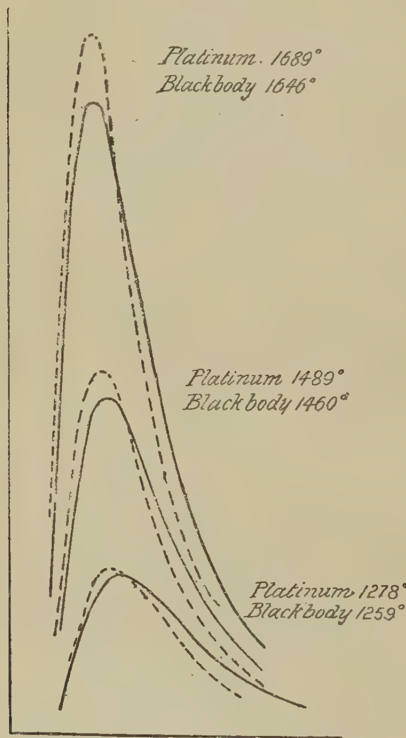
the other hand, argues that because one carbon radiates about twice the power another does, it must be hotter. I submit that Prof. Bottomley's experiments are inaccurate as to the light. If his two platinum wires were put in a closed hot chamber and examined through a small hole, they would be at the same temperature. The black one would be emitting and absorbing four times the power of the bright one. They would both receive the same radiation, but the black would absorb most of what it received and reflect the rest. The bright would reflect a much larger proportion of the radiation falling upon it. If Professor Bottomley were right, the black and bright body would only emit the same light; but as the bright body reflects very much more, it would appear enormously brighter in the hot enclosure. It is quite well known that all such bodies in a hot enclosure look equally bright.

Professor Bottomley's results are also contradicted by the well-known experiment of electrically heating a strip of platinum with a spot of ink on one side. The spot looks brighter than the platinum round it; but the other side of the platinum strip shows a dull spot where the ink is, showing that the spot, though giving more light, is really colder than the rest.

Elaborate tests on the emission of the perfectly black body and polished platinum have been made by Lummer and Jahnke. Lummer has treated the question of emission very fully in his *Die Ziele der Leuchtechnik* (Oldenburg, Munich and Berlin, 1903). He gives (p. 69) a set of curves of a black body. He does not merely measure the light with a photometer; he measures the power radiated at each wave-length. The curves are taken for seven different temperatures, and each curve shows the individual observations and calculations, and they agree and give smooth curves. On p. 79 similar curves are given for seven different temperatures for polished platinum. These curves are particularly interesting as they are taken apparently with great care and thoroughness, and Dr. Lummer is a strong believer in the higher efficiency of special surfaces. There is one black-body curve for 1646° and a polished-platinum for 1689°. The platinum curve is much lower, as the ordinates represent the power per unit

surface for the wave-length represented by the abscissæ. As the emissivity of platinum is very much smaller than that of the black body, we ought to increase the surface of the platinum till it radiates the same power, by increasing the ordinates in proportion till the areas of the platinum and black radiation-curves are equal. If my theory is correct,

Fig. 1.



Curves of Radiation for perfectly black body (full lines) and polished platinum (broken lines).

however, it will do as well to raise all the ordinates till the tops of the curves correspond. In the figure the platinum curves are raised, so as to allow roughly for their higher temperatures, and are shown dotted, while the perfectly black body's curves are shown in full lines. All the curves are in the heat region, where $\lambda > 1\mu$. The curves in the

original have been distorted in reproduction, and the two sets are not to the same scale. The curves now produced are, however, accurate enough for the present purpose. They show the platinum to be somewhat more efficient than the black body. Of course the measurements may be wrong, as they are delicate, and the methods for the platinum and "black body" are different; but so much care has been taken that it seems more likely that there is a flaw in my reasoning. It must either prove absolutely equal efficiency at a given temperature, or break down altogether. Lummer's experiments seem to show a greater efficiency in the case of platinum. All the same, considering that the platinum is in each case at a higher temperature, and that the difference in efficiency is comparatively small, though polished platinum and a "black body" are very far apart, it may still be open to say the selective emissivity of platinum is "not proven." Nichols, in the 'Physical Review,' gives results according to which grey carbon is more efficient than black.

There is, however, another consideration. A body may be supposed to have selective emissivity, so that at a given temperature it gives out the same rays as a black body at a higher temperature; but it might be said that the efficient radiator did not do this, that it gave out the same radiation as a black body at that temperature as far as short wave-lengths go, but did not emit long waves in the same proportion. The curious thing about such a theory as this is that it would not at all lead to grey carbon or platinum being efficient; quite the reverse. If the body is to radiate light well when hot, it should be absorbent when cold, that is to say black. It should be a bad absorber of heat when cold; so that to a large animal, with an optical mechanism worked by what we call heat rays, it would look white or polished, while to us it would look black. It does not follow rigidly that a good absorber when cold is necessarily a good emitter when hot. Platinum appears to get blacker when hot. This seems to be Lummer's idea of an economical source of light, yet he apparently expected platinum to be economical. The idea seems generally to be that white bodies are economical, such as grey rather than black carbons, and the white

oxides used in the Nernst electric and the Welsbach mantle. Electric lamps are now made of osmium and tantalum. If bodies with selective emissivity can exist, whose economy depends on their failing to give out low frequency radiation, we may also have bodies which fail to give out high frequency radiation. They would be less efficient than the perfectly black body. It may be argued that such bodies are quite possible thermodynamically. Thus, suppose as an extreme case we have an economical body L which at a given temperature gives out visible light and omits to emit heat, while B is a black body, and H gives out nothing but heat. In a reflecting enclosure of a high temperature it may be said, L gives out and absorbs the same light, and therefore keeps at the same temperature as B. It may reflect some of the light that falls on it. It reflects all the heat that falls on it and gives out none. Against this it may be argued that a thermodynamic radiation-engine, such as is used in proving Boltzmann's, and dealing with Wien's laws, may be used which will take radiation from L, get work out of it, and return it degraded into heat to H ; which would be absurd. Again, imagine a block of H radiating heat only at a temperature at which L and B are luminous. Imagine H surrounded by a very thin vessel whose inner side is lined with H, while the outer is coated with L. This shell will then take in energy as heat, and give it out as light ; in other words, it may be said to fluoresce rays of higher refrangibility than it receives, which may be impossible.

A great deal has been written about the incandescent gas-mantle, and all sorts of theories have been evolved. It is generally assumed that the temperature of the flame in the region where the mantle is hung is measurable with a platinum thermocouple, and therefore below the melting-point of platinum. The oxides are therefore said to be endowed with the power of selective radiation, performing an office analogous to Maxwell's demon, but dealing with the radiation from them, instead of with the molecules. There seems to be a sort of attractive mystery about the rare earths. Erbia is stated to give a discontinuous radiation spectrum. Cerium has two oxides, and it has been urged that it takes in oxygen and then parts with it again to the gas, thus

producing a higher temperature than that of the direct combination. The energy developed is obviously the same with or without ceria. If by intervention of the cerium oxides, with the same final energy liberation we get a higher temperature, we must have produced energy. Putting the matter another way, ceria cannot add energy continually, so the final temperature of combustion cannot be altered by any rapidly alternating changes of the ceria. Apart from such reasonings, there is no ground for supposing such change of oxidation, as oxides of metals with single valencies will do instead of ceria.

The matter admits of very easy explanation on the ground that white bodies have less emissivity than black. In the first place, the part of the flame at the mantle is much above the melting-point of platinum, as a small wire is fused instantly. As a melting wire is absorbing heat from the hot gases and radiating rapidly, it must be much below the temperature of the gases. The gases are much above the melting-point of platinum, and any temperature determinations made with thermocouples are necessarily entirely wrong. If a comparatively large wire, such as used for a thermocouple, is put into the flame, it radiates so well, in proportion to the heat it can get, that it keeps comparatively cool. While a bunsen will only heat a two-inch ball to a low red heat, it will heat a thermocouple yellow-hot, and it will melt fine platinum and volatilise it rapidly if still finer. We may therefore take it that the active zone, at least, of a bunsen is at a very high temperature. If a small body is immersed in it, it will get hotter until it radiates as much as it takes in. The rate at which it takes in heat depends on the difference of the temperatures of the body and the gas and the flow of gas past it. The rate at which it radiates depends on its emissivity and its temperature. Suppose the emissivity is increased, the body, if left at the same temperature, would radiate more, and would need a greater intake of heat; but this would demand a greater difference of temperature between it and the gas, the temperature of the body would thus fall, increasing the intake of heat, and decreasing the radiation, until they balanced at some lower temperature. A body with no emissivity would be

at the temperature of the flame, and would of course give no light. If the emissivity is very small, it will be nearly as hot as the flame, and will give a little light. This will be of high average refrangibility. If the emissivity is increased a little more, the light will increase but will be redder because the body must fall to a temperature at which the intake of heat is enough. As the emissivity is increased the radiation increases, but gets redder, till it is no longer luminous. A black mantle is only red-hot; and a mantle of "perfectly black" material would be duller still. As the temperature of the flame is not high enough to give a light much whiter than that wanted from the mantle, there is not much margin. As soon as the mantle is made emissive enough to give a reasonable amount of light in proportion to the gas consumed, the luminous part of the radiation begins to decrease, so there is very little margin. Pure zirconia, or thoria, or alumina gives very little light. Zirconia was therefore mixed with a little oxide of the yttria group, thus introducing a trace of coloured oxide so as to increase the emissivity. Thoria again has ceria introduced. It is not because ceria has any special properties due to there being two oxides. Any coloured body would do as far as emissivity goes; but what is needed is an emissive body that will not volatilise. Any coloured additions that will stay at all will give the light as long as they are there, but they evaporate. Platinum, for example, evaporates very quickly. Alumina was largely used in conjunction with chromium oxide. At first the chromium oxide, being free, makes a mantle that is quite green when cold and very emissive. It therefore gives a useless red light. The chromium oxide soon combines with the alumina, or goes into solution, forming a very light pink ruby. The fall of emissivity lets the mantle get hot enough to give a good light.

Purely scientific, or non-technical writers on the theory of mantles often assume that the various blends are made with the light alone in view. This is not so: the makers have to consider toughness, durability, flexibility, absence of shrinkage and price. They sometimes have to consider one another's patents. It is found that the oxide must not be quite white,

and that if the threads are very fine they can take in heat relatively well, and can therefore stand more ceria and higher emissivity. On the other hand, they are then less durable, and the total radiating surface is less, unless the fine fibres are made more numerous, and they then tend to impede the flame, and cannot get their heat so well. The best practice in mantle-making is to get a good compromise. Fineness means that the mantle gets heat easily, and can therefore stand more ceria, without getting too cold to emit well. It also means less surface, greater fragility, and more rapid change with time. Too high a temperature means volatilisation of the ceria, and falling off of light. Too many fibres, or a fine mesh tend to check the gases, and prevent the rapid supply of heat to the mantle. The technical practice in making mantles, as far as I am acquainted with it—and it is a difficult subject—completely bears out the theory that oxides radiate merely because they are hot, and have a reasonable emissivity, the emissivity being raised and the temperature therefore lowered to the point which gives the maximum light. The maker has also to make sure that the mantle is tough and flexible, strong when cold, shapes well in seasoning, stands collodion, and can be made economically.

The Nernst lamp was at first supposed to give its high efficiency because the oxides used were specially efficient radiators, and this theory is, no doubt, held in some quarters; but there is no evidence to show that its efficiency is not due simply to its being very hot.

Discussion.

The Secretary read a letter from Dr. J. T. Bottomley, in which he stated that the Author's paper seemed to be to a large extent an attack on his work on radiation. He could not see that Mr. Swinburne had made a single experiment in support of his proposition that "In the case of pure temperature-radiation, the light-efficiency is a function of the surface-temperature only." No amount of discussion such as the Author asked for could do anything to prove or disprove this proposition. He failed to see that the Author had worked out any logical proof of the proposition. The

Author had assumed that he (Dr. Bottomley) was incapable of matching the light-giving properties of two platinum strips placed side by side and properly arranged for this sort of comparison. The matching is exactly the same in principle as in shadow photometry, and in all the cases cited in the paper to which Mr. Swinburne takes exception, Dr. Bottomley's determinations agreed with those of his assistant, Mr. Evans.

Dr. J. A. HARKER expressed his interest in the paper, and said that the question of radiation was of practical importance in the measurement of high temperatures by electrical resistance methods. He asked if two platinum thermometers, one bright and the other blackened, but identical in other respects, were transferred from an ice-bath into a sulphur-bath, would the final temperatures of the two thermometers be the same, and also would the thermometers approach their final temperatures at the same rate? Dr. Harker also drew attention to some interesting experiments on black-body radiation by Messrs. Waidner and Burgess, published in the Bulletin of the American Bureau of Standards. They determined by means of a Holborn & Kurlbaum optical pyrometer the black-body temperature of a meldometer strip calibrated by known melting-points in the usual way, and from these experiments plotted curves showing the difference between the true and black-body temperatures of the strip over the range 700°–1500° C. They also determined the black-body temperature for 1780° C., the value they assumed for the melting-point of platinum. The difference between real and black-body temperature at this point they found to be 239°, 203°, and 168° C. for three wave-lengths in the red, green, and blue respectively. By extrapolation of the meldometer experiments through 280° a second set of values for this difference at the platinum melting-point may be obtained, namely, 153°, 115°, and 90° respectively, numbers quite out of harmony with those just quoted. Dr. Harker pointed out that by taking instead of 1780° C. as the melting-point of platinum, the value obtained in his own recent determination, 1710° C., the discrepancy was reduced in each case to within the limits of error of such experiments.

Prof. H. L. CALLENDAR, referring to Dr. Harker's questions,

stated that if the sulphur-bath was in an isothermal enclosure, the two wires would reach the same ultimate temperature, although the blackened one would reach it quicker. The difference, however, would be small. With fine wires the loss of heat by conduction through the gas was more important than the radiation loss. The loss by radiation was proportional to the surface but the loss by conduction was not, and it therefore became relatively more and more important with the fineness of the wire. With regard to the action of incandescent mantles, he thought the Author's theory was reasonable and satisfactory.

Mr. M. Solomon said that the Author's main contention was that there was no such thing as selective emissivity, but that all bodies at the same temperature radiated light of the same efficiency, and that any departure from this rule was inconsistent with the second law of thermodynamics and consequently with the theory of exchanges. This did not seem correct as selective emissivity existed in the case of incandescent vapours, and if it was consistent with the laws of thermodynamics in such a case, it could not be inconsistent in the case of an incandescent solid. There was no reason why two incandescent bodies both raised to the same temperature, one radiating as a black body the other selectively, should not coexist inside a perfectly reflecting shell. As regards the incandescent mantle, the explanation put forward by the Author did not seem satisfactory. It was difficult to believe that the very small differences in colour between a thoria, a thoria-ceria, and a ceria mantle could account for the enormous differences in their candle-powers. Mr. Solomon remarked that he thought catalytic action seemed a more reasonable hypothesis than that suggested by the Author, and he was inclined to think that the thoria-ceria mantle was actually at a much higher temperature than the flame. It was possible that the average temperature of the mantle was no higher than that of the flame, but that exceedingly high temperatures existed locally in the neighbourhood of the ceria particles. The problem was an extremely difficult one, and was complicated by the confusion which existed as to what constituted selective emission. If two bodies were at the same temperature, one might

radiate more light than the other because it was a better radiator, but the ratio of the energy radiated at any one position in the spectrum to that radiated at any other position might be the same in both cases. Of two such bodies, the one giving more light was a better emitter but it was not a selective emitter. It would only be selective if the ratio of the energy radiated at different positions in the spectrum was different from that for some particular body chosen as standard for comparison—"the perfectly black body."

Prof. W. E. Ayrton referred to experiments which he had carried out which tended to show that the temperature of a Bunsen flame was lowered by an incandescent mantle. He had also found that small pieces of mantle heated in a platinum boat to various temperatures just below the melting-point of platinum never looked brighter than the platinum. If, however, a small hole was made in the bottom of the boat and coal-gas passed in, the mantle immediately became brilliant.

Mr. Swinburne, replying to Dr. Bottomley, said that his main argument was that he had not made experiments. In the investigation of a question such as that considered in the paper, it was possible to prove much by means of experiment, but it was also possible to attack the problem by thermodynamic reasoning. He expressed his interest in the fact that Prof. Callendar had referred to his views as reasonable. As to Mr. Solomon's statement that incandescent vapours show selective emissivity, it required confirmation if he meant vapours heated by application of heat; if they were heated by degradation of work, for instance by electric discharge, they did not bear on the question. Catalytic action is sufficiently mysterious to be generally dragged in to explain anything, but catalysis is generally hastening an action that would otherwise go on slowly or not at all; it tends to produce equilibrium. To make the mantle at a higher temperature than the gas it is in involves supplying energy, a feat which catalytic action cannot perform.

III. *The Dielectric Strength of Air.*

By ALEXANDER RUSSELL, *M.A., M.I.E.E.**

TABLE OF CONTENTS.

* Read November 24, 1905.

1. *Introduction.*

Physicists generally attempt to deduce the dielectric strength of air, at a given barometric pressure, from the results of experiments on the disruptive voltages between equal metal electrodes at given distances apart. They calculate the maximum value of the electric intensity between the metal electrodes on the assumption that the electric field round them is similar to that existing at low voltages. Figures obtained in this way were found, greatly to the disappointment of the early experimenters, to vary widely with the distance apart of the electrodes. Lord Kelvin, however, as far back as 1860 *, deduced from the results of his experiments with large electrodes that it was "most probable" that the numbers obtained in this way at higher voltages would be "sensibly constant." An examination of the results, which are given below, obtained recently by electricians will show that experiment has amply justified Lord Kelvin's conclusion. The author finds, by considering experimental results obtained both with direct and alternating pressures, that the limiting value to which the numbers approach is 38 kilovolts per centimetre.

When the electrodes are small, or when the disruptive voltages are only a few kilovolts, the numbers obtained in this way differ largely from 38 kilovolts per centimetre. It is therefore necessary to explain why this is the case. It will be shown in what follows that when the electrodes are small, the air surrounding them may have broken down and become a conductor at voltages which are only a fraction of the disruptive voltage. In this case luminous effects are generally observed at the electrodes. When a high alternating pressure, less than the disruptive voltage, is maintained between small electrodes a few inches apart, each electrode, when the P.D. is sufficiently high, is seen surrounded by a faintly luminous enveloping cloud of a bluish colour, which apparently does not touch the conductor it envelopes. We shall call this cloud the corona. As the pressure is increased, short violet streamers are seen issuing outwards from the

* Proc . Roy. Soc. April 12, 1860; 'Reprint,' p. 259.

corona, the space immediately outside it being the seat of great electrical activity. At higher pressures the streamers are longer, and a hissing noise is heard. When the potential-difference between the electrodes approaches the disruptive value, sparks take place between them, and finally, when all the air is broken down, an arc is suddenly established.

Now, when luminous effects make their appearance, it is obvious that the boundaries of the Faraday-tubes are altered, and, consequently, that the electric field is different from what it is at low voltages. We cannot apply formulæ, therefore, which have been obtained on the assumption that the distribution of the tubes is the same as that for low pressures. We have not attempted to deduce formulæ which will give the dielectric strength of air from the disruptive voltage between two electrodes surrounded with coronæ, as the space occupied by the brush discharges is not clearly defined. There are many cases, however, when there are no luminous effects and where a disruptive discharge ensues the moment that the dielectric stress attains the breaking-down value. We have deduced the dielectric strength of air from the experimental results obtained in these cases.

Many electricians consider that a disruptive discharge always occurs the moment the electric stress at any point of the dielectric between the two electrodes attains a certain maximum value. In what follows, however, we show that in many cases, when some of the air round an electrode breaks down, the new value of the "maximum electric intensity" at the boundary of the broken-down air is less than the old value at the boundary of the metal, and so there is equilibrium, a corona being formed.

The explanation of the varying numbers obtained when large electrodes are used and the disruptive voltages considered are small, is more difficult. When the minimum distance x between the electrodes is less than 3μ, the sparking potentials are practically independent of the nature of the gas between the electrodes *. Since the material of which the electrodes is made exerts an important influence on the

* G. M. Hobbs, "The Relation between P.D. and Spark-length for Small Values of the latter." Phil. Mag. [6] x. p. 617 (Dec. 1905).

sparking potential V, at these small distances, it is highly probable that the carriers of the discharge come from the metal and not from the gas. For a certain distance greater than 3μ, G. M. Hobbs finds in some cases that V remains constant and equal to the minimum spark-potential which in air is about 350 volts *. For slightly greater distances V increases uniformly with x.

It is obvious, therefore, that when the electrodes are very close together, we cannot assume that we have a homogeneous medium bounded by rigid equipotential surfaces. Hence, as the equipotential surfaces are unknown, we cannot apply the ordinary electrostatic equations. For these reasons we have in the following paper only considered experimental results obtained for values of x greater than one millimetre. If we had only considered distances greater than half a centimetre (one fifth of an inch), it would have been unnecessary to make any assumptions about the actions that take place at the end of the tube subjected to the maximum electric stress, as the maximum values of the electric intensity, at the instant of discharge, are found to be in satisfactory agreement. In order, however, to include in our formulæ the sparking potentials for values of x lying between 0·1 and 0·5 cm., we have found that it is necessary to make the following assumption. At the moment of the disruptive discharge, the pressure on the ends of the Faraday tube subjected to the maximum stress is $V-\epsilon$, where ϵ represents what we shall call the lost volts. When the electrodes are surrounded with coronæ an assumption of this nature must be made †, but in this case ϵ will be a function of V and x. In the cases we consider we assume that ϵ is constant and equal to 0·8 of a kilovolt. Making this assumption and, for reasons given above, only considering experiments with large electrodes, at appreciable distances apart, we find that the maximum values of the electric intensity at the moment

* The Hon. R. J. Strutt, "On the Least Potential-Difference required to produce Discharge through Various Gases." Phil. Trans. vol. 193. A. p. 377 (1899–1900).

† H. J. Ryan, "The Conductivity of the Atmosphere at High Voltages." Trans. Am. Inst. El. Eng. vol. xxiii. p. 101 (1904).

of the disruptive voltage is practically constant for distances varying from a millimetre up to 15 centimetres, and for voltages varying between 4 and 160 kilovolts.

2. *Historical.*

Nearly all experimenters have used equal spherical electrodes. It is therefore necessary to be able to write down at once the value of the electric intensity between two spheres whatever may be their potentials. Kirchhoff * in a very valuable paper has shown how to obtain from Poisson's † equations an expression for the maximum value of the electric intensity in the form of an infinite series. Unfortunately this paper can only be understood by those who are thoroughly familiar with Jacobi's theorems in Elliptic Functions, and so the important results contained in it are known to few physicists. In 1890, Professor A. Schuster ‡ published a table giving the value of the maximum electric intensity between two spheres when one was at potential V and the other at potential zero. He gives, however, no proof of the formula, merely referring to Kirchhoff's work. He reduces the infinite series formula for the case of two spheres close together, given by Kirchhoff, into a remarkably simple algebraical form, and shows that, when the spheres are at potentials V and 0, it applies with sufficient accuracy for practical purposes up to a distance between them equal to one-fifth of their radius. In what follows it will be shown that this Kirchhoff-Schuster formula applies with very considerable accuracy, when the potentials are $+V/2$ and $-V/2$, up to a distance apart equal to their radius. This is proved by actually calculating the values of the series, as it is difficult to see from Kirchhoff's method of proof what are the limitations of his formula. By considering the equipotential surfaces round two particles having equal and opposite

* Crelle's *Journal*, 1860, "Ueber die Vertheilung der Elektricität auf zwei leitenden Kugeln," p. 89; *Gesammelte Abhandlungen*, p. 78.

† *Mémoires de l'Institut Impérial de France*, "Sur la Distribution de l'Electricité à la Surface des Corps Conducteurs." Read 9th May and 3rd Aug. 1812.

‡ A. Schuster, "The Disruptive Discharge of Electricity through Gases," Phil. Mag. vol. xxix. p. 192 (Feb. 1890).

charges, the author shows how the first two terms of the Kirchhoff-Schuster formula can be found very simply.

Professor A. Heydweiller* carried out a valuable set of experiments on sparking distances in 1892. He also uses Kirchhoff's formula without, however, proving it. Tables of the numerical values of the electric intensity when the spheres are various distances apart and when they are at equal and opposite potentials are calculated. The formulæ are applied to his own experimental results, but as he does not discriminate between the cases when they are and when they are not applicable, and neglects the 'lost volts,' the results vary widely. The experimental results analysed in Table VI. below are taken from this paper. In Mascart and Joubert's 'Leçons sur l'Electricité et le Magnetisme,' vol. ii. p. 610 (1897), a neat proof of a series formula for the maximum electric intensity between two unequal spheres is indicated. Kirchhoff's results are also quoted, and the formulæ are applied with, however, indifferent success.

In this paper the author gives a simple proof by Kelvin's method of images of Kirchhoff's series formula. He shows by elementary algebra that it can be expressed quite approximately enough for all practical purposes by a simple formula. He has also calculated complete tables which enable any one to write down at once the maximum value of the electric intensity between two equal spheres whatever may be their potentials.

In some of the experiments analysed below cylindrical electrodes are used; it is therefore necessary to get the formula for this case also. It will, however, be more instructive to consider the very simplest mathematical cases first, and thus we shall be able to form a clearer picture of the phenomena that happen in the more difficult practical cases.

3. *The Electric Intensity between Two Concentric Spheres.*

In the case of a spherical condenser we have a metallic sphere concentric with a metallic spherical envelope. If the

* Wiedemann's *Annalen*, vol. xlviii. p. 785 (1893).

radius of the inner sphere be a and the inner radius of the outer sphere be b, we have

$$-\frac{dv}{dr}=\frac{q}{r^2},$$

where v is the potential at a distance r from their common centre, and q is the charge on the inner sphere. Hence we easily find that

$$-\frac{dv}{dr}=\frac{Vab}{r^2(b-a)},$$

where V is the P.D. between the spheres. Now dv/dr has obviously its maximum value R_m when $r=a$, and thus we have

$$R_m=\frac{Vb}{a(b-a)}.$$

If we suppose that b and V are fixed and a is a variable, we see that R_m increases from $a=0$ to $a=b/2$ and diminishes for greater values of a. Hence, if a be greater than $b/2$ and we gradually increase V, the moment the electric intensity attains a certain value the air immediately in contact with the inner sphere breaks down and becomes conducting. The electric intensity at the surface of this stratum of conducting air round the inner sphere will be greater than the old maximum electric intensity, and hence a new stratum will be broken down. It is unlikely that the boundaries of the strata successively broken down will be exactly spherical, but any lack of symmetry will accelerate the discharge and an arc between the two spheres will certainly be established. Thus when a is greater than $b/2$ the sparking voltages between the two spheres may be used to calculate $R_{max.}$ the dielectric strength of air.

On the other hand, if a be less than $b/2$, an increase in its value will diminish R_m, and thus equilibrium is possible with a conducting stratum of air round the inner sphere. The outside of this stratum is what we call the corona. As the voltage V is increased the corona grows until its radius is nearly equal to $b/2$, when a disruptive discharge will ensue. We see therefore that the size of the inner sphere has no practical effect on the disruptive voltage provided that its radius be less than $b/2$.

When the radius is greater than $b/2$ we should expect no luminous effects until the final discharge took place. This would occur at the instant when

$$R_{\text{max.}} = \frac{V-\epsilon}{b-a} \cdot \frac{b}{a},$$

where ϵ represents the lost volts.

4. *The Electric Intensity inside a Concentric Main.*

Let us now consider the important practical case of a concentric main. A hollow conducting cylinder of inner radius b contains a coaxial conducting cylinder of radius a. If the cylinders be separated by air, the electric intensity R at a point P in the air at a distance r from the axis of the cylinders is given by*

$$R = -\frac{dV}{dr} = \frac{V}{r \log (b/a)},$$

where V is the P.D. between the cylinders. R has obviously its maximum value R_m when $r=a$. We also have

$$\frac{dR_m}{da} = \frac{V}{\{a \log (b/a)\}^2} \{1-\log (b/a)\},$$

if V and b remain constant. Hence, if a be less than b/ϵ, where ϵ is the base of Neperian logarithms, R_m will diminish as a increases. In this case a corona will be formed. When the radius of the corona is nearly equal to b/ϵ a disruptive discharge will ensue.

If the radius of the inner cylinder be greater than b/ϵ, a disruptive discharge ensues whenever the intensity at the surface of the inner cylinder equals $R_{\text{max.}}$. This was verified roughly by Gaugain †.

The same formulæ apply when the dielectric coefficient of the insulating material between the conductors is not unity. We see, therefore, that the "factor of safety" of concentric mains is not necessarily increased by diminishing the radius of the inner conductor. This result is of considerable practical importance.

* Russell, 'Alternating Currents,' vol. i. p. 95.

† *Annales de Chimie et de Physique*, viii. p. 75 (1866).

5. *The Corona round a Cylinder.*

The luminous effects produced when a cylinder is maintained at a very high alternating potential from earth have been investigated experimentally by E. Jona*. He found that a cylindrical wire supported by high-tension insulators becomes luminous when the voltage between it and surrounding objects attains a definite value, which depends mainly on the diameter of the cylinder. The corona in this case is practically a concentric cylinder, the diameter d of which varies with the voltage V. In the following table V is in kilovolts and d is in millimetres.

TABLE I.—E. Jona's experiments on the diameter of the corona round a thin wire at various voltages.

V...	12·3	18·2	28	42	57	81	106	127	152	185	196
d...	0·12	0·25	0·50	1·00	2·00	4·00	6·00	7·60	10·0	12·5	15·0

From 1 to 15 millimetres V is given roughly by the equation

$$V=30+12d.$$

It will be seen that the diameter of the corona for a pressure of 18 kilovolts is 0·025 cm. Thus if the diameter of the wire is less than 0·025 cm., in a dark room it will be seen surrounded with a corona when the pressure between it and the earth is greater than 18 kilovolts. Approaching an earthed conductor to the wire will increase the luminous effects. E. Jona found that the diameter of the corona was 1·5 cm. whether the wire were 0·01 or 1·4 cm. in diameter, when the pressure was 196 kilovolts.

6. *The Stress in the Dielectric round Two Particles having equal and opposite charges of Electricity.*

Let there be a charge $+q$ of electricity at P (fig. 1) and a charge $-q$ at N. The potential v at a point distant r_1 and r_2 from P and N respectively is given by

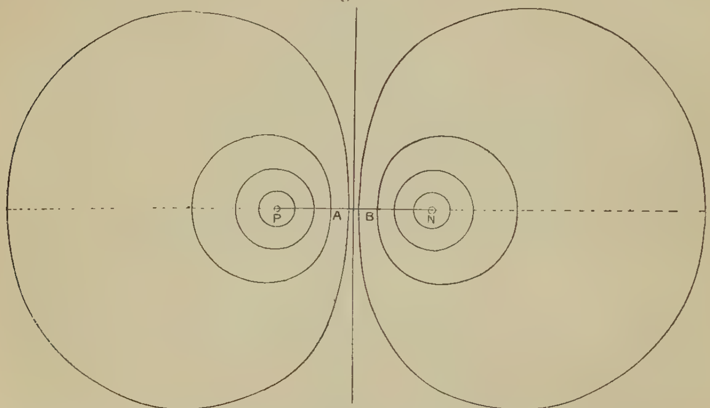
$$v=q/r_1-q/r_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The locus of the points, the bipolar coordinates of which

* E. Jona, *Elettricistà*, Rome, xiii. pp. 113–115, April 15, 1904; Science Abstracts, vol. vii. B, p. 605.

satisfy the equation (a) for a given value of v, will give the surface on which the potential is v. Hence these surfaces (fig. 1) can be easily constructed. We see that the equipotential surfaces near P and N are practically spheres.

Fig. 1.



Equipotential surfaces round two points having equal and opposite charges.

Hence the equipotential lines shown in the figure are approximately the same as those of two spheres at a considerable distance apart.

Let V/2 and −V/2 be the potentials on any two equal equipotential surfaces surrounding P and N respectively. We shall find a formula to determine the maximum value of the electric intensity between these two surfaces. From symmetry the maximum value of the electric intensity R_m in the space between the two will be at the points A and B, where the line joining P and N cuts the surfaces. If $PN=d$ and $PA=a$, we have

$$\frac{V}{2}=\frac{q}{a}-\frac{q}{d-a},$$

and therefore
$$\frac{q}{V}=\frac{a(d-a)}{2(d-2a)}.$$

We also have
$$R_m=q\left\{\frac{1}{a^2}+\frac{1}{(d-a)^2}\right\},$$

and hence
$$R_m=(V/x)f, \qquad (b)$$

where x, which equals $d-2a$, is the minimum distance between the two surfaces and f is given by

$$f=\tfrac{1}{2}(1+x/a)+\frac{1}{2(1+x/a)}. \quad . \quad . \quad . \quad . \quad (c)$$

Now V/x is the average value of the electric intensity along the line joining the nearest points of the surfaces, and is the number which electricians ordinarily give as a measure of the dielectric stress on the insulating medium. We see that f is the factor required to convert this number into the maximum electric intensity.

When x/a is large the surfaces are very approximately spheres of radius a, and (c) can therefore be used to calculate the value of f for two spheres when their distance apart is large compared with the radius of either.

When x/a is small we can show that

$$f=1+x/3\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

approximately, where ρ is the radius of curvature of the equipotential surfaces at the points where the intensity is a maximum. We should expect therefore that, if we had two spheres the radius of each of which was ρ, (d) would give the value of f approximately when x/ρ was small. We shall show later on that (d) gives the value of f in this case, to an accuracy of one in a thousand when x/ρ is 0·1 or less. Even when x/ρ is unity the error is only about 2 per cent.

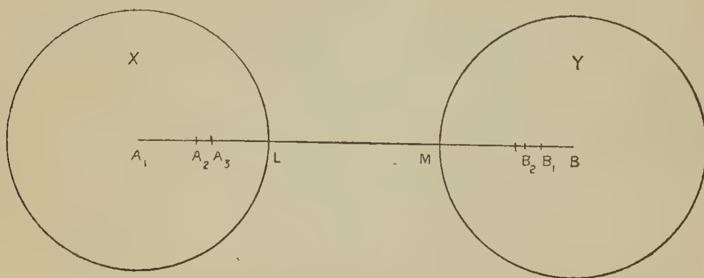
The two terms given on the right-hand side of (d) are the first two terms in the important Kirchhoff-Schuster formula quoted below.

7. *Proof of the Series-Formula for the Maximum Electric Intensity between Two Equal Spheres.*

Let us suppose that the radii of the conducting spheres X and Y (fig. 2) are each equal to a, that the distance between their centres A_1 and B is d, and that the minimum distance LM between them is x, so that $d=x+2a$. Let us suppose also that these spheres are at potentials V_1 and V_2. We picture Faraday-tubes starting from their surfaces. If their potentials are of opposite sign some of these tubes connect the two spheres and others connect them with neigh-

bouring conductors. We suppose that these other conductors are so far away that they do not appreciably affect the distribution of the tubes in the field between the two spheres. Now, if the spheres be removed we shall show that this field can be exactly reproduced by a series of point charges placed at definite points on the lines AL and BM (fig. 2). The

Fig. 2.



$A_1B=d$; $A_1L=BM=a$; $LM=x=d-2a$.

point charges will have the spherical surfaces X and Y for the equipotential surfaces V_1 and V_2 respectively. We can therefore write down at once the potentials and the electric intensities at all external points.

We shall first consider the series of points A_1, A_2 . . . B_1, B_2 . . . (fig. 2) which are connected by the following relations,

$$BA_1 \,.\, BB_1 = a^2 = A_1A_2 \,.\, A_1B_1$$

$$BA_2 \,.\, BB_2 = a^2 = A_1A_3 \,.\, A_1B_2.$$

. . . .

We see that the points A_2, B_1 ; . . . A_{n+1}, B_n, are conjugate with respect to the sphere X and the points B_1, A_1 ; . . . B_n, A_n, are conjugate with respect to the sphere Y. Let

$$A_1\,A_{n+1}=u_{n+1} \text{ and } B\,B_n=u'_n,$$

then the above equations may be written

$$(d-0)u_1' = a^2 = u_2(d-u_1')$$

$$(d-u_2)u_2' = a^2 = u_3(d-u_2')$$

. . . .

In general, we have

$$(d-u_{n-1})u'_{n-1} = a^2 = u_n(d-u'_{n-1}),$$

and thus

$$u_n\{d-a^2/(d-u_{n-1})\} = a^2,$$

or

$$u_n u_{n-1}-\{(d^2-a^2)/d\}u_n-(a^2/d)u_{n-1} = -a^2.$$

This form of difference equation is well known* and is readily solved by assuming that $u_n = v_{n+1}/v_n+(d^2-a^2)d$. Making this assumption we find that

$$v_{n+1}+\{(d^2-2a^2)/d\}v_n+(a^4/d^2)v_{n-1} = 0,$$

a linear difference equation with constant coefficients. Hence solving in the ordinary way † we get

$$v_n = \mathrm{A}a^n(a/d-q)^n+\mathrm{B}a^n(a/d-1/q)^n,$$

where A and B are constants and

$$2q = d/a - \sqrt{d^2-4a^2}/a, \quad . \quad . \quad . \quad . \quad (1)‡$$

and

$$2/q = d/a + \sqrt{d^2-4a^2}/a, \quad . \quad . \quad . \quad . \quad (2)$$

so that

$$1/q+q = d/a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$1/q-q = \sqrt{d^2-4a^2}/a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now when n is unity $u_1=0$, and thus

$$v_2/v_1 = -(d^2-a^2)/d = -a(1+q^2+q^4)/\{q(1+q^2)\}.$$

Substituting for v_2 and v_1 their values, in terms of A and B, in this equation we find that $\mathrm{B} = -\mathrm{A}q^2$. We thus find on substituting for v_{n+1} and v_n their values and simplifying, that

$$u_n = aq\frac{1-q^{4n-4}}{1-q^{4n-2}}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$u_n' = \frac{a_n}{d-u_n}$$

$$= aq\frac{1-q^{4n-2}}{1-q^{4n}}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

* Boole's 'Finite Differences,' 3rd ed. p. 233.

† Boole's 'Finite Differences,' chap. xi.

‡ I have called this expression q so as to introduce elliptic function notation. It is a pure number and has nothing to do with an electric charge.

Let us now suppose that charges $Q_1, \ldots Q_n$, are placed at the points $A_1, \ldots A_n$ (fig. 2), and that charges $Q_1', \ldots Q_n'$, are placed at the points $B_1, \ldots B_n$. We shall find the values of these charges so that the potential of the spherical surface X is V_1 and that of the spherical surface Y is zero.

Consider the potential at a point P' at a distance a from B. The potential at this point will obviously be

$$\Sigma\left(\frac{Q_n}{P'A_n} + \frac{Q_n'}{P'B_n}\right).$$

If therefore we choose the ratio of Q_n to Q_n' so that $Q_n/Q_n' = -P'A_n/P'B_n =$ a constant for every point on Y, the potential at P' will be zero, and therefore also the potential of the spherical surface Y will be zero. Since A_n and B_n are conjugate points with respect to the sphere Y we have *

$$\frac{Q_n'}{Q_n} = -\frac{P'A_n}{P'B_n} = -\frac{BB_n}{a} = -\frac{u_n'}{a}. \quad . \quad . \quad . \quad (7)$$

Again, the potential at a point P distant a from A is given by

$$\frac{Q_1}{a} + \Sigma\left(\frac{Q_{n+1}}{PA_{n+1}} + \frac{Q_n'}{PB_n}\right).$$

This will be V_1 if we make $Q_1 = V_1 a$ and

$$\frac{Q_{n+1}}{Q_n'} = -\frac{PA_{n+1}}{PB_n} = -\frac{A_1A_{n+1}}{a} = -\frac{u_{n+1}}{a}. \quad . \quad . \quad (8)$$

Hence if we determine the charges Q_n and Q_n' by means of (7) and (8) the potential of the spherical surface X will be V_1 and that of the spherical surface Y will be zero.

From (7) and (8), we have

$$\frac{Q_{n+1}}{Q_n} = \frac{u_{n+1}u_n'}{a^2}$$

$$= q^2\frac{1-q^{4n-2}}{1-q^{4n+2}},$$

and thus, since $Q_1 = V_1 a_1$, we have

$$Q_n = aV_1\frac{(1/q-q)q^{2n-1}}{1-q^{4n-2}}. \quad . \quad . \quad . \quad (9)$$

Hence also

$$Q_n' = -aV_1\frac{(1/q-q)\,q^{2n}}{1-q^{4n}}. \quad . \quad . \quad . \quad (10)$$

* Russell, 'Alternating Currents,' vol. i. p. 101.

The electric intensity between the two spheres will obviously have its maximum values R_m at L and M, and thus,

$$R_m = \frac{Q_1}{a^2} + \frac{Q_2}{(a-u_2)^2} + \ldots + \frac{Q_{n+1}}{(a-u_{n+1})^2} + \ldots$$
$$- \frac{Q_1'}{(d-a-u_1')} - \ldots - \frac{Q_n'}{(d-a-u_n')^2} - \ldots$$

Now by (8)

$$\frac{Q_n'}{(d-a-u_n')^2} = -\frac{Q_{n+1}(u_{n+1}/a)}{(a-u_{n+1})^2},$$

and hence

$$R_m = \frac{Q_1}{a^2} + \Sigma \frac{Q_{n+1}}{a} \cdot \frac{a+u_{n+1}}{(a-u_{n+1})^2}.$$

Substituting for u_{n+1} and Q_{n+1} their values from (5) and (9) we get

$$R_m = \frac{V_1}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-3}}{(1+q^{4n-3})^2} q^{2n-2}. \quad . \quad . \quad (11)$$

The value of the electric intensity R_m' at M is given by

$$R_m' = \frac{Q_1}{(d-a)^2} + \ldots + \frac{Q_n}{(d-a-u_n)^2} + \ldots$$
$$- \frac{Q_1'}{(a-u_1')^2} - \ldots - \frac{Q_n'}{(a-u_n')^2} - \ldots$$

Noticing that $d-u_n = a^2/u_n'$ and that $Q_n/Q_n' = -a/u_n'$, we find that

$$R_m' = -\frac{V_1}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-1}}{(1+q^{4n-1})^2} q^{2n-1}. \quad . \quad (12)$$

We can write down the values of R_m and R_m' when the spheres X and Y are at potentials 0 and V_2 in a similar manner. Hence by the principle of superposition we find that the electric intensity at L when the spheres are at potentials V_1 and V_2 is given by

$$R_m = \frac{V_1}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-3}}{(1+q^{4n-3})^2} q^{2n-2}$$
$$- \frac{V_2}{a}\frac{(1+q)^2}{1-q}\sum_1^\infty \frac{1-q^{4n-1}}{(1+q^{4n-1})^2} q^{2n-1} \quad . \quad . \quad (13)$$

The most important case is when

$V_1 = -V_2 = V/2$, and in this case

$$R_m = \frac{V}{2a} \cdot \frac{(1+q)^2}{1-q} \sum_1^\infty \frac{1-q^{2n-1}}{(1+q^{2n-1})^2} q^{n-1} \quad . \quad . \quad (14)$$

$$= \frac{V}{x} \cdot f,$$

where

$$f = \frac{x}{2a} \frac{(1+q)^2}{1-q} \sum_1^\infty \frac{1-q^{2n-1}}{(1+q^{2n-1})^2} q^{n-1}, \quad . \quad . \quad . \quad (15)$$

and x is the minimum distance between the two spheres. It is convenient to tabulate f for various values of x/a. We see that f is the factor which converts the average electric intensity in the line joining the centres of the two spheres into the maximum electric intensity. In measuring dielectric strengths electricians as a rule merely give the average electric intensity, assuming that f is unity whatever may be the shape of the electrodes.

Another practical case is when one of the spheres is maintained at zero potential. In this case let us suppose that $V_1 = V$, and $V_2 = 0$. Hence by (13)

$$R_m = \frac{V}{x} f_1,$$

where

$$f_1 = \frac{x}{a} \frac{(1+q)^2}{1-q} \sum_1^\infty \frac{1-q^{4n-3}}{(1+q^{4n-3})^2} q^{2n-2}, \quad . \quad . \quad . \quad (16)$$

In practice it is very difficult to make certain that one sphere is at zero potential, and so this method of testing dielectric strengths is not advisable.

We may write (13) in the form

$$R_m = \frac{V_1}{x} f_1 - \frac{V_2}{x}(2f - f_1)$$

$$= \frac{V_1 - V_2}{x} f_1 + 2\frac{V_2}{x}(f_1 - f). \quad . \quad . \quad (17)$$

Thus if we can calculate the values of f and f_1 for any given value of x/a we have completely solved the problem.

When V_2 is zero or negative we see, since $f_1 - f$ is always

positive, that for a given value of $V_1 - V_2$, R_m has its greatest value when V_2 is zero and has its least value when $V_2 = -V_1$.

We shall now give methods and formulæ for calculating f and f_1, and we shall also give tables of their values.

8. *Approximate Formulæ for the Maximum Electric Intensity between Two Equal Spheres.*

Formula (15) may be written in the form

$$f = \frac{x}{2a}\left\{1 + \frac{(1+q)^2}{1-q}\cdot\frac{1-q^3}{(1+q^3)^2}q + \dots \right.$$
$$\left. + \frac{(1+q)^2}{1-q}\cdot\frac{1-q^{2n+1}}{(1+q^{2n+1})^2}q^n + \dots\right\}.$$

It is easy to see that the $(n+1)$th term of the series in the brackets equals

$$\frac{1+q+q^2+\dots+q^{2n}}{(1-q+q^2-\dots+q^{2n})^2}q^n = \frac{q^n+1/q^n+q^{n-1}+1/q^{n-1}+\dots}{\{q^n+1/q^n-(q^{n-1}+1/q^{n-1})+\dots\}^2}.$$

By formula (3)

$$q + 1/q = d/a = 2 + x/a = y \text{ (say)},$$

and thus *

$$q^n + 1/q^n = y^n - ny^{n-2} + \frac{n(n-3)}{\lfloor 2}y^{n-4} - \dots$$
$$+ (-1)^r\frac{n(n-r-1)\dots(n-2r+1)}{\lfloor r}y^{n-2r}\dots$$

We can thus easily express f in terms of y. Substituting and simplifying we find that

$$f = \frac{y-2}{2}\left\{1 + \frac{y+1}{(y-1)^2} + \frac{y^2+y-1}{(y^2-y-1)^2} + \frac{y^3+y^2-2y-1}{(y^3-y^2-2y+1)^2}\right.$$
$$+ \frac{y^4+y^3-3y^2-2y+1}{(y^4-y^3-3y^2+2y+1)^2}$$
$$+ \frac{y^5+y^4-4y^3-3y^2+3y+1}{(y^5-y^4-4y^3+3y^2+3y-1)^2}$$
$$\left. + \quad \dots\dots\dots \right\}.$$

Now y cannot be less than 2. Hence expanding by the binomial theorem and neglecting $1/y^9$ and higher powers of

* Todhunter's 'Theory of Equations,' 3rd ed. p. 183.

$1/y$, we find that

$$f=\frac{y-2}{2}\left\{1+\frac{1}{y}+\frac{4}{y^2}+\frac{9}{y^3}+\frac{17}{y^4}+\frac{33}{y^5}\right.$$
$$\left.+\frac{64}{y^6}+\frac{126}{y^7}+\frac{252}{y^8}+\ldots\right\}. \quad . \quad . \quad . \quad . \ (18)$$

It will be seen that the coefficients of $1/y$ are rapidly getting larger, but it has to be remembered that f must equal unity when $y-2$ is zero. We therefore alter the above formula so as to make $f=1$ when y is 2, and yet make the expanded form of the altered formula agree with (18) as far as the coefficient of $1/y^8$. By this means we secure that the formula (19) gives the correct value of f when y is 2, and again when we can neglect the ninth term in the series formula (15). Expanding $y/(y-2)$ in powers of $1/y$ as far as the term containing the eighth power, and substituting in (18) we get

$$f=\frac{y-2}{2}\left\{-\frac{1}{y}+\frac{y}{y-2}+\frac{1}{y^3}+\frac{1}{y^4}+\frac{1}{y^5}-\frac{2}{y^7}-\frac{4}{y^8}\right\}$$

approximately, or

$$f=\tfrac{1}{2}(y-1)+\frac{1}{y}+\frac{(y-2)}{2}\left\{\frac{1}{y^3}+\frac{1}{y^4}+\frac{1}{y^5}-\frac{2}{y^7}-\frac{4}{y^8}\right\}. \quad . \ (19)$$

Substituting $2+x/a$ for y, we get

$$f=\tfrac{1}{2}(x/a+1)+\frac{1}{x/a+2}+\frac{x/a}{2(x/a+2)^3}+\frac{x/a}{2(x/a+2)^4}$$
$$+\frac{x/a}{2(x/a+2)^5}-\frac{x/a}{(x/a+2)^7}-\frac{2x/a}{(x/a+2)^8}. \quad . \quad . \ (20)$$

The values of f are easily computed by this formula. For values of x/a less than 0·1 or greater than 0·7 the error is less than 1 in 1000, whilst for values of x/a between 0·1 and 0·7 the error is never as great as 2 in 1000. For practical purposes therefore the formula (20) gives the values of f with sufficient accuracy. We could have made it more accurate by taking more terms into account in the expansion (18), but we have not done so, as we have found by actual computation that the Kirchhoff–Schuster formula

$$f=1+\frac{1}{3}\cdot\frac{x}{a}+\frac{1}{45}\cdot\frac{x^2}{a^2}+\frac{73}{53760}\cdot\frac{x^3}{a^3} \quad . \quad . \quad . \ (21)$$

sums the series with a most gratifying accuracy until x/a gets

greater than 0·7. It therefore completely covers the part of the scale of our formula which is slightly inaccurate. The formulæ (20) and (21) therefore give the complete practical solution. It is not easy to give a simple proof of (21), but we have found above by elementary considerations the first two terms. If we expand the expression (20) in powers of x/a we get

$$f=1+\frac{11}{32}\cdot\frac{x}{a}-\frac{3}{256}\cdot\frac{x^2}{a^2}+\frac{11}{256}\cdot\frac{x^3}{a^3}\cdot \quad . \quad . \quad . \quad (22)$$

The difference between the values of f given by (21) and (20) when x/a is small is roughly the hundredth part of x/a, and as f is greater than unity it will be seen that the percentage error made by using (22) instead of (21) is small.

In Table III. below the values of the column headed f have been found directly from the series-formula (15). In calculating this column I have to acknowledge the help I received from four of my pupils, Messrs. Hewitt, Hoggett, Ritter, and Taylor. In the second column the numbers are calculated by (21), and in the third column by (20).

I am indebted to Mr. Arthur Berry, of King's College, Cambridge, for showing me how the direct calculation can be greatly simplified. The formula (15) may be written

$$f=\frac{x}{2a}\cdot\frac{(1+q)^2}{(1-q)\sqrt{q}}\left\{\frac{\mathrm{K}k}{2\pi}-\frac{2}{q^{3/2}}\sum_1^\infty\frac{q^{3n}}{(1+q^{2n-1})^2}\right\},$$

for*

$$\frac{\mathrm{K}k}{2\pi}=\sum_1^\infty\frac{q^{(2n-1)/2}}{1+q^{2n-1}}.$$

We have used this theorem to check several of our results. For instance, when x/a is 0·5, q is also 0·5 by formula (1). Also

$$\dagger \ \sqrt{2k\mathrm{K}/\pi}=2q^{\frac{1}{4}}(1+\sum_1^\infty q^{n^2+n})$$

$$=2q^{\frac{1}{4}}(1+q^2+q^6+q^{12}+q^{20}+\ldots)$$

$$=2^{\frac{3}{4}}(1+0{\cdot}25+0{\cdot}015625$$

$$+0{\cdot}000244+\ldots)$$

$$=2^{\frac{3}{4}}(1{\cdot}26587).$$

* A. Enneper, *Elliptische Functionen*, p. 179.

† A. G. Greenhill, 'Elliptic Functions,' p. 303.

Therefore
$$\frac{k\mathrm{K}}{2\pi}=2^{-\frac{1}{2}}(1\cdot26587)^2$$
$$=1\cdot1331.$$

Also
$$\sum_1^\infty \frac{q^{3n}}{(1+q^{2n-1})^2}=0\cdot07001.$$

We thus find that $f=1\cdot1726$, when $x/a=0\cdot5$.

Knowing the values of f we can find the values of f_1 easily by means of an elliptic integral series which is quoted in Kirchhoff's paper. It can be shown that *

$$\sum_1^\infty (-1)^{n-1} q^{\frac{2n-1}{2}} \frac{1-q^{2n-1}}{(1+q^{2n-1})^2}=\frac{kk'\mathrm{K}^2}{\pi^2}.$$

Hence it follows from (15) and (16) that

$$f_1=f+\frac{x}{2a}\frac{(1+q)^2}{(1-q)\sqrt{q}}\left(\frac{kk'\mathrm{K}^2}{\pi^2}\right)$$
$$=f+\sqrt{\frac{x}{a}}\cdot\frac{x+4a}{2a}\cdot\frac{kk'\mathrm{K}^2}{\pi^2}. \quad . \quad . \quad . \quad (23)$$

The values of k, k', and K can easily be found by well-known formulæ. Let us suppose, for instance, that we wish to find the value of f_1 when x/a is $0\cdot5$. We have already found that f is $1\cdot1726$ and q is $0\cdot5$.

Now †
$$\sqrt{2\mathrm{K}/\pi}=1+2\sum_1^\infty q^{n^2}$$
$$=1+1+0\cdot125+0\cdot003906$$
$$+0\cdot000031+\ldots$$
$$=2\cdot1289.$$

We also have

$$\ddagger\ \sqrt{2\mathrm{K}k'/\pi}=1+2\sum_1^\infty(-)^n q^{n^2}$$
$$=1-1+0\cdot12503-0\cdot00391+\ldots$$
$$=0\cdot12112.$$

Thus $\qquad k'=(0\cdot12112/2\cdot1289)^2=0\cdot0032369$

and $\qquad k=\sqrt{1-k'^2}=0\cdot99999.$

Hence $\qquad kk'\mathrm{K}^2/\pi^2=0\cdot01662.$

* A. Enneper, *Elliptische Functionen*, p. 180.
† A. G. Greenhill, 'Elliptic Functions,' p. 303.
‡ Greenhill, p. 303.

Thus finally by (16)

$$f_1 = 1{\cdot}1726 + \{9/(4\sqrt{2})\}(0{\cdot}01662)$$
$$= 1{\cdot}1990.$$

This agrees with the value of f_1 found by direct calculation from (16).

For values of x/a greater than unity the values of f_1 can be computed by the remarkably simple formula

$$f_1 = x/a + \frac{1}{x/a+1} + \frac{1}{(x/a+1)(x/a+2)^3}. \quad . \quad . \quad (24)$$

Hence it is unnecessary to tabulate the values of f_1 when x/a is greater than 4. The first row in the following table is taken from Schuster's paper, the second row is calculated by the formula

$$f_1 = x/a + \frac{1}{x/a+1}, \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and the third row by (24).

TABLE II.—Values of f_1.

x/a.	4.	5.	6.	7.	8.
Schuster's values ...	4·200	5·172	6·144	7·126	8·111
f_1 by (25)	4·200	5·167	6·143	7·125	8·111
f_1 by (24)	4·201	5·167	6·143	7·125	8·111

The values of f_1 given in the last row are the correct values.

For values of x/a greater than 1·5 the values of f given in the last column are correct to four decimal figures. We have shown above by direct calculation that the value of f when x/a is 0·5 is 1·1726. The Kirchhoff-Schuster formula makes it 1·1724. This formula is therefore very accurate for values of x/a less than 0·5, and these are the values which it is so laborious to find by direct computation from (15).

TABLE III.—Values of f.

x/a.	q by (1).	f by (15).	f by (21).	f by (20).
0·0	1·0000	1·000	1·0000	1·0000
0·1	0·7298	1·034	1·0336	1·0343
0·2	0·6417	1·068	1·0676	1·0686
0·3	0·5821	1·102	1·1020	1·1032
0·4	0·5367	1·137	1·1370	1·1384
0·5	0·5000	1·173	1·1724	1·1735
0·6	0·4693	1·208	1·2083	1·2095
0·7	0·4431	1·245	1·2447	1·2460
0·8	0·4202	1·283	1·2814	1·2832
0·9	0·4006	1·321	1·3190	1·3210
1·0	0·3820	1·359		1·3594
1·5	0·3139	1·559		1·5594
2·0	0·2680	1·770		1·7704
3·0	0·2087	2·214		2·2149
4·0	0·1716	2·677		2·6777
5·0	0·1459	3·151		3·1513
6·0	0·1270	3·632		3·6317
7·0	0·1125	4·117		4·1165
8·0	0·1010	4·604		4·6044
9·0	0·0917	5·095		5·0946
10·0	0·0838	5·586		5·5865
100·0	0·0098	50·51		50·5098
1000·0	0·0012	500·5		500·5010

In the following table for the values of f_1 the first column is taken from Table III. The next column is calculated by the equation

$$\Delta=\sqrt{\frac{x}{a}}\cdot\frac{x+4a}{2a}\cdot\frac{kk'K^2}{\pi^2},$$

and the last column for f_1 is got by the equation

$$f_1=f+\Delta.$$

TABLE IV.—Values of f_1.

x/a.	f from Table III.	Δ.	f_1.
0	1·000	0·00000	1·000
0·1	1·034	0·00001	1·034
0·2	1·0676	0·0008	1·068
0·3	1·102	0·004	1·106
0·4	1·137	0·013	1·150
0·5	1·173	0·026	1·199
0·6	1·208	0·045	1·253
0·7	1·245	0·068	1·313
0·8	1·283	0·095	1·378
0·9	1·321	0·125	1·446
1·0	1·359	0·158	1·517

The values of f_1 given in this table are in exact agreement with the numbers given by Professor Schuster *.

* Phil. Mag. vol. xxix. p. 192.

9. *The Disruptive Discharge between Two Spherical Electrodes.*

The formulæ and tables given above enable us to find the maximum value R_m of the intensity of the electric field round two spherical electrodes provided that the electrodes are not enveloped by coronæ; that is, provided that none of the air surrounding them is broken down. If no coronæ are formed before the disruptive discharge ensues, then we can calculate R_m at this instant, and so find $R_{max.}$ the dielectric strength of the air. As in the case of a concentric main or two concentric spheres, it is of importance to know in what cases coronæ can be formed. The problem is now much more difficult as the coronæ are only approximately spherical, the maximum thickness of the stratum of conducting air round each electrode being on the line joining the centre of the two spheres.

If we make the assumption that the surrounding air is broken down to the same depth at every point on the surface of either electrode, we can find whether the value of R_m increases or diminishes with this depth. In the former case a disruptive discharge will certainly ensue, and *a fortiori* it will ensue in the actual case of two spherical electrodes, as the actual breakdown begins at the centre of the spherical face, raising, as it were, a small blister at that point, and so R_m must be greater owing to the greater curvature.

When the distance between the spheres is greater than the radius a, we have, to an accuracy of 1 in a 1000,

$$R_m = \frac{V}{x}\left\{\tfrac{1}{2}(1+x/a) + \alpha/(2+x/a)\right\},$$

where α is 1·077, provided that x/a is less than 7.

Hence

$$R_m = \frac{V}{2}\left\{\frac{1}{d-2a} + \frac{1}{a} + \frac{2a\alpha}{d(d-2a)}\right\}$$

$$= \frac{V}{2}\left\{\frac{1+\alpha}{d-2a} + \frac{1}{a} - \frac{\alpha}{d}\right\},$$

and therefore

$$\frac{dR_m}{da} = \frac{V}{2}\left\{\frac{2(1+\alpha)}{(d-2a)^2} - \frac{1}{a^2}\right\},$$

when V and d are constants. Hence, for values of d less than $a(2+\sqrt{2(1+\alpha)})$, that is, for values of d less than 4·04a, R_m increases as a increases, and thus, on our assumption, a disruptive discharge will ensue.

If the spheres be not further apart than twice their diameter we should therefore expect a disruptive discharge to ensue the moment R_m became R_{max}. For large spheres, experiment shows that this is the case up to a distance apart equal to about three times their diameter. For greater distances apart, the moment R_m attains the value $R_{max.}$ the air in the neighbourhood of that point is broken down and a partial corona is formed, the value of R_m at the surface of the corona being less than R_{max}. In these cases, as the equipotential surfaces are no longer spheres, we cannot apply our formulæ.

10. *The Maximum Electric Intensity between a Sphere and a Plane.*

When the plane is at zero potential, we see, by taking the image of the sphere in the plane, that

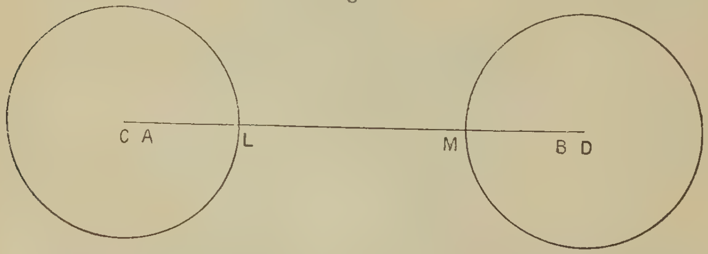
$$R_m = (V/x)\, f_p, \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where f_p is the value of the factor f given above corresponding to $2x/a$; x being the least distance of a point on the sphere from the plane and a being its radius.

11. *The Maximum Electric Intensity between two infinitely long parallel Cylinders.*

Let us consider the value of the electric potential at points

Fig. 3.



CD$=d=$the distance between the axes of the two parallel cylinders.
LM$=x=$the minimum distance between the cylinders.
$a=$the radius of either cylinder.
A and B are inverse points. $CA\,.\,CB=CL^2=DA\,.\,DB$.

between the two cylinders, the sections of which by the plane of the paper are shown in fig. 3. If q and $-q$ be the

charges per unit length on the cylinders the axes of which pass through A and D respectively, the potential v at any point P external to them is given by

$$v=-2q\log(\mathrm{AP}/\mathrm{BP}),$$

where A and B are the inverse points of the circular sections. The maximum values R_m of the electric intensity will be at L and M. The potential at any point p on CD will be

$$v=-2q\log r+2q\log(c-r),$$

where r is Ap and c is the distance AB. Hence

$$\mathrm{R}=\frac{2q}{r}+\frac{2q}{c-r}.$$

Now R has its maximum value R_m when r is AL.

Hence $$\mathrm{R}_m=\frac{2qc}{\mathrm{AL}(c-\mathrm{AL})}=\frac{2qc}{a(d-2a)}=\frac{2qc}{ax},$$

where x is the minimum distance between the cylinders. Now, we have *

$$q=\frac{\mathrm{V}}{4\log\{(d+c)/2a\}},$$

where V is the potential-difference between the cylinders and $c^2=d^2-4a^2$.

Hence we have

$$\mathrm{R}_m=\frac{\mathrm{V}}{x}\frac{c/2a}{\log\{(d+c)/2a\}}$$

$$=\frac{\mathrm{V}}{x}f,$$

where $$f=\frac{y}{\log(1+x/2a+y)}, \quad . \quad . \quad . \quad . \quad (27)$$

and $$y=\{x/a+(x/2a)^2\}^{\frac{1}{2}}.$$

* Russell, 'Alternating Currents,' vol. i. p. 102.

Values of f are given in the following table:—

TABLE V.

x/a ...	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09
f	1·00	1·00	1·00	1·01	1·01	1·01	1·01	1·01	1·015

x/a ...	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
f	1·02	1·03	1·05	1·065	1·08	1·10	1·11	1·13	1·14

x/a ...	1	2	3	4	5	6	7	8	9
f	1·16	1·315	1·46	1·61	1·74	1·88	2·01	2·14	2·24

x/a ...	10	20	30	40	50	100	1000	10,000
f	2·39	3·56	4·62	5·62	6·58	11·0	72·5	587

12. *The Disruptive Discharge between Two Parallel Cylinders.*

It is well known in practice that when we have two parallel wires with a high P.D. between them, then in certain cases coronæ envelope the wires. When they are close together, however, this effect is not produced, a disruptive discharge occurring directly the P.D. attains a certain value. It is important therefore to know what distance apart the wires must be in order that coronæ can be formed.

If we assume that $\sec\theta = d/2a$, we find that

$$R_m = \frac{V}{d(1-\cos\theta)} \cdot \frac{\tan\theta}{\log(\tan\theta + \sec\theta)}.$$

Let d be constant and let a vary, then, solving the equation $dR_m/d\theta = 0$, we find that

$$\log\tan(\pi/4 + \theta/2) = \frac{\sin\theta}{\sin^2\theta - \cos\theta}.$$

When θ is nearly 70° this equation is satisfied, and in this case $d = 5{\cdot}85\,a$ nearly.

Hence making the assumption that the coronæ are cylindrical in shape, we see that R_m diminishes as a increases when d is greater than $5{\cdot}85\,a$. In practice, therefore, we should not expect coronæ to be formed when the wires were at a less distance apart than about three times their diameter.

13. *The Application of the Formulæ to Experimental Results.*

I. WITH DIRECT PRESSURES.

(i.) *Lord Kelvin's tests with large electrodes.*

Lord Kelvin * was the first to make accurate tests on the disruptive voltages between electrodes in air. He found that the apparent dielectric strength of a thin stratum of air was much greater than that of a thick one. The apparent dielectric strength in our notation being Vf/x, we have

$$Vf/x = R_{\text{max.}} + 0{\cdot}8\, f/x,$$

where V is in kilovolts, and $R_{\text{max.}}$, the dielectric strength of air, is a constant. In Kelvin's experiments f was practically equal to unity at all distances, and thus V/x increases rapidly as x diminishes.

From his experimental results† Lord Kelvin concludes that a battery of 5510 Daniell cells could produce a spark between two slightly convex electrodes when the minimum distance between them was 1/8th of a centimetre. Taking the E.M.F. of a Daniell cell as 1·07 volts, this makes the dielectric strength $R_{\text{max.}}$ of air to be 40·8 kilovolts per centimetre; a result which is only about 6 per cent. higher than the number we give as the average value of R_{max}.

(ii.) A. Heydweiller. 5 *cm. spheres* (2·5 *cm. radius*).

In the valuable paper by A. Heydweiller, published in the *Annalen der Physik und Chemie*, vol. xlviii. p. 785 (1893), there are many tables of sparking-distances given both between equal and unequal electrodes. We consider merely the last table he gives, and we choose the 5 centimetre spheres as being likely to give the most accurate results.

* Proc. Roy. Soc. 1860, or 'Reprint,' p. 24.

† Proc. Roy. Soc. April 12, 1860, p. 259.

The height of the barometer was 74·5 cms., and the temperature 18° C. during the test. The columns headed x and V are taken from Heydweiller's paper, f_1 is calculated by the formulæ given above, and $R_{max.}$ is found by

$$R_{max.} = \{(V - 0{\cdot}8)/x\} f_1.$$

We have assumed that the potentials of the spheres are V and 0 at the instant of the discharge. The results seem to indicate that this was not the case when the electrodes were at their greatest distances apart.

TABLE VI.

Heydweiller's test with 5 cm. spheres ($a = 2{\cdot}5$).

x=distance apart in cms. V=disruptive pressure in kilovolts.

x.	x/a.	f_1 (calc.).	V (observed).	$R_{max.}$ (calc.).
0·5	0·2	1·068	18·36	37·5
0·6	0·24	1·081	21·60	37·5
0·7	0·28	1·102	24·54	37·3
0 8	0·32	1·116	27·33	37·0
0·9	0·36	1·132	30·09	36·9
1·0	0·40	1·150	32·85	36·9
1·1	0·44	1·169	35·58	37·0
1·2	0·48	1·188	38·31	37·0
1·3	0·52	1·209	41·01	37·4
1·4	0·56	1·231	43·68	37·7
1·5	0·60	1·253	46·23	37·9
1·6	0·64	1·277	48·66	38·2

The mean value of the numbers in the last column is 37·5, and none of them differ from the mean value by as much as 2 per cent. Hence this experiment gives 37·5 kilovolts per centimetre as the dielectric strength of air.

(iii.) J. Algermissen. 5 *cm. spheres* ($a = 2{\cdot}5$ *cm.*).

In the following table the values of x and V are taken from Dr. Zenneck's work 'Elektromagnetische Schwingungen und Drahtlose Telegraphie,' 1905, p. 1011. They are due to J. Algermissen, and are deduced from the average of the values obtained on different days under varying conditions. We have assumed that the potentials of the electrodes were $+V/2$ and $-V/2$ respectively at the instant of the discharge.

As the results in the last column are very approximately constant our assumption is justified.

TABLE VII.

J. Algermissen. 5 cm. spheres ($a=2·5$).

x is measured in cms. and V in kilovolts.

x.	x/a.	f(calc.).	V (obs.).	$R_{max.}$ (calc.).
1·5	0·6	1·208	46·2	36·6
1·6	0·64	1·223	48·6	36·5
1·7	0·68	1·238	51·0	36·6
1·8	0·72	1·253	53·4	36·6
1·9	0·76	1·268	55·8	36·7
2·0	0·80	1·283	58·2	36·8
2·1	0·84	1·298	60·6	37·0
2·2	0·88	1·312	62·8	36·9
2·3	0·92	1·326	65·0	37·0
2·4	0·96	1·342	67·0	37·0
2·5	1·00	1·360	69·0	37·1
2·6	1·04	1·374	70·8	37·0
2·7	1·08	1·390	72·6	37·0
2·8	1·12	1·406	74·4	37·0
2·9	1·16	1·421	76·2	37·0
3·0	1·20	1·437	78·0	37·0
3·1	1·24	1·452	79·7	37·0
3·2	1·28	1·469	81·3	37·0
3·3	1·32	1·484	83·0	37·0
3·4	1·36	1·500	84·7	37·0
3·5	1·40	1·515	86·4	37·1
3·6	1·44	1·533	88·0	37·1
3·7	1·48	1·549	89·6	37·2
3·8	1·52	1·566	91·2	37·3
3·9	1·56	1·583	92·7	37·3
4·0	1·60	1·599	94·2	37·4
4·1	1·64	1·616	95·7	37·4
4·2	1·68	1·632	97·2	37·4

The mean of the values of $R_{max.}$ in the last column gives the dielectric strength of air as 37·0 kilovolts per centimetre, and the greatest difference between any of the calculated numbers and this value is only about one per cent. It will be seen therefore, that the agreement between theory and experiment is quite satisfactory. The three final values for $R_{max.}$ obtained in this table closely agree with the mean of the values we deduced from Heydweiller's test. Considerable weight, therefore, must be attached to the results of these experiments in determining the value of $R_{max.}$.

(iv.) J. Joubert and G. Carey Foster. 1 *cm. and* 2 *cm. spheres* ($a=0{\cdot}5$ & 1).

In Foster and Porter's (Joubert's) 'Electricity and Magnetism,' p. 135, tables of the sparking-distances between 1 centimetre and 2 centimetre spheres are given. The results for the 1 cm. spheres are taken from Joubert's *Traité élémentaire d'électricité* (2nd edit.), and those for the 2 cm. spheres were obtained by G. Carey Foster. An analysis of the table for the 1 centimetre spheres shows that if we calculate $R_{max.}$ for sparking-distances of 5, 10, and 15 cms., on the assumption that the air round the electrodes is not broken down to any appreciable depth before the discharge occurs, the values are much too large. This is in accord with the conclusion of § 9. The mean of the values up to a distance of 2 cms. apart makes $R_{max.}$ 42·8. The mean of the values for the 2 cm. spheres makes $R_{max.}$ 42·9.

(v.) É. Hospitalier. 1 *cm. spheres.*

An analysis of the experimental results given by É. Hospitalier in the *Formulaire de l'Électricien*, 21st year, 1904, p. 289, for the sparking-distances between two electrodes, each one cm. in diameter, shows that the potentials of the spheres are not $+V/2$ and $-V/2$ at the instant of the discharge. The values of $R_{max.}$ calculated on this assumption diminish steadily from the maximum value 44·1 when the spheres are 0·6 of a cm. apart to 40·0 when they are 2 cms. apart. The values of f, however, are little affected by the absolute values of the potentials of the electrodes, provided that x/a is not greater than 0·3. Taking, therefore, the mean of the first three results given, we find that $R_{max.}$ is 42·2.

(vi.) Compagnie de l'Industrie Électrique. *Plate and sphere.*

The Compagnie de l'Industrie Électrique et Mécanique have published tests * on the disruptive voltages between a plate and a ball.

* Turner and Hobart, 'Insulation of Electric Machines,' p. 33 (1905).

TABLE VIII.

Compagnie de l'Industrie Électrique. Plate and 2 cm. ball.

x.	x/a.	f_p (calc. by § 10).	V (obs.).	$R_{max.}$
0·5	0·5	1·36	18	46·8
1·0	1·0	1·77	26	44·6
1·5	1·5	2·21	31	44·5
2·0	2·0	2·68	35·5	46·5
2·5	2·5	3·15	39	48·1
3·0	3·0	3·63	42·5	50·4
4·0	4·0	4 60	48·0	54·2
5·0	5·0	5·59	54·0	59·5
6·0	6·0	6·57	58·0	62·5

It will be seen that $R_{max.}$ is beginning to increase rapidly (see § 9). The mean of the first four values gives 45·6 kilovolts per centimetre as the dielectric strength of air. In practice the plates used are not large, and so we are only justified in using our formula for f_p when the plate and the ball are close together. We do not attach much importance to this test.

II. WITH ALTERNATING PRESSURES.

(i.) C. P. Steinmetz. *2 inch spheres.*

In a paper on the "Dielectric Strength of Air," published in the Transactions of the American Institute of Electrical Engineers, vol. xv. p. 281, Professor C. P. Steinmetz gives the results of an elaborate and careful research on the disruptive voltages between pointed, spherical, and cylindrical electrodes. Alternating voltage was used of frequency 125, and the shape of the wave was practically identical with a sine curve when a particular smooth-core alternator was used. The ratio of the maximum to the effective voltage in all his experiments with this machine was practically 1·42. The spherical and cylindrical electrodes were put in nitrate of mercury and then rubbed with a clean cloth. When this was done it was found that the disruptive discharge, for a given distance apart of the electrodes, always took place at the same voltage. If the electrodes were merely polished, then at small distances apart the results were very erratic. The accuracy of the results obtained probably lies well within

4 per cent. in most cases. In the experiments the barometer varied from 75·2 to 76·2 cms. This variation introduces an uncertainty of about one per cent. The voltmeter readings may be one per cent. out, and there may be a one per cent. error in determining the ratio of the maximum to the effective potential-difference. An error is also due to the moisture in the air. This, however, was found to be small. When the electrodes were immersed in "live" steam at atmospheric pressure, the effect of the steam was to *increase* the apparent dielectric strength of the air, a greater voltage being required to produce the disruptive discharge. As pressures up to 160 effective kilovolts were employed, the sparking-distances were large and could be measured with great accuracy. We should expect that, with these high voltages, our formulæ would apply with considerable accuracy, as the disturbing effect of the cathode glow would be small and the field would be approximately symmetrical.

In the following table the results of tests when the electrodes were spheres 2 inches in diameter are analysed. The column headed $R_{max.}$ gives the values of the dielectric strength in kilovolts per centimetre, calculated by the formula

$$R_{max.} = \{(1{\cdot}42V - 0{\cdot}8)/x\}f.$$

TABLE IX.

C. P. Steinmetz. 2 inch spheres ($a = 2{\cdot}54$ cms.).
$\sim = 125$. $E/V = 1{\cdot}42$, where E is the maximum and V the effective value of the alternating voltage.

No. of Experiment.	x.	x/a.	f(calc.).	V (obs.).	$R_{max.}$(calc.).
1	0·318	0·125	1·04	8·95	39·0
2	0·635	0·25	1·08	15·9	37·1
3	1·25	0·49	1·17	26·7	34·7
4	2·74	1·08	1·39	51·0	36·2
5	3·69	1·45	1·54	65·2	38·3
6	4·29	1·69	1·63	70·8	37·9
7	5·72	2·25	1·88	83·8	38·9
8	7·62	3·00	2·21	94·0	36·7
9	8·74	3·44	2·42	102·0	39·9
10	10·0	3·95	2·66	101·5	38·0
11	12·9	5·08	3·19	108·0	37·7
12	14·2	5·60	3·44	114·5	39·1

The mean value of $R_{max.}$ obtained from the figures in the last column is 37·8. Considerable importance is attached to this test as the numbers actually observed are given.

The curve in fig. 4 (p. 82) gives the relation between V and x on the supposition that $R_{max.}$ is 38. Steinmetz's experimental results are plotted in this figure for purposes of comparison.

(ii.) Compagnie de l'Industrie Électrique. 2 *cm. spheres.*

The Compagnie de l'Industrie Électrique et Mécanique of Geneva have published* a curve giving the sparking-distances between two spherical electrodes, each one centimetre in radius. The frequency of the alternating pressure employed was 50, and the ratio of the maximum to the effective voltage was 1·26. Calculating $R_{max.}$ by the formula

$$R_{max.} = \{(1{\cdot}26V - 0{\cdot}8)/x\}f$$

for values of x from 0·5 cm. to 5 cms., we find that the mean value of $R_{max.}$ is 37·9, which practically agrees with Steinmetz's result for 2 inch spheres.

(iii.) C. P. Steinmetz. 1, 0·5, *and* 0·25 *inch spheres.*

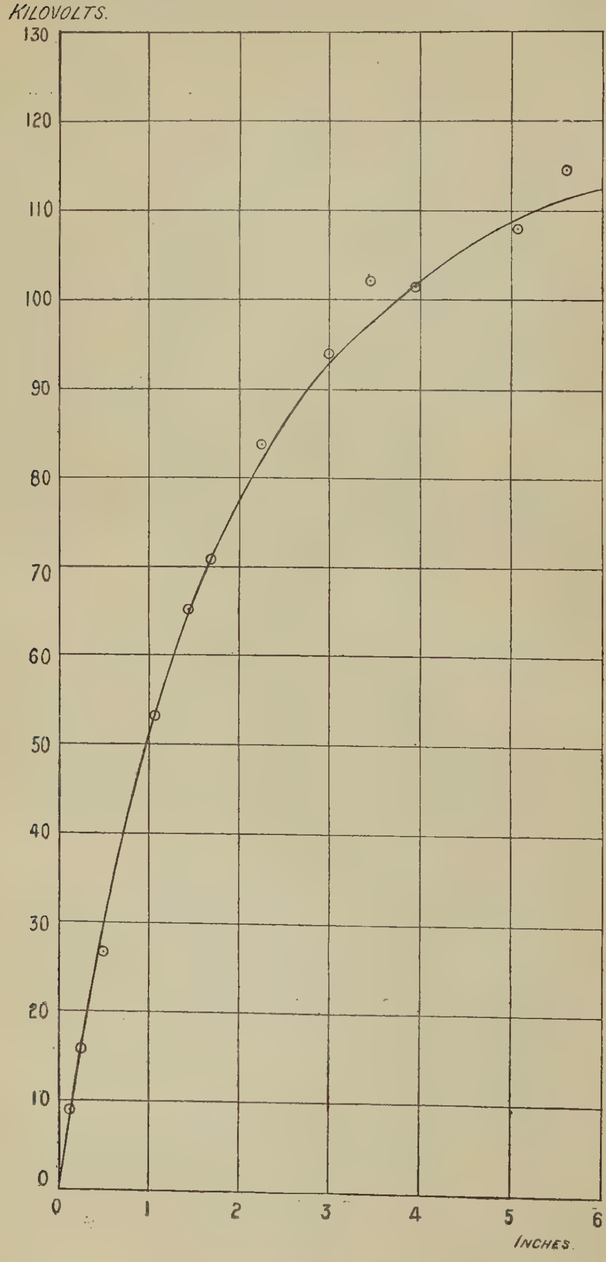
The analysis of Steinmetz's experiments with 1, 0·5, and 0·25 inch spheres are instructive, but for reasons explained in § 9 they do not give much assistance in obtaining $R_{max.}$ With the 1 inch spheres the mean of the values of $R_{max.}$ obtained up to pressures of 63·7 effective kilovolts is 41·3 kilovolts per centimetre. With the half-inch spheres the mean of the values for pressures up to 31·3 effective kilovolts is 43·1.

When the quarter-inch spheres were 28 cms. apart, the disruptive pressure was 112 effective kilovolts. If we calculate $R_{max.}$ for this pressure as if the spheres were in a vacuum, we find that it is more than six times the dielectric strength of air. In the experiment there must have been coronæ round each of the electrodes after the pressure was about 17 effective kilovolts.

In these experiments the frequency was 125 and E/V was equal to 1·42.

* Turner and Hobart, 'The Insulation of Electric Machines,' p. 35.

Fig. 4.—Sparking Voltages between 2 inch spherical electrodes.
Points marked ⊙ are Steinmetz's experimental results.



(iv.) E. Jona. *Point and Plate. Two spheres.*

E. Jona has published * a table giving the sparking-distances between a point and a plate and also between two equal spherical electrodes each of 2 cms. diameter, for pressures varying from 15 to 240 kilovolts. When the electrodes are far apart it is obvious that the distribution of the Faraday-tubes is considerably affected by the supporting rods connecting the electrodes with the transformer terminals. An important result proved by these experiments is that for all distances greater than 23 cms. the sparking-voltages, with the electrodes used by Jona, were the same in the two cases. For instance, when the maximum value of the applied P.D. was 240 kilovolts ($\sim = 42$), the sparking-distance was 47 cms., whether the point and the plate or the two spherical electrodes were used. Faraday anticipated this result in his 'Experimental Researches,' § 1499:—"But as has long been recognized, the small body is only a blunt end, and, electrically speaking, a point only a small ball; so that when a point or blunt end is throwing out its brushes into the air, it is acting exactly as the small balls have acted in the experiments already described, and by virtue of the same properties and relations."

(v.) C. P. Steinmetz. 0·795 *cm. cylindrical electrodes* ($a=0{\cdot}3975$).

Professor Steinmetz in the paper referred to above also describes tests on the sparking-distances between cylindrical electrodes. In one case the electrodes were two copper rods 0·795 cm. in diameter and 71 cms. long. The rods were slightly curved, so that the sparks ensued across the minimum distance between them. The radius of curvature at this part was 198 cms., and so, provided the rods are not further apart than about 2 cms., we can neglect the curvature and assume that the field is very similar to that between two infinitely long parallel straight rods. We can therefore use the formula (27) for f given in § 11.

* E. Jona, *Atti dell' Associazione Elettrotechnica Italiana*, vol. vi. p. 3, "Distanze Esplosive nell' Aria, negli olii ed altri Liquidi Isolanti."

TABLE X.

C. P. Steinmetz. Two cylindrical rods slightly curved. Diameter of rods 0·795 cm. $\sim=125$, $E/V=1{\cdot}42$.

No. of Experiment.	x.	x/a.	f (calc.).	V (obs.).	$R_{max.}$
1	0·165	0·42	1·07	4·78	38·9
2	0·203	0·51	1·08	5·70	38·8
3	0·433	1·15	1·18	10·85	38·9
4	0·881	2·21	1·35	18·55	39·1
5	1·115	2·81	1·43	23·7	42·2
6	1·194	3·01	1·46	22·9	38·8
7	1·435	3·61	1·55	23·0	34 3
8	1·753	4·40	1·66	30·5	40·3
9	2·134	5·38	1·80	34·3	40·4
10	2·362	5·96	1 88	33·1	36·8

The mean of the values of $R_{max.}$ given in the last column makes the dielectric strength of air 38·8 kilovolts per centimetre.

(vi.) C. P. Steinmetz. 1·11 *inch cylindrical electrodes.*

Experiments were also made with large cylinders 1·11 inch in diameter and 20 inches long. Up to a distance apart of about one-third of an inch we may assume that our formula applies approximately.

We have neglected therefore the experimental results for greater distances. The mean of the values of $R_{max.}$ deduced from the first five experiments is 34 kilovolts per centimetre. We do not attach so much importance to this result as to the preceding as our formula does not apply so accurately.

14. *Table of the Numbers obtained for the Dielectric Strength of Air from the Direct Pressure Experiments.*

TABLE XI.—Direct Pressures.

	Nature of Electrodes.	Authority.	$R_{max.}$
Table VII...	5 cm. spheres.	J. Algermissen.	37·0
Table VI. ...	5 cm. spheres.	A. Heydweiller.	37·5
§ 13, I., i. ...	Slightly convex surfaces.	Lord Kelvin.	40·8
§ 13, I., v....	1 cm. spheres.	É. Hospitalier.	42·2
§ 13, I., iv...	1 ,, ,,	J. Joubert.	42·8
,, ,,	2 ,, ,,	G. Carey Foster.	42·9

Of the above tests the first three seem to be the most accurate. The mean of the results obtained from these three tests makes the dielectric strength of air 38·4 kilovolts per centimetre.

15. *Table of the Numbers obtained for the Dielectric Strength of Air from the Alternating Pressure Experiments.*

Table XII.—Alternating Pressures.

	Nature of Electrodes.	Authority.	$R_{max.}$
§ 13, II., vi.	1·11 inch cylinders.	C. P. Steinmetz.	34
§ 13, II., ii.	2 cm. spheres.	Comp. de l'Ind. Élect.	37·9
Table IX.	2 inch spheres.	C. P. Steinmetz.	37·8
Table X.	0·313 inch cylinder.	„	38·8
§ 13, II., iii.	1 inch spheres.	„	41·3

An examination of the experimental results obtained and the methods of calculating $R_{max.}$ from them shows that the second, third, and fourth of the above tests are the only really satisfactory ones. The mean of the results obtained in these three tests is 38·2 kilovolts per centimetre, and this agrees closely with the number we obtained from the direct pressure experiments.

16. *Conclusion.*

We conclude therefore that the dielectric strength of the air at ordinary atmospheric pressures lies between 38 and 39 kilovolts per centimetre, which is about 30 per cent. greater than the value ordinarily given. J. J. Thomson * gives the value as approximately 30 kilovolts per centimetre and M. O'Gorman † as 27 kilovolts per centimetre.

The confidence of electricians who are responsible for the working of high-pressure net-works for the distribution of electric power ‡ on the working of their spark-gap safety-

* J. J. Thomson, 'Electricity and Magnetism,' p. 59 (1904).

† M. O'Gorman, "Insulation on Cables," Journal of the Inst. of Elect. Eng. vol. xxx. p. 666 (1901).

‡ Dusaugey, *Soc. Int. Elect.*, Bull. 5, pp. 109–132. "Méthode de Protection contre les Surtensions actuellement employée dans les Réseaux de Transport d'Energie," Feb. 1905.

valves at the moment the pressure attains a definite value, and the extensive use they make of micrometer spark-gaps * for measuring high voltages, prove that under ordinary working conditions they find that the dielectric strength of air is approximately constant. In ordinary work we may take its value as 38 kilovolts per centimetre.

Appendix I.

The Disruptive Voltages for Large Spherical Electrodes.

Table XIII. gives the disruptive pressures in kilovolts between equal spherical electrodes when their radii are 1, 10, 100, and 1000 cms. respectively. The dielectric strength of air has been taken as 38 kilovolts per centimetre, and V and V′ are calculated by the formulæ

$$V = 0{\cdot}8 + R_{\text{max.}}(x/f) \text{ and } V' = V/\sqrt{2},$$

respectively. V′ therefore gives the effective value of the disruptive voltage when the pressure is alternating and sine-shaped.

Table XIII.

Calculated Values of the Disruptive Voltages between large Spherical Electrodes.

V = kilovolts (direct pressure). V′ = effective kilovolts (alternating pressure) when $V' = V/\sqrt{2}$.

x in cms.	2 cm. spheres.		20 cm. spheres.		200 cm. spheres.		2000 cm. spheres.	
	V.	V′.	V.	V′.	V.	V′.	V.	V′.
0·1	4·5	3·2	4·6	3·25	4·6	3·25	4·6	3·25
0·5	17·0	12·0	19·4	13·9	19·8	14·0	19·8	14·0
1·0	28·8	20·3	37·7	26·7	38·8	27·4	38·8	27·4
5·0	61·1	43·3	163	115	187	132	191	135
10·0			280	198	370	262	381	269
50·0			604	427	1625	1150	1860	1320
100·0					2795	1980	3690	2610
500·0					6030	4270	16200	11500
1000·0							28000	20000
5000·0							60000	42500

* P. H. Thomas, Amer. Inst. Electr. Engin. Proc. xxiv. pp. 705–742. "An Experimental Study of the Rise of Potential on Commercial Transmission Lines due to Static Disturbances caused by Switching, Grounding, etc.", July, 1905.

If the electrodes were infinite planes, the direct pressure required to produce the disruptive discharge when they were 50 metres apart would be 190 million volts.

With spherical electrodes of 10 metre or less radius, about 60 million volts would be sufficient to spark over the same distance. We suppose of course that the P.D.'s are established sufficiently slowly to allow the Faraday-tubes to attain their positions of statical equilibrium approximately before the discharge occurs.

Appendix II.

The Capacity Currents to the Electrodes.

The formulæ (9) and (10) enable us to find at once the analytical expressions for the electrostatic coefficients of two equal spheres. If Q, Q′ denote the charges and V_1 and V_2 the potentials of the electrodes, we have *

$$Q = K_{1\cdot 1} V_1 + K_{1\cdot 2} V_2$$

and

$$Q' = K_{2\cdot 2} V_2 + K_{2\cdot 1} V_1.$$

In our case $K_{1\cdot 1} = K_{2\cdot 2}$. By making $V_2 = 0$, we get by (9)

$$K_{1\cdot 1} = a\frac{1-q^2}{q}\sum_1^\infty \frac{q^{2n-1}}{1-q^{4n-2}},$$

and by (10)

$$K_{1\cdot 2} = -a\frac{1-q^2}{q}\sum_1^\infty \frac{q^{2n}}{1-q^{4n}}.$$

Denoting Lambert's series $\sum_1^\infty \frac{q^n}{1-q^n}$ by $F(q)$ we get

$$K_{1\cdot 1} = a\frac{1-q^2}{q}\{F(q) - 2F(q^2) + F(q^4)\}$$

and

$$-K_{1\cdot 2} = a\frac{1-q^2}{q}\{F(q^2) - F(q^4)\}.$$

By a known transformation due to Clausen † we have

$$F(q) = \sum_1^\infty q^{n^2}\frac{1+q^n}{1-q^n};$$

and this series can be very readily computed.

* Russell, 'Alternating Currents,' vol. i. p. 89 *et seq.*

† Crelle's *Journal*, vol. iii. p. 95, quoted in Jacobi's *Fundamenta Nova*, pp. 187–188.

For instance, when x is 0·5 we find by (1) that q is also 0·5, and

$$F(0\cdot5)=1\cdot6067,\ F(0\cdot25)=0\cdot4210,\ \text{and}\ F(0\cdot0625)=0\cdot0709.$$

Hence we find that

$$K_{1\cdot1} = 1\cdot2534a \text{ and } K_{1\cdot2} = -0\cdot5252a.$$

The capacity between the two spheres * is

$$(K_{1\cdot1} - K_{1\cdot2})/2 = 0\cdot8893a$$

and the capacity † for equal potentials is

$$2(K_{1\cdot1} + K_{1\cdot2}) = 1\cdot4565a.$$

To reduce these values to microfarads we divide by 900,000, a being measured in centimetres.

When the potentials follow the harmonic law we have

$$A_1 = (K_{1\cdot1} V_1 + K_{1\cdot2} V_2)\omega$$

and

$$A_2 = (K_{1\cdot1} V_2 + K_{1\cdot2} V_1)\omega,$$

where A_1 and A_2 are the effective values of the capacity currents flowing to the two spheres respectively; V_1, V_2 the effective values of their potentials, and $\omega/2\pi$ the frequency of the alternating pressures.

For instance, suppose that we have two 20 cm. spherical electrodes 5 cms. apart, and suppose that the effective value of the P.D. between them is 90 kilovolts. Then, if ω be 1000, so that the frequency is nearly 160, and the potentials of the electrodes be equal and opposite at every instant, we have

$$\begin{aligned} A = \omega KV &= 1000 \times 0\cdot889 \times 10 \times 90 \times 10^3/(9 \times 10^5 \times 10^6) \\ &= 0\cdot000889 \text{ ampere.} \end{aligned}$$

If the potentials of the electrodes be not equal and opposite at every instant, the difference between the effective values of the capacity currents to the electrodes equals the effective value of the current in the earth connexion. Our formulæ are not applicable when the electrodes are surrounded with coronæ. In this case the capacity between them is considerably increased.

* Russell, 'Alternating Currents,' vol. i. p. 92.

† Russell, 'Alternating Currents,' vol. i. p. 393.

The author is indebted to Mr. Arthur Berry for suggesting the use of Clausen's theorem as an aid in calculating the capacity coefficients of two equal spheres.

Postscript.

I have received from Principal G. Carey Foster the results of experiments made in his laboratory in 1876 on the sparking distances between 2·6 cm. brass knobs. As these results are of considerable interest I have obtained his permission to publish them.

G. Carey Foster.—2·6 cm. spheres ($a=1·3$).

x.	x/a.	f.	V (obs.).	$R_{max.}$
0·05	0·0385	1·013	3·09	46·4
0·1	0·0769	1·026	5·04	43 5
0·2	0·1538	1·051	8·43	40·0
0 3	0·2307	1·077	11·46	38·3
0·4	0·3076	1·105	14·61	38·1
0·5	0·3846	1·131	17·49	37·7
0·6	0·4614	1·159	20·43	37·9
0·7	0·5383	1·185	23·37	38·2
0·8	0·6152	1·213	26·25	38·5
0·9	0·6921	1·242	29·13	39·0

A 'home-made' absolute electrometer was used to measure the voltage. The attracted disk was hung from an ordinary balance and the attraction weighed directly. Up to 30 kilovolts it gave trustworthy readings.

Neglecting the first value of $R_{max.}$ as the formula given in the paper is only roughly applicable when the distances are less than one millimetre, we find that the mean value of $R_{max.}$ is 39. If we neglect the first two readings the mean value of $R_{max.}$ is 38·5. Both of these results agree very closely with the final conclusions at which we arrived. It will be seen that the experimental results obtained during the last thirty years on the sparking distances in air at ordinary pressures, when no coronæ are formed, could have been predicted with considerable accuracy from the above results.

Principal Carey Foster also suggested the formula

$$V = 2·13 + 30·6\,x,$$

for the sparking potentials between 2·6 cm. knobs. It is

interesting to notice that Baille and many other experimenters subsequently suggested linear formulæ for the relation between V and x.

In reply to a question by Professor Poynting, I have worked out the values of $R_{max.}$ for the case of Heydweiller and Algermissen's tests on the assumption that the ordinary electrostatic equations hold, without modification, at the instant of breaking down. In this case we have

$$R'_{max.} = (V/x)f,$$

where f can be found by formula (20) given above. The values of $R_{max.}$ are those found in Tables VI. and VII.

Heydweiller's Test.

	Maximum value.	Minimum value.	Mean value.
$R_{max.}$	37·5+0·7	37·5−0·6	37·5
$R'_{max.}$	38·35+0·85	38·35−0·55	38·35

Algermissen's Test.

	Maximum value.	Minimum value.	Mean value.
$R_{max.}$	37+0·4	37−0·5	37·0
$R'_{max.}$	37 4+0·3	37·4−0·3	37·4

The last result is so remarkable that I give the complete table.

Algermissen's Test.

x.	$R'_{max.}$	x.	$R'_{max.}$	x.	$R'_{max.}$	x.	$R'_{max.}$
1·5	37·2	2·2	37·5	2·9	37·3	3·6	37·5
1·6	37·1	2·3	37·5	3·0	37·4	3·7	37·5
1·7	37·1	2·4	37·5	3·1	37·3	3·8	37·6
1·8	37·2	2·5	37 5	3·2	37·3	3·9	37·6
1·9	37·2	2·6	37·4	3·3	37·3	4·0	37·7
2·0	37·3	2·7	37·4	3·4	37·4	4·1	37·7
2·1	37·5	2·8	37·4	3·5	37·4	4·2	37·5

Hence, Algermissen's experimental results give us the ratios of all the values of f, from x equal to $1{\cdot}5\,a$ to x equal to $4{\cdot}2\,a$, with a maximum inaccuracy of less than 1·6 per cent. To fully appreciate this result it is necessary to try and sum the series (15) for any two values of x lying between the given limits.

For sparking distances greater than half a centimetre (one fifth of an inch), therefore, when no coronæ, and consequently no brush discharges, are formed, the error made in assuming that the boundaries of the electrodes form the equipotential surfaces is negligibly small. The disruptive discharge ensues as soon as the maximum value of the electric intensity attains a definite value which is the measure of the dielectric strength of the air between the electrodes under the given atmospheric conditions.

Discussion.

Dr. H. A. Wilson expressed his interest in the Author's explanation of the brush discharge and the formation of coronas. When the distance between the electrodes was not too small, it was known that the sparking P.D could be expressed as $V=\alpha+\beta d$, where d was the distance between the electrodes and α and β were constants. This constant β the Author had called the dielectric strength of air, but he did not think he was justified in doing so.

Prof. J. H. Poynting, referring to Table V. in the paper, asked if the rise in value of the dielectric strength as the distance apart of the electrodes increased was due to the formation of coronas.

Mr. Russell, in reply, remarked that Mr. Strutt had shown that the P.D. between the cathode and the negative glow was 341 volts whatever the atmospheric pressure. We were therefore quite justified in assuming that at ordinary pressures the electric pressure on the Faraday tube subject to the maximum stress is $V-\epsilon$, where ϵ is greater than 341. The experimental results analysed in the paper indicate that ϵ is 0·8 of a kilovolt. In answer to Prof. Poynting, he stated that the slight rise in the values of the dielectric strength in Table V. was probably due to the potentials of the electrodes not being V and zero at the instant of discharge.

IV. *The Comparison of Electric Fields by means of an Oscillating Electric Needle.* *By* DAVID OWEN, *B.A.* (*Camb.*), *B.Sc.* (*Lond.*), *Lecturer in Physics, Birkbeck College, London**.

EXPERIMENTS on fields of force due to electric charges suffer as compared with the corresponding measurements in magnetic fields by reason of the impracticability of obtaining an "electric needle" corresponding to the oscillating magnetic needle. But while it is true that a permanently electrified needle cannot be obtained, it is possible to have one that has equal charges of opposite sign induced by the field. This paper contains an account of experiments on the use of such a needle for the measurement of electric fields, steady or alternating, and for the experimental illustration of some of the laws of electrostatics.

Theory.

Consider a cylindrical needle supported at its centre by a fibre whose torsional control is negligible, placed with its axis horizontal in a uniform horizontal field of strength F. The needle will move so as to set itself parallel to the field. If disturbed from that position a restoring couple will come into play. This couple will be proportional, both to the induced charge, *i. e.* to the "pole-strength," and to the field. As the former is proportional to the latter, it follows that the couple will vary as the square of the field-strength. It will diminish as the angle between the axis of the needle and the direction of the field increases, and will clearly be zero when the two are at right angles. In the case of an ellipsoid the couple is proportional to the sine of twice the angle θ between the long axis and the field. Assuming that law to hold for a cylindrical needle, we may write the couple equal to $a\mathrm{F}^2 \sin 2\theta$, where a is some constant. If θ is small we have

$$\frac{\text{Couple}}{\text{Angular displacement}} = \frac{a\mathrm{F}^2 \sin 2\theta}{\theta} = 2a\mathrm{F}^2 \text{ (constant)},$$

* Read June 30, 1905.

in which case the vibrations of the needle will be isochronous, the time being given by

$$T=2\pi\sqrt{\frac{I}{2aF^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)}$$

where I = moment of inertia of the needle.

Writing $T=\frac{1}{N}$, we may denote the frequency of vibration by

$$N=bF, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

where b is a constant depending on the form, size, and mass of the needle. Thus it appears that the strengths of uniform fields are directly in proportion to the frequencies of such a needle oscillating in them.

Of course if the controlling couple due to the suspending fibre is sufficient to take into account, this may be done by observing the frequency N_0 when the electric field is nil. Denoting the frequency actually observed in field by N', we have

$$N=\sqrt{N'^2-N_0^2}=bF. \quad . \quad . \quad . \quad . \quad \text{(II)}'$$

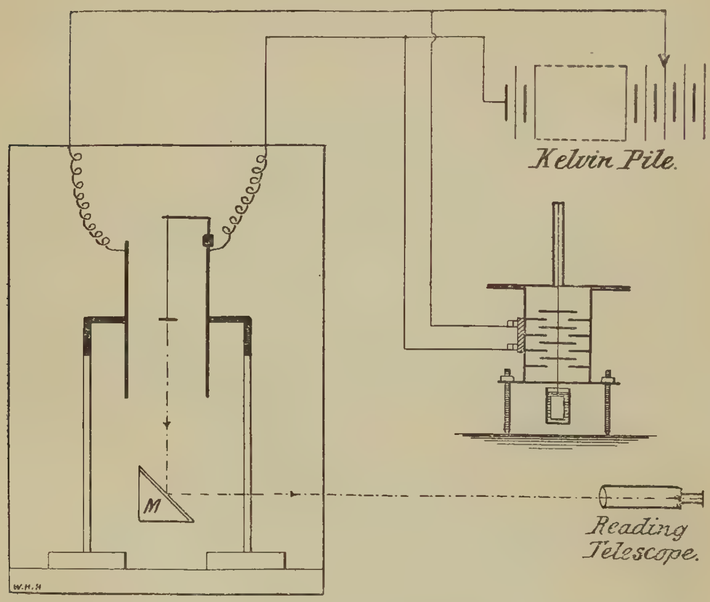
This law was tested by establishing an electric field between a pair of parallel circular plates, each 12 cms. in diameter, kept at a constant distance apart. The difference of potential between them was obtained by means of a Kelvin voltaic pile, whereby any voltage from 400 volts downwards could be applied. The volts were measured by a Kelvin multicellular electrostatic voltmeter reading to 400 volts. The vibrations were observed by the help of a plane mirror placed below the needle, as shown in fig. 1, which represents the apparatus used. The image of the needle in the mirror was viewed through a telescope, in the focal plane of which an image of the needle appears vertical in the position of rest. The vertical cross-wire is set to bisect it.

The time of a counted number of oscillations was obtained by means of a stop-watch reading to fifths of seconds, sufficient passages being allowed to admit of the whole period of time measured being from one to two minutes.

The apparatus was enclosed in a balance-case in order to avoid draughts. Observations were restricted within an

amplitude of oscillation of about 5°, in which case the error due to finite amplitude is negligible.

Fig. 1.



The needle used was of copper wire, 1·4 cms. long and 2 mms. in diameter, suspended by a quartz fibre about one-tenth of a mm. in diameter and 8 centimetres long. The plates were at a distance of 3 cms. apart. The results gave as expected a straight-line law connecting V and N.

Disturbance of the Field due to the presence of the Needle.

The effect of the presence of such a needle in an electric field is in every case to produce a concentration of the field near the needle. Further, if the charged conductor forming one of the boundaries of the field is sufficiently near the needle, the distribution of the charge on that boundary will be altered. This of course prevents the application of the electric needle for the accurate comparison of the electric forces *near* the surfaces of conductors of different forms and curvatures. It was considered desirable before proceeding

further to investigate the nature of the disturbance caused, and to determine its bearing on the applicability of the needle for exact quantitative measurements of field.

A case of importance is that of the field between two parallel metal plates. The question may be put thus: if an electric needle be placed in an infinite uniform field, at what distance does the disturbing influence of the needle cease to be appreciable? I know of no mathematical treatment for the case of a short cylindrical needle such as was used above. The following experiment was therefore performed. The parallel plates already described were used, and the arrangement of apparatus was as in fig. 1. The plates could be moved perpendicular to themselves so as to alter their distance *d* apart. A constant potential-difference of 400 volts was maintained between them. The needle used was of aluminium, of the following dimensions:—

length $=1{\cdot}526$ cms.,
diameter $=\ {\cdot}101$,,
and mass $=\ {\cdot}0329$ grams.

The distance *d* was varied between the limits of 6·04 cms. and 1·92 cms., and values of N for these and several intermediate values of *d* were determined. The needle, suspended by a quartz fibre, was set centrally between the plates in each case. The results may be tabulated as follows:—

$T_0 = 18{\cdot}80$ secs.

No. of exp.	*d* in cms.	T in secs.	N′ per sec.	N $= \sqrt{N'^2 - N_0^2}$.	N×*d*.
1	6·04	7·20	·1392	·1282	·775
2	5·33	6·40	·1562	·1468	·784
3	4·82	5·82	·172	·163	·785
4	4·40	5·34	·1866	·178	·783
5	4·03	5·00	·200	·193	·778
6	3·00	3·69	·271	·2656	·797
7	2·53	3·04	·329	·325	·822
8	2·325	2·72	·368	·364	·844
9	2·07	2·32	·433	·430	·885
10	1·92	2·04	·490	·487	·916

These numbers are represented graphically in fig. 2, where products of frequency × distance apart of plates (N × *d*) are

plotted against distance apart of plates (d). From this curve it may be seen that beyond $d=4\cdot5$ cms., or say

Fig. 2.

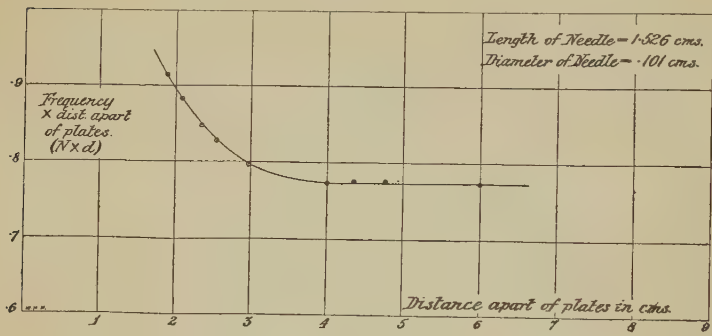
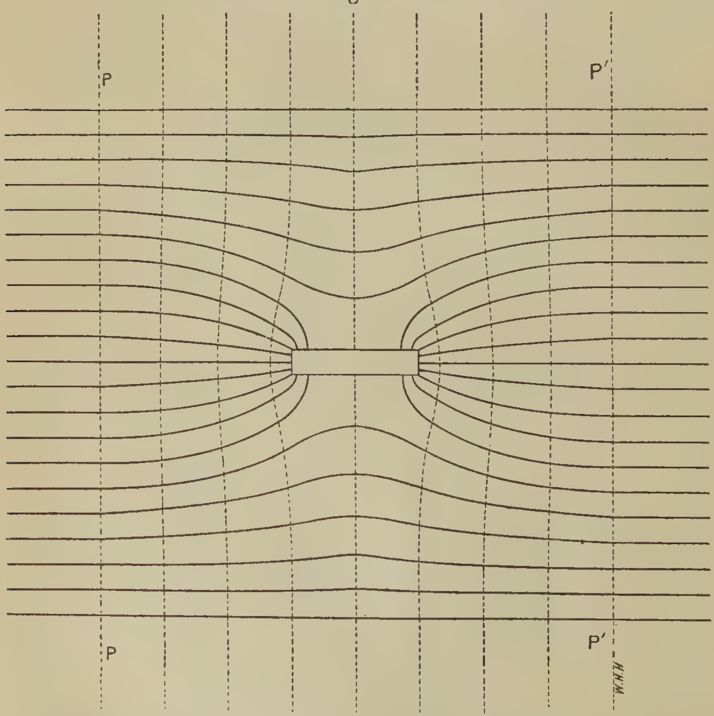


Fig. 3.



exceeding three times the length of the needle, the disturbing effect of the needle is inappreciable, and the product $N \times d$

remains constant. This may be seen further by the aid of a diagram (fig. 3) representing the lines of force and the equipotentials near the needle when the axis of the latter is along the direction of the undisturbed field.

The lines of force are shown as full lines, the equipotentials as broken lines. The equipotentials near the needle are much curved, but the effect diminishes rapidly on going away from the needle, so that the equipotentials PP and P′P′ are practically straight. Beyond these the equipotential surfaces are a series of parallel planes, and the field is uniform and of the same value as if the needle were not present.

The result suggests the possibility of determining by the use of an electric needle the specific inductive capacity K of a dielectric. For if a pair of parallel plates kept at fixed potential-difference be used to establish a uniform field, their distance apart well exceeding the limiting distance at which the disturbing effect of the needle is appreciable; and if between one of the plates and the critical plane a parallel slab of the dielectric be introduced, the effect is to increase the field in the air space by an amount involving K in a simple and well-known way. Of this nothing further will be said in this paper, as careful experiments are being undertaken dealing with the determination of specific inductive capacity by such a method.

In regard to the disturbance introduced in other cases, it is important to notice that for a given arrangement of conductors and given position of the needle, variations of field due to alteration of potential will be measured by the frequency of the needle, the ratio of frequencies being unaffected by the distortion of the field. For example, in the case of the field between a pair of parallel plates at a distance apart only slightly exceeding the length of the needle; though the frequency of the needle is greater, as shown above, than in an infinite field of the same strength, yet it increases exactly in proportion to the difference of potential between the bounding plates.

The choice of Needles.

It is desirable that the needle should be as small as possible, to reduce the disturbing effect; and sensitive, for the measurement of as small fields as possible.

To investigate this latter condition six cylindrical needles were made of aluminium, with dimensions as follows:—

Three needles of diam. of 1 mm.; lengths $1\frac{1}{2}$ cms., 1 cm., ·5 cm.
" " " ·5 mm.; " $1\frac{1}{2}$ " 1 " ·5 "

A similar set was afterwards made in brass.

These needles were drilled centrally at right-angles to their length with a hole of about ·3 mm. in diameter, into which was fitted a suspending wire of the same diameter, and length varying from 5 to 10 mms. To this wire was cemented a quartz fibre about $\frac{1}{15}$ mm. in diameter, and about 8 cms. long, the upper end of the fibre being cemented to a supporting piece attached to one of the plates. These needles were supported in succession between the plates, which were kept at the same distance apart and at the same difference of potential throughout the experiment. The time of vibration of each was thus determined:—The dimensions and mass of the needles being known, the moments of inertia could be calculated, and by equation (I) the couple on each when in a specified field could be determined.

No high degree of accuracy was to be expected where the moment of inertia of such small needles was involved. The results given below, however, show sufficiently the effect of alteration of length and of diameter. The frequency came out less for the brass than for the aluminium needles, just as calculated from their relative density. The values of the frequency in column 4 are for needles of aluminium.

Dependence of Frequency on Dimensions of Needle.
Electric field = 100 volts per cm.

Length (cms.).	Diameter (cms.)	1000 × Couple (dyne-cms.)	Frequency (per sec.) N.	Relative couples.	Relative frequencies.
1·526	·101	9·82	·197	100	100
1·103	·1025	4·42	·233	45	118
·510	·1025	·643	·275	6·54	140
1·500	·050	·693	·337	70·5	171
1·000	·050	·253	·371	25·7	178

These numbers show that although the couple on the needle increases rapidly with length and slowly with diameter, yet the effect on frequency is more than counterbalanced by the increase in the moment of inertia. Thus whilst, as the first three rows of numbers above show, the length of needle decreased in the proportion of about 3·1, the couple decreased in the proportion of about 15 : 1, whilst the frequency went up in the proportion of only 1 : 1·4.

Hence, while as the needle gets smaller the disturbing effect on the field rapidly diminishes, its frequency of vibration increases, though slowly. This is a favourable circumstance to be borne in mind in experiments such as that suggested above for the determination of specific inductive capacity.

Where the disturbance is not of importance, and it is desired to show the vibrations, either directly or by projection in a lantern, the needle may with advantage be large. Aluminium needles of $1\frac{1}{2}$ cms. length and 1 mm. diameter have been found very satisfactory; brass needles should be of smaller diameter. For strong fields the needles may be still heavier, and the suspension may be of silk threads consisting of several single-fibres. There is thus little danger of breakage of the suspension, where, as for purposes of lecture demonstration, the needle has to be moved about and may be more or less roughly used.

Applications.

The electric needle may be regarded as a measurer of *electric force*, and be applied to test the field-strength due to any arrangement of conductors, effect of alteration of their relative position, or of the potential, &c. In some cases exact results will not be obtained owing to the disturbance introduced by the needle itself. In others the disturbing effect of the needle does not enter. Of the first class the following example may be cited :—

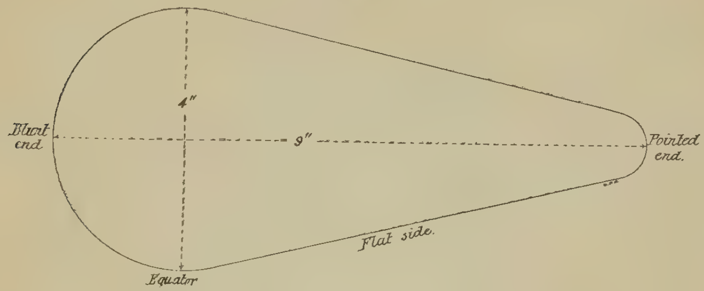
Comparison of the electric field at different points on the surface of any conductor.

The case of a pear-shaped conductor (fig. 4) was examined.

Four positions on it were chosen, namely, at the pointed end

and blunt end, at the equator, and opposite the middle of the flat side (where the curvature is single). The needle

Fig. 4.



used was $1\frac{1}{2}$ cms. long. Its nearer end was set in each case at a distance from the conductor equal to half the length of needle. The results were as follows :—

Position.	Time of oscillation T (secs.).	Relative frequency.
Pointed end	2·55	1
Blunt end	3·20	·8
Equator...............	3·25	·78
Flat side	3·85	·66

For a couple of spherical conductors at a distance from other conductors the following numbers were obtained :—

Diameter of sphere.	T.	Relative frequencies.	Electric force (theoretical).
12 cms.	2·40 secs.	1	1
3 ,,	1·23 ,,	1·96	4

These comparative numbers indicate the increase of electric force F with curvature. By means of a proof-plane the increase of surface-density σ of charge with curvature may be shown. In both cases the numbers suffer in accuracy owing to the disturbance introduced by test needle and proof-plane respectively.

Such experiments as the above are best performed by attaching the conductor by a fine wire to the inner coating of a charged Leyden jar. The potential then changes only very slowly with time.

It may be interesting to add that the potential of the pear-shaped conductor, as determined by an electric needle, was 6200 volts in the case to which the numbers above refer.

In the following two cases the disturbing effect of the needle is of no significance :—

(1) Wherever the variation of electric force with alteration of potential is to be found, no motion of the bounding conductors or of position of needle taking place. The electric force as measured by the frequency of the needle is proportional to the difference of the potential.

(2) To prove that inside a charged conductor (not enclosing an insulated charged conductor) the electric force is zero. Though this theorem is one of fundamental importance, it is not easy to demonstrate it experimentally. In the electric needle we have a means of doing this of a simplicity corresponding to that which the proof-plane affords for showing that the charge inside the conductor is zero.

As the needle is lowered into the conductor, *e.g.* a metal can or a cylinder of wire netting, the frequency of the needle falls until when well inside its value is the same as when the conductor is discharged.

On the Shielding Effect of Dielectrics.

Having mounted a needle within a glass specimen-tube in order to do away with the effects of air-currents, it was observed that even when in the strongest electric fields the needle was quite unaffected. The tube acted as if it were a conductor. As this effect might be due simply to the unclean condition of the surfaces, the experiment was repeated with a carefully cleansed glass lamp-chimney, with the same result. It appears then, that owing to the conductivity of the glass, slight as it is, a closed glass vessel screens electric force from its interior, just as if it were a conductor.

The same result was found for a sheet of mica of thickness $\frac{1}{12}$ mm. bent into a cylinder. A mica cylinder in ordinary air screens the interior completely (when placed in a steady field, as in fig. 5). If heated over a hot plate and quickly

transferred to the field so as to surround a needle swinging in it, there is seen to be electric force at the needle ; but it quickly dies down, and in 20 or 30 seconds the needle once more vibrates in its own natural period, depending on the torsional couple of the quartz suspension.

This conductivity of the mica appears to be a surface effect, probably due to the deposition of moisture from the atmosphere. For if the mica sheet be immersed for some time in melted paraffin-wax at a temperature close on 200° C., then taken out and allowed to cool, the sheet (made into a cylinder as before) is found to be capable of transmitting the field for some time. But even in this case the field within the cylinder falls off with time. Thus in one case a needle within a paraffined mica cylinder gave 12 vibrations in 10 seconds at first, but four minutes later only 8 vibrations were counted in the same time, the fall of frequency being according to the exponential law. Electrical separation has slowly taken place in the mica. If now the plates between which the field was produced be discharged, the cylinder should be left charged, and an electric field should exist in the interior. This is found to be the case. The field dies out with time, as in the experiments in the constant field. Another way of showing this effect of minute conductivity is to introduce the cylinder into a steady field, leave it there for some time, and then turn it about its axis through 180°. The field inside the cylinder is seen to be for the moment greater than that of the original field, for on the latter is superposed the field due to the separated charges on the walls of the cylinder. With cylinders of thicker sheets of mica similarly treated with paraffin-wax, the rate of falling off of the interior field is found to be greater. These experiments indicate an appreciable though exceedingly small conductivity in the mica itself.

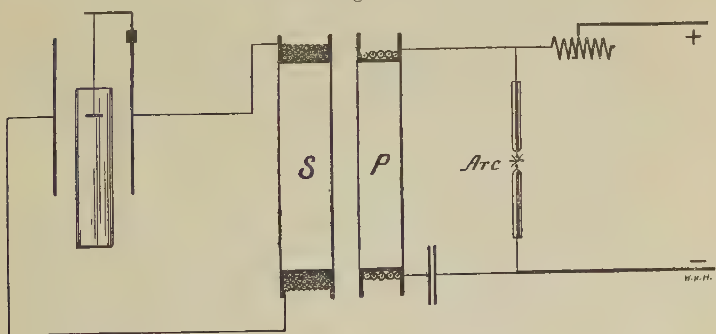
Experiments on the shielding effect of ordinary white paper were made. Cylinders of paper screen the interior completely from the surrounding field. But on heating for a short time over a hot plate the shielding effect vanishes ; in about one minute, however, the frequency of the needle within is found to have fallen considerably, and a minute later is the same practically as in zero field. Dry paper then acts as a perfect insulator. But by the absorption of

moisture when exposed to the open air it quickly gains appreciable conductivity.

With paraffined paper the shielding power was nil, and remained nil for days in the open air of the room. Of all the materials tried this alone permitted for any length of time the undiminished transmission of electric force. Thus by applying a thin coating of paraffin wax to a sheet of dielectric, and then observing its shielding effect on an electric needle, we have a convenient means of testing the conductivity of the dielectric, which removes all doubts as to surface-action. This effect of the high insulating, non-hygroscopic, quality of paraffin-wax is made use of in the experiments on mica above described.

Experiments with alternating fields.—As with steady fields glass and mica proved sufficiently conducting to screen off electric effects, experiments were made with these materials, using alternating electric fields. The alternating voltage was obtained by the use of the Musical Arc. Mr. Duddell, I may say here, very kindly advised me as to the conditions for obtaining high voltages by this means. The connexions were as shown in fig. 5.

Fig. 5.



The alternating voltage on the self-induction P in the shunt circuit to the arc was found to be about 200 volts. This was transformed up by means of a second coil S, to the terminals of which the parallel plates were connected. In this way any voltage up to 1000 volts could be maintained between the plates. The frequency of alternation of this voltage was found, by means of a sonometer and standard fork, to be 1700 per second. The needle was allowed to

vibrate in the field with

(*a*) glass cylinder surrounding it ;
(*b*) cylinder taken away.

The time of vibration was found to be the same in each case, namely 1·83 secs. The glass cylinder had a diameter of 5 cms., the thickness of wall being 1·4 mm. With such dimensions the weakening of the field inside the cylinder due purely to the specific inductive capacity effect should only be small. With a cylinder of thicker glass, but diameter only slightly exceeding the length of the needle (1 cm.), the frequency of vibration was distinctly less in case (*a*) than in case (*b*). With mica cylinders the same result was obtained.

These experiments emphasise the necessity of using alternating fields in all cases where from the measurement of the force experienced by a mass of a dielectric its specific inductive capacity is to be determined. In steady fields, unless the specific conductivity of the material be quite nil, the force will ultimately be the same as for a conductor. This consideration will account for the high values of k obtained in some of the experiments of Boltzmann (see Gray's Abs. Measmts. vol. i. p. 465).

Use of Electric Needle to measure Volts.

A pair of insulated parallel metal plates with a needle suspended centrally between them forms a simple means of measuring high voltages. For we have

volts across plates = frequency of needle × a constant.

The constant may be determined at a low voltage by any voltmeter available. The highest voltage measurable is limited by the difficulty of counting the frequency of the needle if this exceeds 4 or 5 per second. But the range may be extended by opening out the plates ; for as already shown the frequency of the needle for constant difference of potential is inversely as the distance apart of the plates, provided that the minimum distance used exceeds three times the length of the needle.

The constant k is most simply defined as the volts across the plate required to produce a frequency of oscillation of 1 per sec. for a distance apart of the plates equal to 1 cm.

Then if

d = distance apart of plates,
N = frequency,
V = voltage between plates,

we have

$$V = kd \times N. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III)$$

An instance of this application has been given already Another may be added :—

The difference of potential yielded by a Wimshurst machine was measured by charging a Leyden jar, the outer coating being connected to one pole, the inner to the other pole, the poles being kept well apart. The jar was connected with the pair of parallel plates, which were at a distance of 10·5 cms. apart. The frequency of the needle in the field between them was 2·04 per sec. The needle used had a constant (reduced to $d=1$ cm.) of 1300 volts. Applying formula III, we have

$$V = 1300 \times 10{\cdot}5 \times 2{\cdot}04$$
$$= 27800 \text{ volts.}$$

The machine, it may be added, was working badly, giving a spark between balls of 2 cms. and 5 cms. diameter respectively, Leyden jars being in, of something less than 1 cm length.

These experiments were made in the laboratories of the Birkbeck College. I would record my thanks to the Principal for facilities for carrying them out.

V. *Note on Constant-Deviation Prisms.*

By T. H. BLAKESLEY.*

It appears that any prism of three faces can be made to give a spectrum in which the light, that occupies the centre of the field of view of the telescope at any moment, has undergone passage through the two refracting surfaces of the prism in such a way that its original angle of incidence is equal to its final angle of emergence. This condition, which in the ordinary employment of the prism is associated with minimum deviation, must be described as isogonal passage, the property which has the minimum value being not the

* Read November 10, 1905.

deviation, but the rate of passage across the field of view for a given motion of the prism, to which alone in these instruments motion has to be given to bring different parts of the spectrum into the field, the telescope and collimator both remaining fixed. If any triangle having the angles α, β, γ is adopted as the shape of a prism, the telescope must be set to make one of these angles, say γ, with the line of the collimator. Then the prism being placed in the region between them, a position can be found so that any ray selected will be refracted through one of the sides containing the angle γ, reflected at the side opposite γ, and finally refracted through the remaining side containing γ. On emergence it will be parallel to the telescope, and its passage through the refracting faces will be isogonal. The prism will affect the light to the same degree as one used in the ordinary way, of refracting angle $\beta-\alpha$, would do. The sine of the angle of original incidence is equal to $\mu \,.\, \sin(\beta-\alpha)/2$ for every ray occupying the centre of the field of view.

If the prism is turned over but the same angle γ employed, the telescope will remain unaltered but the spectrum will run in an opposite direction to the first. If n similar prisms are piled one upon another, the diameter of the telescope and collimator being large enough to embrace them all, n different parts of the spectrum may be brought into coincidence at the centre of the field of view. Thus such prisms afford an excellent mode of mixing colours. Mr. Blakesley showed the case of two prisms in which the spectra ran in different directions. The top prism was slightly tilted by the insertion of a small piece of silver paper between the prisms. By this means one of the spectra was shifted upward by a small amount, and one could see in the telescope a band, at top and bottom, of the component colours, and in the centre a band of the resulting colours. It was suggested that spectroscopes on this plan could be advantageously employed in measuring the motion on the line of sight of heavenly bodies, as a line brought into coincidence with itself for a terrestrial source in the two spectra would, in the case of such motion, split up into two moving *different* ways in the field of view. It was also explained how such prisms could be placed in trains for increased dispersion.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1905-1906.

February 24th, 1905.

Meeting held at the Royal College of Science.

Prof. POYNTING, President, in the Chair.

The following Candidates were elected Fellows of the Society :—
K. EDGCUMBE and J. E. KINGSBURY.

The following Paper was read :—

The Curvature Method of Teaching Geometrical Optics. By Dr. C. V. DRYSDALE.

Mr. R. J. SOWTER exhibited Dr. Meisling's Colour Patch Apparatus; and Mr. SCHOFIELD showed a Method of Illustrating the Laws of the Simple Pendulum, and exhibited some String Models of Optical Systems.

March 10th, 1905.

Meeting held at the Royal College of Science.

Dr. Glazebrook, Past-President, in the Chair.

The following Candidates were elected Fellows of the Society:—
J. H. Brinkworth and A. Wilkinson.

The following Papers were read:—

1. Direct Reading Resistance Thermometers. By Mr. A. Campbell.

2. The Stresses in the Earth's Crust before and after the sinking of a Bore-Hole. By Dr. C. Chree.

3. The Lateral Vibration of Bars of Uniform and Varying Sectional Area. By Mr. J. Morrow.

March 24th, 1905.

Meeting held at University College.

Prof. Poynting, President, in the Chair.

The following Papers were read:—

1. The Voltage Ratios of an Inverted Rotary Converter. By Mr. W. C. Clinton.

2. The Flux of Light from the Electric Arc with varying Power Supply. By Mr. G. B. Dyke.

3. The Application of the Cymometer, and the Determination of the Coefficient of Coupling of Oscillation Transformers. By Prof. J. A. Fleming.

April 14th, 1905.

Meeting held at the Royal College of Science.

Dr. Glazebrook, Past-President, in the Chair.

The following Candidates were elected Fellows of the Society:—
J. C. Kirkman and P. J. Vinter.

The following Papers were read:—

1. Ellipsoidal Lenses. By Mr. R. J. Sowter.

2. The Determination of the Moment of Inertia of the Magnets used in the Measurement of the Horizontal Component of the Earth's Field. By Dr. Watson.

Dr. WATSON then exhibited a series of Lecture Experiments illustrating the Properties of the Gaseous Ions produced by Radium and other Sources.

May 12th, 1905.

Meeting held at the Royal College of Science.

Dr. CHREE, Vice-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—

J. P. YORKE.

The following Papers were read:—

1. A Simple Method of Determining the Radiation Constant, suitable for a Laboratory Experiment. By Dr. A. D. DENNING.

2. A Bolometer for the Absolute Measurement of Radiation. By Prof. H. L. CALLENDAR.

3. The Resistance of a Conductor the Measure of the Current flowing through it. By Mr. W. A. PRICE.

May 26th, 1905.

Meeting (informal) held at the National Physical Laboratory.

Demonstrations were given on

(1) The Specific Heat of Iron at High Temperatures. By Dr. J. A. HARKER.

(2) The Measurement of Small Inductances. By Mr. A. CAMPBELL.

(3) Two new Optical Benches. By Mr. SELBY.

The Laboratories were thrown open to inspection.

June 16th, 1905.

Meeting held at the Royal College of Science.

Prof. POYNTING, President, in the Chair.

The following Papers were read:—

1. The Ratio between the Mean Spherical and the Mean Horizontal Candle-power of Incandescent Lamps. By Prof. J. A. FLEMING, F.R.S. This was supplemented by a description, by Mr. G. B. DYKE, of the experiments involved.

2. The Electrical Conductivity of Flames. By Dr. H. A. Wilson.

3. Contact with Dielectrics. By Mr. Rollo Appleyard.

4. The Pendulum Accelerometer; an Instrument for the Direct Measurement and Recording of Acceleration. By Mr. F. Lanchester.

5. A New Form of Pyknometer. By Mr. N. V. Stanford.

Mr. Rollo Appleyard exhibited a new form of Refractometer.

June 30th, 1905.

Meeting held at the Royal College of Science.

Dr. Glazebrook, Past-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—
T. Wadsworth.

The following Papers were read:—

1. The Comparison of Electric Fields by means of an Oscillating Electric Needle. By Mr. David Owen.

2. The Fluorescence of Sodium Vapour. By Prof. R. W. Wood.

3. The Magneto-Optics of Sodium Vapour and the Rotary Dispersion Formula. By Prof. R. W. Wood.

October 27th, 1905.

Meeting held at the Royal College of Science.

Prof. Poynting, President, in the Chair.

The following Paper was read:—

The Theory of Phasemeters. By Dr. W. E. Sumpner.

Prof. H. L. Callendar then described an Apparatus for measuring the Coronal Radiation during an Eclipse.

November 10th, 1905.

Meeting held at the Royal College of Science.

Dr. CHREE, Vice-President, in the Chair.

The following Candidates were elected Fellows of the Society:—
S. J. GUNNINGHAM and J. I. P. THOMAS.

The following Papers were read:—

1. The Question of Temperature and Efficiency of Thermal Radiation. By Mr. JAMES SWINBURNE.

2. A Note on Constant-Deviation Prisms. By Mr. T. H. BLAKESLEY.

November 24th, 1905.

Meeting held at the Royal College of Science.

The PRESIDENT, and subsequently Dr. CHREE, in the Chair.

The following Papers were read:—

1. The Dielectric Strength of Air. By Mr. A. RUSSELL.

2. The Electrical Conductivity of Flames containing Salt Vapours for Rapidly Alternating Currents. By Prof. H. A. WILSON and Mr. E. GOLD.

3. The Lateral Vibration of Loaded and Unloaded Bars. By Mr. J. MORROW.

December 15th, 1905.

Meeting (informal) held at the Royal College of Science.

An exhibition of apparatus was given by the following firms:— Messrs. R. & J. Beck; Cambridge Scientific Instrument Co.; Crompton & Co.; Elliott Bros.; Everett, Edgcumbe & Co.; Fricker & Miller; Peter Heele; A. Hilger; Isenthal & Co.; Marconi Wireless Telegraph Co.; Nalder Bros. & Thompson; Newton & Co.; R. W. Paul; Pitkin & Co.; Rumney & Rumney; The Synchronome Co.; and Carl Zeiss.

Prof. FLEMING also exhibited his latest form of Cymometer.

January 26th, 1906.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Candidates were elected Fellows of the Society:—
H. S. ALLEN, A. RUSSELL, and W. SHAW.

The following Papers were read:—

1. The Isothermal Distillation of Nitrogen and Oxygen and of Argon and Oxygen. By Mr. J. K. H. INGLIS.

2. The Use of Chilled Cast-Iron for Permanent Magnets. By Mr. A. CAMPBELL.

3. Experiments on the Propagation of Longitudinal Waves of Magnetic Flux along Iron Wires and Rods. By Prof. LYLE and Mr. BALDWIN.

Annual General Meeting.

February 9th, 1906.

Meeting held at the Royal College of Science.

The PRESIDENT in the Chair.

The following Report of the Council was read by the Secretary:—

SINCE the last Annual General Meeting eleven ordinary Scientific Meetings and two informal Meetings of the Society have been held. Of these eleven were held at the Royal College of Science; one, that on March 24th, was held at University College, and one, on May 26th, at the National Physical Laboratory. The average attendance at the Meetings, omitting the informal Meeting at the National Physical Laboratory when no record of the attendance was taken, has been 48.

During the past year the Council departed from the usual programme in setting apart one evening exclusively to an exhibition of apparatus. This innovation was well supported, manufacturers showing a readiness to exhibit, and the attendance of Fellows and Visitors was greater than was anticipated, amounting to 240.

The number of Fellows now on the roll is 427, an increase of 2 over the number last year. Fourteen new Fellows have been elected. There have been eight resignations, and the Society has to mourn the loss by death of four Fellows, namely, W. Ackroyd, Rev. H. P. Gurney, Frank McClean, and H. R. Noble.

The number of Students now on the roll is 4, the same number as last year. Two new Students have been elected. There has been one transfer, and one resignation.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The following Candidates were elected Fellows of the Society:—

G. F. Phillips and T. Simons.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. J. Perry, F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. Foster, F.R.S.; Prof. W. G. Adams, M.A., F.R.S.; The Lord Kelvin, D.C.L., LL.D., F.R.S.; Prof. R. B. Clifton, M.A., F.R.S.; Prof. A. W. Reinold, M.A., F.R.S.; Prof. W. E. Ayrton, F.R.S.; Prin. Sir Arthur W. Rücker, M.A., D.Sc., F.R.S.; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S.; Shelford Bidwell, M.A., LL.B., F.R.S.; Prin. Sir Oliver J. Lodge, D.Sc., F.R.S.; Prof. S. P. Thompson, D.Sc., F.R.S.; R. T. Glazebrook, D.Sc., F.R.S.; Prof. J. H. Poynting, M.A., D.Sc., F.R.S.

Vice-Presidents.—C. Chree, Sc.D., LL.D., F.R.S.; H. M. Elder, M.A.; Prof. J. A. Fleming, M.A., D.Sc., F.R.S.; J. Swinburne.

Secretaries.—W. R. Cooper, M.A.; Prof. W. Cassie, M.A.

Foreign Secretary.—Prof. S. P. Thompson, D.Sc., F.R.S.

Treasurer.—Prof. H. L. Callendar, M.A., LL.D., F.R.S.

Librarian.—W. Watson, D.Sc., F.R.S.

Other Members of Council.—T. H. Blakesley, M.A.; A. Campbell, B.A.; W. B. Croft, M.A.; W. Duddell; J. A. Harker, D.Sc.; W. A. Price, M.A.; S. Skinner, M.A.; S. W. J. Smith, M.A.; W. Watson, D.Sc., F.R.S.; Prof. H. A. Wilson, M.A., D.Sc.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Board of Education.

The President then delivered his Address.

Treasurer's Report.

It has become increasingly difficult during the past two or three years to collect arrears of subscriptions, which now stand at a higher figure than usual, presumably in consequence of general depression and excessive taxation. With this exception the position of the Society remains practically unchanged. There is no unusual expenditure to report, except for binding periodicals and for refreshments at the Soirée. At the present rate the Society is perhaps leaving rather too narrow a margin for contingencies. Allowing for liabilities the balance is about £10 less than last year, but we may hope that conditions will improve, before the balance is dangerously reduced.

HUGH L. CALLENDAR,
Hon. Treasurer.

The Treasurer in Account with the Physical Society, from January 1st, 1905, to December 31st, 1905.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1905. Balance in Bank				84	3	0
Entrance-Fees	10	10	0			
Subscriptions	359	0	6			
Extra Subscriptions	32	9	0			
Arrears	23	2	0			
Abstracts etc.	8	12	6			
Life Composition	21	0	0			
Students	2	2	0			
				456	16	0
Dividends:—						
Furness Debenture Stock	15	3	10			
Metropolitan Board of Works	6	13	0			
Lancaster Corporation Stock	11	8	0			
New South Wales	8	9	0			
Midland Railway	38	0	0			
Great Eastern Railway	19	0	0			
London, Brighton, & South Coast Railway	26	2	7			
				124	16	5
Sales of Publications				37	6	6
				£703	1	11

Cr.	£	s.	d.	£	s.	d.
Science Abstracts				278	6	6
International Catalogue				6	11	2
Taylor and Francis:—						
Proceedings	144	4	9			
Abstracts	72	7	4			
Distribution	21	5	10			
Separate Copies	20	15	0			
				258	12	11
Periodicals				3	9	0
Attendance				6	0	0
Royal Asiatic Society, 1904				2	2	0
Commissions etc.					5	11
Binding				10	10	0
Petty Cash, Secretaries				9	8	7
Refreshments Dec. 15th				7	18	8
Balance in Bank				119	17	2
				£703	1	11

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

15*th January*, 1906.

ED. F. HERROUN, } *Auditors.*
G. L. ADDENBROOKE, }

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1905.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Subscriptions due, Treasurer's estimate	40	0	0	Taylor and Francis, Printing	66	15	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	463	0	0	Cheques not presented:—			
£1600 Midland Railway $2\frac{1}{2}$ per cent. Preference Stock	1200	0	0	Reporting	25	16	8
				Petty Cash, Secretary	2	0	6
£200 Metropolitan Board of Works $3\frac{1}{2}$ per cent. Consolidated Stock	208	0	0	Royal Asiatic Society, 1905	2	2	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	344	0	0	Balance	4020	3	0
£254 2*s.* 9*d.* New South Wales $3\frac{1}{2}$ per cent. Inscribed Stock	252	0	0				
£500 London, Brighton, and South Coast Railway Ordinary Stock	695	0	0				
£500 Great Eastern Railway 4 per cent. Debenture Stock	595	0	0				
Publications in Stock, estimated	200	0	0				
Balance in Bank	119	17	2				
	£4116	17	2		£4116	17	2

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

15*th January*, 1906.

ED. F. HERROUN,
G. L. ADDENBROOKE, } *Auditors.*

OBITUARIES.

The Rev. Henry Palin Gurney was educated at the City of London School and at Clare College, Cambridge, where he graduated B.A. as fourteenth wrangler in 1870, and was placed in the first class of the Natural Science Tripos in the same year. Soon after taking his degree he was elected a Fellow of Clare College and took Orders, becoming Curate to Canon Beck, Rector of Rotherhithe. While there he began the work with Mr. Wren to which he subsequently devoted himself as a partner in the well-known firm of Wren and Gurney, which has prepared many young men to enter the public service at home and abroad.

In 1894 he accepted the office of Principal of the Durham University College of Science at Newcastle-on-Tyne. With the Principalship he also undertook the duties of Professor of Mathematics and Lecturer on Mineralogy. To this work he devoted the remainder of his life with such success that he presided over the long-delayed completion of the College building erected as a memorial to Lord Armstrong, whose name it bears. Dr. Gurney received the degree of D.C.L. from the University of Durham, and was one of the Chaplains of the Bishop of Newcastle and Warden of the Diocesan Penitentiary. He was also a zealous and accomplished Mineralogist, and for some time acted as Deputy Professor of Mineralogy in the University of Cambridge. On 13th August, 1904, he fell a victim to his love of mountaineering.

Dr. Frank McClean was the son of Mr. J. R. McClean, F.R.S., M.P., a well-known Engineer. Born in 1837, he was educated at Westminster, Glasgow University, and Trinity College, Cambridge, of which he was a Scholar. After graduating in 1859 as a Wrangler he adopted the profession of Engineering, and became apprenticed to Sir John Hawkshaw. Three years later he entered into partnership in the firm of Messrs. McClean & Stileman. In 1865 he married Ellen, daughter of Mr. John Greg, of Escowbeck, Lancaster, with whom he leaves three sons and two daughters.

In 1870 he retired from his profession and devoted himself to the extensive and valuable series of researches upon which his reputation rests. The chief of these were his comparison of the spectra of high and low sun, the comparison of the spectra of

the sun and metals, and, most important of all, a series of photographs of the spectra of all stars of magnitudes above $3\frac{1}{2}$ in both Hemispheres. The photographic survey of the southern stars was done at Cape Town; but with that exception his work was done at his house at Tunbridge Wells.

In 1895 Dr. McClean was elected a Fellow of the Royal Society, and received the degree of LL.D. from the University of Glasgow. In 1899 he was awarded the gold medal of the Royal Astronomical Society. He died at Brussels on November 8, 1904, after a short illness.

Presidential Address to THE PHYSICAL SOCIETY OF LONDON,

9th February, 1906.

PROFESSOR JOHN PERRY, D.Sc., F.R.S.

I HAVE had so much work on hands for some years that I have been able to attend your meetings only occasionally. Hence, when I was informed that the Council wished to nominate me as your President, my pleasure was accompanied by no regret. You see, I forgot that changes might have occurred in your procedure since those golden years when I had the honour of acting as a secretary of this Society. We are too apt to think that our own individual vote at an election does not count, our own absence from church produces no evil effect on the community, in our own absence from meetings of the Society things will go on properly. This is all a mistake, as I now find to my great consternation; and I have made up my mind that for some time to come I will attend every council and committee and general meeting of the Physical Society, giving on every necessary occasion my vote, and even my two votes if possible, on every motion. In the early days, when Guthrie and Foster, and Kelvin and Fitzgerald, and other good men were presidents, no addresses of this kind were delivered. Are we not overdoing this business of requiring general addresses which must almost always have as their theme the progress of Science? Every president or president of a Section of the British Association; every president of every Scientific or quasi-scientific Society; every president of every Society that looks after any application of Science, is expected to give a yearly address. Seldom do we find in such addresses new accounts of important original work. I have given a presidential address myself on several occasions, but never have I felt its inappropriateness so much as now, when I see before me the members of a Society whose proceedings are more intense with original work of the best kind than any other Society known to me, with the exception of the Royal Society. I feel sure that the existence of the Physical Society has induced more young men to take up original investigation of a Physical kind than any other institution of the country.

Why should I try to put before you the progress of Physical Science during the last year or any number of years? Is there a member of the Society who does not read? Why even the members of the general public who, like the people of the City of Nineveh, know not their right hand from their left, are told in those newspapers which cater for them, all the wonderful fresh discoveries in Science, by the very cleverest of their writers; writers whose literary genius is so great that it has triumphed over their ignorance of the subject.

I have often spoken of the importance of a training in Physics in creating scientific habits of thought, so important in every human study or pursuit, but here again you know as much as or more than I do. You would use much the same arguments as I if you were addressing the Philistine: we are all of one mind in regard to educational reform.

There is a subject about which I often talk to my students, and about which I have written occasionally, and my views upon it may be new to many of you, so that possibly it may not be altogether inappropriate on such an occasion as this, namely, the unscientific use which is made by many men of the scraps of knowledge of Physics which they think they possess.

Every young reader of a paper before this or any other Scientific Society, makes the mistake of assuming that his audience knows a great deal of the subject which is so familiar to himself, and hence his paper is not understood. Writers of books on Physics assume their readers to be all truly logical students; they use words properly in a technical sense and forget that many of their readers may use them in the newspaper writers' sense. For example, take the expression 'Adiabatic expansion.' There are people who insist on finding that Rankine, Maxwell, and all others of our most exact writers, are not only inconsistent with one another in the use of the expression, but that each is inconsistent with himself.

If a portion of fluid expands slowly without gain or loss of heat, we know the way in which its p, v, and t alter as it changes state: this was originally called 'adiabatic expansion,' and the term has become a technical term for that kind of alteration of p, v, and t however it may occur. Steam or air may be throttled through a non-conducting reducing valve, but the expansion is not adiabatic although there is no gain or loss of heat. Steam or air passing along a pipe with friction, if it can only be made to lose heat through the metal of the pipe at exactly the proper rate at every

place, is expanding abiabatically. When we assume that steam or air flows without friction from a vessel through an orifice, we say that the expansion is adiabatic although it is rapid. We usually assume that portions of the atmosphere when moved about follow the adiabatic law, and this will be so if there is no friction or changes in kinetic energy. Stuff expanding even slowly behind a steam-engine piston, even if the cylinder were non-conducting and there were no leakage, does not as a whole usually fulfil the condition of adiabatic expansion, because we cannot speak of its v, p, and t, the temperature not being the same throughout. If you tell me that ten horse-power is being generated by some four-legged animals drawing a vehicle, I do not object, but you must not object to my speaking of ten horse-power being applied to a vehicle by means of a petrol motor. If I say that 3000 horse-power is being generated in a certain room, you have no right to contradict me merely because the room is not large enough to hold the bodies of 3000 horses. The horse-power is a technical term, and has only a remote historical association with that animal the horse. When authorities with questionable wisdom ordain that every medical student must pass an examination in that kind of general physics which is supposed to be orthodox, I do not care how slipshod the teaching may be, the main thing is that every student shall pass a silly examination. The subject is utterly dull and uninteresting, and he learnt nothing of Physics at home or in school, and he can learn nothing at college.

But when we have to teach any man who is about to apply his science in a profession, slipshod teaching is a crime. Now I affirm that the teaching of Physics to engineers, nearly all of it, the teaching of the theory of their profession to engineers, is almost always slipshod.

Many men enter a Science College at the age of 18 or more knowing nothing of Physical Science or knowing something which is worse than nothing. I consider that in the case of 99 per cent. of such men, it is impossible that they should acquire the scientific habit of thought, however high their mental powers may be rated. It is because we deal with so much of this kind of material that much of our teaching is slipshod. Every pupil entering a Science College ought to have been experimenting and working graphically and numerically on Physical Science problems from a very early age, and then our College Science classes, even if they began with the A B C of subjects, would deal with them in a scientific way. It

is terrible to think of the mental state of the average young man who is supposed to know something of Chemistry and Physics and Mechanics and Mathematics, because he has absorbed, although he cannot digest, a session's information. The causes of the unfitness of the average student are two: one, that his instincts and habits of thought were not trained from early youth; the other, that his teachers in Science Colleges have absurd and uninteresting courses of study for him. The second of these causes I will refer to later.

And now for some of the interesting results. The student begins to specialize in his professional subject, and he finds that his knowledge (yes, he calls it knowledge) of Chemistry or Physics or Mechanics or Mathematics is of no use to him. This true opinion he holds till he dies, and unfortunately he holds also the untrue opinion that all study of these subjects is useless.

A marvellous phenomenon is to be observed in all Technical Schools, particularly in America, greatly in this country and in Germany, of students being taught the theory of some kind of Engineering without having anything but the vaguest knowledge of those sciences by which that branch of Engineering has been built up. The Professor of Engineering adopts his teaching to his students, and necessarily it must be of the pocket-book rule of thumb order. Everything is formula! His classes are mere trade classes.

The Engineer deals with problems very much more complex than what we have in the Physics laboratory. We cannot hope to thoroughly solve any of them, but laboratory knowledge gives us a wonderful power to understand the Engineer's complex problems and to arrive at approximate solutions. If one of these solutions gets into Engineering books and becomes part of an engineering syllabus, that solution is taught as part of the theory; it is usually called a 'proof.' The teacher knows all the orthodox proofs, and faithfully gives them year after year to the Strassburg geese of whom he has charge. May I give you a few examples?

What occurs in a loaded beam is a very complex business, but if a beam is long in comparison with its lateral dimensions, there is an approximate solution used by engineers which is sufficiently correct for all practical purposes. It is based on the assumption that plane cross sections remain plane. A few cases of curiously loaded prisms of isotropic and truly elastic material are all that we have exact solutions for, and by appealing to these we find that our approximate theory is correct enough for our practical purposes. In arriving at these solutions and in his discovery of the law of

equipollent loads, M. St. Venant performed a very great service to practical people. In truth, however, the ultimate appeal in all such cases must be to experiment, and such experiments as have been made seem to indicate that the engineer's approximate solution of beam problems may be relied upon.

But very few Professors of Engineering teach their students anything about the scientific bases of their beam theory, and in numerous cases we find the students, and indeed their teachers, using their formulæ to short beams where they are quite inaccurate. A dam may be regarded as a short beam, and we usually assume that a plane cross section remains plane; but the scientific engineer uses the ordinary formula with great caution and appeals to experiment whenever this is possible. I have in this building a large model in indiarubber of part of the Assuan dam: it can be headed by mercury pressure, and some of my students will attempt from it to find out what occurs in such a structure. Plane cross sections will certainly not remain plane, but we must be cautious in applying our results because of the curious value of Poisson's ratio in indiarubber.

When a man has been taught in the unscientific way I have described, he is capable of taking up any complex problem himself and ignoring its complexities, and his proof if plausible is given to students year after year by teachers who certainly ought to know better; it is given, I believe, in many cases by men who know it to be wrong, but who know that it may be required in examinations.

There is hardly any proof of any formula used by Hydraulic Engineers which is not quite wrong as given in many books and classes and as required in many examinations, the fundamental simple principles of Mechanics being ignored. The loss of head at the sudden enlargement of a pipe and the flow from a rectangular gauge notch may be particularly mentioned. The well-known theory as to the strength of a shaft subjected at the same time to both bending and twisting, is one of many other unsafe proofs which are only useful in the examination room.

What occurs inside a steam-engine cylinder? This is one of the most complex of problems. From work done in the Physical laboratory we know the properties of water-stuff at many pressures and temperatures, but the conditions are all statical; our fluid is all at the same pressure and temperature. Well, on these statical assumptions the help of simple thermodynamic principles enables us to get some general notion of what goes on. We know that one

portion of the stuff may be evaporating and another part condensing at the same time. There may be a film of water over the metal; there may be only drops or there may be pockets of water. Again, it makes a very great difference from any of these conditions, if the water present is mixed with the steam as a mist. Our laboratory and mathematical knowledge of heat conductivity enables us to speculate with more or less vagueness depending upon our common sense upon the taking and giving of heat to the stuff from the cylinder walls. Look at the complexity of this part of the business. The area of the surface is always changing and the emissivity of most of it is always changing. Now, in spite of all this complexity the knowledge of Physics possessed by the late Mr. Willans enabled him to get what we know to have been a fairly good knowledge of what goes on, and particularly did it enable him to arrange experiments to test his approximate solution of the very complex problem. He was very fond of Mr. MacFarlane Gray's $\theta\phi$ diagram, not only in learning the properties of steam and water but for his speculations as to cylinder phenomena. Indeed, the pv and $\theta\phi$ diagrams enable us to get some good notions as to what goes on. Unfortunately, most teachers and writers of books, having to deal with pupils who are quite unscientific, pupils merely preparing for examinations, men who never were and never can be students, use these diagrams without knowing, or at all events without telling their pupils, on what curious assumptions they are treating a complex problem as if it were simple. In using a pv or a $\theta\phi$ diagram it is assumed that there is a known weight of water-steam whose p and v or θ and ϕ are known throughout a cycle. Here are two large assumptions. The first, that we have the same quantity at all times of the cycle, is not nearly so necessary as the other, that at any time at all we know this quantity. It might easily be argued that the quantity of water present is many times as great as what is usually assumed, and yet professors in their lectures and writers in their books and papers, and students in their exercises, work on placidly without referring to the fact that everything they do is based on what is almost certainly a quite wrong assumption.

Almost none of the pupils of such teachers who are disgusted with all such theory seek for better knowledge; the others scorn all theory. As for those who are not disgusted, they pass examinations, they are cocksure of the exactness of their knowledge ever after, and they write books!

The Electrician is in a very different position from Civil or

Mechanical Engineers, and as time goes on this difference must become greater and greater. As I have said, much Civil and Mechanical Engineering theory can only be used properly by men who have had a scientific training: the phenomena are too complex for mere formulas. The successful Civil or Mechanical Engineer is scientific to his finger tips; but the phenomena studied by the Electrician are very much more that of the laboratory, and not only can exact rules be given but exact measurements can be made to test every detail of the machinery and arrangements when finished. The professional practice of an Electrician is not nearly so likely therefore to teach him the importance of a general scientific training, because his formulas are correct. No problem is complex, and every new problem is like some old problem that somebody has already solved—a thing that seldom happens in Civil Engineering.

The principles taught by a study of Physics are few and simple: many people therefore think that only a very short study of Physics is sufficient. Newton's Law of Motion—I never speak of any but one law—is very easy to state:—Force is rate of change of momentum, or, to be more precise, the vector called *Force* is the time rate of change of the vector called *Momentum*. Is it not a simple statement? and yet how long it is before a man can apply the idea to the hundreds of different-looking cases that come before him. He is long ignorant of Newton's law of motion. State mathematically the first and second laws of thermodynamics or Ohm's law, or any of the simple far-reaching principles which the most elementary student can get full marks for in an examination, and how great a distance there is between such exact statements and a true comprehension of what they mean.

The fact is, in Physics we are dealing with ideas which are not familiar to any young animal, ideas which can only become familiar in the laboratory. Why even such a simple mathematical idea as that of a decimal cannot be given in elementary schools in less than five years—so they say—but I do not believe them; one week of weighing and measuring would give young children great familiarity and quite clear ideas about decimals.

I once was witness in a law case in which during forty days the cleverest scientific men endeavoured to give an elementary knowledge of some electrical phenomena to one of the cleverest of Judges; it was a costly farcical failure. In another case, where the laws of friction were most important, after ten days' coaching

the learned Court in a weighty judgment announced three several times that the force of friction between solids is proportional to the square of the speed *. I could give an endless number of examples to prove that the principles of Physics cannot be understood unless there has been early experimental training; and this is the reason why the professors of science in Colleges of University rank and the professors in Technical Colleges obtain such poor reward for all their labour.

But, some of you will say, see how every year many hundreds take Science degrees at the universities, some thousands pass the difficult London University Matriculation examinations. Of course, if this is your standard of excellence, my address can serve no useful purpose whatsoever: I say that nothing is really essential, that is, nothing ought to be *compulsory* in Schools except the study of English and of Natural Science. The object of a Matriculation examination is to test whether a student entering a college will be able to benefit by the courses of study there. The only language a knowledge of which ought to be compulsory in the Science department of a university examination is English.

Holding these views more and more strongly the longer I live, it is just possible that I ought to make no reference to that great choker of education, London University, whose arrangements are almost every one of them opposed to all my views of education.

If you give a professor of eminence an adequate salary, say twice what professors usually receive, why not trust him? Let him teach his subject in the way that seems best to him, and let him examine his students himself. Hedge him round with rules and regulations framed by a Board of Studies; tie him down to a syllabus, and surely the work he will do might be much better, certainly much more cheaply, done by a grinder at low wages.

There is a thing which I have very much at heart, and I am afraid almost to mention it as this may alienate from me many dear friends. The professors of chemistry and experimental physics of our country are, as a rule, not only famous for their knowledge but they are also good teachers. The professor of mathematics or mechanics is usually—there are some splendid exceptions—a very

* People often talk of the importance of great reforms in our Patent Law. The only necessary reform is to have Judges who know the simple principles of Natural Science. The theory of English law is that Judges should decide merely on the evidence brought before them, it does not provide for a Judge's not being able to understand the evidence.

poor teacher indeed. I know of many cases where the teacher of Mathematics is doing much valuable original work in pure or applied mathematics, but there are only one or two of his pupils to whom his teaching is of any value whatsoever. The time has come when we must recognize that the average student pays money and often gets no return whatsoever, because, although the professor knows his subject, he is quite unable to give any ideas to the average student; and before all other things, this is what the professor is paid to do. Surely, without discouraging the good work of these men of genius, we can find some remedy for such an absurd and dishonest state of things! The money spent in their salaries is no great loss to the nation; what is lost by their incompetence is of enormous value.

But now, supposing we had professors of mathematics and mechanics who were as competent to teach as the professors of chemistry and experimental physics usually are, we should still be compelled to object to their usual courses of instruction. All these subjects are becoming very large and we must be eclectic. Let us then, I say, select our courses in such a way that the student shall be interested. The course of study in Chemistry which best interests a Chemist is, much of it, quite uninteresting to the Physicist, the Engineer, the Geologist, the Biologist, or the Medical Student. If medical students were taught chemistry by a man who knew something of physiology and medicine, they would all of them learn because they would be interested in their work. Their teacher would use medical illustrations; he would bring into his elementary course many things that are now given only to advanced chemical students; he would leave out much that makes the present elementary courses wearisome because uninteresting and useless to medical students. He would also overlap the work of the teacher of physics as the teacher of physics would overlap the work of the teacher of chemistry.

If the teacher of chemistry to a class of men training for mechanical engineering had himself some engineering knowledge, he would dwell greatly upon the simple things like mere oxidation, mere combustion, mere solution. He would give them working notions on the dissociation of CO_2 and H_2O; he would dwell more upon the chemical properties of the materials used by the engineer, and hardly mention the obscure salts, the description of the preparation and properties of which make the usual course of study in chemistry so uninteresting.

I object to the idea that there is one general elementary course in Physics which all students ought to take; neither by their previous training nor from the uses which they will make of the principles of physics are they fit to be taught together. I want more classes, more rooms, and more men, each with more money.

In many of the elementary schools of this country they are now making "accomplishments" an important part of the curriculum. I have no objection to the teaching of French and the violin, but I resent the neglect of what is essential. The majority of the common people in this country cannot write a letter and cannot compute; our cooks are so little accustomed to weigh things that if you ask them whether a certain quantity of flour appears to be 2 lb. or 8 lb. they do not know. Above all, these people are never taught so that they shall be fond of reading.

And we can say just the same thing about our public schools. The average boy does not care to read anything, he is unscientific in an age of science. His Latin and Greek and even his French and German are surely unimportant if we compare them with a study of his own language. A man fond of reading is always studying those subjects that are important to his profession, his education is never finished.

Everywhere we find that it is the unessential that is taught; whatever is essential to the average student is neglected. No man has a right to call himself a scientific man unless he knows something of Biology; but where has a specialist in Physics or Chemistry, or in applications of these, a chance of getting such an interesting short course of study in Biology as he needs? In this Science School where we have met, a Chemist or Physicist or Engineer, if he takes Geology at all, must take up an elaborate course on Geology as if he were going to be a Geologist. Few Mining Engineers are to be prospectors; they are in charge of mines filled with mechanical and electrical contrivances: surely they do not need the elaborate course on Geology which is necessary for the Geologist or Prospecting Engineer.

You will see that I have illustrated my views from such subjects as Chemistry; I have been afraid to say much to you about the teaching of Physics, but truly all my remarks apply there also. The Professor of Engineering finds that his men know practically nothing of Thermodynamics, and if he knows that subject himself he can teach his students very well—but then he has no time left for the teaching of Engineering. He usually ignores their

ignorance and gives them formulas without knowledge. A curious result of this is that courses of instruction are getting to be far too long, and men are entering on professional work too late in life. The Americans found a three years' course too short, and it is now everywhere four years where it is not five. There is now the belief that a five years' course is necessary. The fact is, attempts are being made to teach everything that an Engineer is likely to need in his profession. These students are supposed to learn everything at college; nobody can depend upon their private study afterwards, for they are not and never will be fond of reading. It has become forgotten that a technical college is no mere trade school, its function is to prepare a man for the continued study of the science of his profession all through his life; to make him anxious to study all such subjects as touch his profession in any way; to interest him even in the study of many subjects which hardly touch his profession at all, subjects interesting to him as a citizen, as an intelligent inhabitant of a very wonderful world.

VI. *On the Lateral Vibration of Loaded and Unloaded Bars.*

By JOHN MORROW, *M.Sc.* (*Vict.*), *M.Eng.* (*Liverpool*), *Lecturer in Engineering, University College, Bristol* *.

CONTENTS.

Section I. *Introduction and Notation.*

§ 1. A METHOD of calculating the frequency of the lateral vibration of bars has been described in a recent paper, "On the Lateral Vibration of Bars of Uniform and Varying Sectional Area" (see Philosophical Magazine, July 1905).

It is an important feature of this method that it gives, in a simple form, the equation of the elastic central line of the displaced bar, and thus provides data from which the stresses and strains in all parts may be readily calculated. The method lends itself to many cases of loaded bars which have not hitherto been solved, and the present paper, after dealing with some cases of unloaded bars under different end conditions, and bars of negligible mass carrying concentrated loads, gives more particularly the solutions for some important problems of loaded bars of appreciable mass.

It will be seen that a full consideration of the simpler cases treated first very materially lessens the labour involved in solving the more complex problems in which both the load and the mass of the bar itself are taken into account.

These solutions are, in general, obtained by a process of continuous approximation. Each approximation depends on the principle that, at any point in the length of the bar, the curvature is equal to the couple due to the reversed effective forces divided by the flexural rigidity.

To estimate the value of the couple a vibration-curve must be assumed. The above principle then gives an expression for the curvature at all points.

* Read November 24, 1905.

The process of continuous approximation to the exact solution is based on the fact that the expression for the deflexion, as obtained from that for the curvature, is a much closer approximation to the truth than is that originally assumed for the purpose of calculating the effective forces.

A reference should be made here to the important papers by Professor Dunkerley * and by Dr. Chree † on the "Whirling" of Shafts. The relationship of the whirling to the vibrational problem was very clearly brought out by Chree, and under certain circumstances the two problems are identical.

§ 2. The notation used is similar to that of the paper previously cited; x and z are taken parallel, and y perpendicular, to the undisturbed position of the axis, bending occurring in the xy plane.

E = Young's Modulus for the material (assumed homogeneous and isotropic),

ρ = density of material of bar,

ω = sectional area,

$\mathrm{I} = \omega k^2$ = Geometrical Moment of Inertia of cross-section about the neutral axis,

l = length or span of bar,

$\mathrm{U} = \sqrt{\dfrac{\mathrm{E}}{\rho}}$ = Velocity of transmission of longitudinal vibrations in the bar,

$y_1\, y_a$ &c. = displacements of given points in the length of the bar,

$\mathrm{M}_0\, \mathrm{M}_1$ = bending couples required to fix the ends.

We have also

$$\frac{\ddot{y}_1}{y_1} = \frac{\ddot{y}}{y};$$

and for the frequency

$$\mathrm{N} = \frac{1}{2\pi}\sqrt{-\frac{\ddot{y}_1}{y_1}}.$$

The value of N is, however, not always recorded, as it is sufficient to find the expression for $\dfrac{\ddot{y}_1}{y_1}$.

* Phil. Trans. Roy. Soc. A 1894, p. 279.

† Phil. Mag. May 1904, p. 504.

Section II. *Unloaded Bars.*

§ 3. *Uniform Bar Clamped at Both Ends.*—Taking the origin at one end, the terminal conditions are

$$x=0,\ y=\frac{dy}{dx}=0;\quad x=l,\ y=\frac{dy}{dx}=0;\quad x=\frac{l}{2},\ y=y_1,\ \frac{dy}{dx}=0;$$

and hence the equation to be assumed as the first approximation to the type of displacement is

$$y=\frac{16y_1}{l^2}\left(x^2-\frac{2x^3}{l}+\frac{x^4}{l^2}\right).$$

The ordinary Euler-Bernoulli theory leads to

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{M}_0+\rho\omega\int_0^x(x-z)\,\ddot{y}_z\,dz-\tfrac{1}{2}\rho\omega x\int_0^l\ddot{y}\,dx.$$

Substituting for y, integrating, and determining M_0 by the condition that $\frac{dy}{dx}=0$ when $x=l$, we find

$$-y=\frac{\cdot16\rho\omega\ddot{y}_1}{\mathrm{EI}l^2}\left(\cdot178{,}571\,l^4x^2-\cdot2\dot{7}\,l^3x^3+\cdot2\dot{7}\,x^6-\cdot238{,}095\frac{x^7}{l}\right.$$
$$\left.+\cdot059{,}524\frac{x^8}{l^2}\right);$$

$$\therefore\quad \frac{\ddot{y}_1}{y_1}=-494{\cdot}74\frac{\mathrm{EI}}{\rho\omega l^4}.$$

Similarly, using these values of y and $\frac{\ddot{y}_1}{y_1}$, we arrive at a second approximation, in which

$$-y=\frac{\cdot079158\,\rho\omega\ddot{y}_1}{\mathrm{EI}l^6}\left(\cdot35572\,l^8x^2-\cdot55113\,l^7x^3+\cdot49603\,l^4x^6\right.$$
$$\left.-\cdot33069\,l^3x^7+\cdot05511\,x^{10}-\cdot03006\frac{x^{11}}{l}+\cdot00501\frac{x^{12}}{l^2}\right),$$

and

$$-\frac{\ddot{y}_1}{y_1}=500{\cdot}4\,\frac{\mathrm{EI}}{\rho\omega l^4},$$

agreeing well with the value of 500·6 obtained for the constant by the exact solution.

§ 4. *Clamped-supported Bar.*—In this case let y_1 be the displacement at such a distance from the fixed end (taken as

origin) that $\frac{dy}{dx}=0$ there. The initial type is then

$$y = 7\cdot6931\, y_1\left(\frac{3}{2}\frac{x^2}{l^2} - \frac{5}{2}\frac{x^3}{l^3} + \frac{x^4}{l^4}\right),$$

and

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \frac{\mathrm{M}_0(l-x)}{l} + \rho\omega\frac{\ddot{y}_1}{y_1}\int_0^x (x-z)\,y_z dz - \rho\omega\,\frac{x}{l}\frac{\ddot{y}_1}{y_1}\int_0^l y(l-x)dx.$$

Integrating and determining M_0 and the arbitrary constants by the terminal conditions

$$-y = \frac{\cdot076931\,\rho\omega\ddot{y}_1}{\mathrm{EI}\,l^4}(\cdot565476\,l^6x^2 - \cdot744048\,l^5x^3 + \cdot41\dot{6}\,l^2x^6 - \cdot297619\,lx^7 + \cdot059524\,x^8),$$

and

$$-\frac{\ddot{y}_1}{y_1} = 235\cdot9\frac{\mathrm{EI}}{\rho\omega l^4}.$$

A small error in the value of x for which $y=y_1$ makes no appreciable difference in the final result.

The next approximation gives

$$-y = \frac{\cdot018146\,\rho\omega\ddot{y}_1}{\mathrm{EI}\,l^8}(2\cdot36692\,l^{10}x^2 - 3\cdot10194\,l^9x^3 + 1\cdot57077\,l^6x^6 - \cdot88577\,l^5x^7 + \cdot08267\,l^2x^{10} - \cdot03758\,lx^{11} + \cdot00493\,x^{12})\;;$$

and putting

$$x = \cdot5805\,l,$$

$$-\frac{\ddot{y}_1}{y_1} = 238\cdot0\frac{\mathrm{EI}}{\rho\omega l^4}$$

and

$$\mathrm{N} = \frac{15\cdot42}{2\pi}\frac{k\mathrm{U}}{l^2}.$$

§ 5. *Clamped-Free Bar of Circular Section, Diameter varying as distance from Free End.*—Taking, as previously in similar cases, the origin at the free end,

$$\frac{d^2y}{dx^2} = \frac{\rho}{\mathrm{EI}}\frac{\ddot{y}_1}{y_1}\int_0^x \omega y_z(x-z)dz.$$

Let

$$y_z = y_1\left(1 - \frac{2z}{l} + \frac{z^2}{l^2}\right),$$

and the diameter $d = Ax$. Then

$$-\frac{d^2y}{dx^2} = \frac{16\rho\ddot{y}_1}{EA^2x^4}\int_0^x z^2(x-z)\left(1-\frac{2z}{l}+\frac{z^2}{l^2}\right)dz,$$

$$-y = \frac{16\rho\ddot{y}_1}{EA^2}\left(\cdot 01\dot{6}\, l^2 - \cdot 0\dot{4}\, lx + \cdot 041\dot{6}\, x^2 - \cdot 01\dot{6}\frac{x^3}{l} + \cdot 002\dot{7}\frac{x^4}{l^2}\right),$$

and $$-\frac{\ddot{y}_1}{y_1} = 3\cdot 75\,\frac{EA^2}{\rho l^2}.$$

In the second approximation,

$$-\frac{\ddot{y}_1}{y_1} = 4\cdot 5324\frac{EA^2}{\rho l^2};$$

and in the third approximation,

$$-y = \frac{\cdot 696176\,\rho\ddot{y}_1}{EA^2 l^4}\Big(\cdot 304931\, l^6 - \cdot 874742\, l^5x + \cdot 957615\, l^4x^2$$
$$- \cdot 542880\, l^3x^3 + \cdot 192{,}901\, l^2x^4 - \cdot 044091\, lx^5 + \cdot 006889\, x^6$$
$$- \cdot 000656\frac{x^7}{l} + \cdot 000033\frac{x^8}{l^2}\Big),$$

$$-\frac{\ddot{y}_1}{y_1} = 4\cdot 7106\frac{EA^2}{\rho l^2}, \quad \text{and} \quad N = \frac{2\cdot 170}{2\pi}\frac{AU}{l}.$$

Kirchhoff* has obtained the solution for the frequency in this case, and his result may be written

$$N = \frac{2\cdot 179}{2\pi}\frac{AU}{l}.$$

In the Euler-Bernoulli theory of beams it is assumed that the greatest diameter is small compared with the total effective length. When the diameter varies as the distance from one end, therefore, it is necessary that the variation should be small. In other cases, so far as the mathematical theory is concerned, the solution can only be looked upon as a probable approximation.

Section III. *Loaded Bars of Negligible Mass.*

§ 6. When the bar carries a load at some point in its length, and the mass of the bar itself is negligible, it is not necessary to assume a type of displacement. The method then

* See *Berliner Monatsberichte*, 1879, p. 815.

becomes an exact one, and gives at once the true type and frequency of the vibration.

In the following paragraphs, m is the mass with which the bar is loaded, and a is the distance of its centre of gravity from the point chosen as origin.

§ 7. *Clamped-Free Bar.*—If the origin be at the load and y_1 be the displacement there,

$$-\mathrm{EI}\frac{d^2y}{dx^2} = m\ddot{y}_1 x,$$

and the vibration-curve is

$$-y = \frac{m\ddot{y}_1}{\mathrm{EI}}\left(\frac{x^3}{6} - \frac{l^2x}{2} + \frac{l^3}{3}\right),$$

where l is the distance between the load and the fixed end.

$$-\frac{\ddot{y}_1}{y_1} = \frac{3\mathrm{EI}}{ml^3}.$$

On the other side of the load the bar is straight.

§ 8. *Bar supported at Both Ends.*—Origin at one end. Mass divides length of bar into segments a and b.

Here, for $x < a$,

$$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{b}{l} m\ddot{y}_a x;$$

whilst, for $x > a$,

$$\mathrm{EI}\frac{d^2y'}{dx^2} = m\ddot{y}_a\left(\frac{b}{l}x - x + a\right).$$

The constants of integration, expressing the inclinations of the bar at its ends, are obtained from the consideration that both $\frac{dy}{dx}$ and y must have the same value in each equation when $x = a$. The inclinations are, therefore,

$$-\frac{m\ddot{y}}{\mathrm{EI}}\frac{ab}{l}\left(\frac{a}{6} + \frac{b}{3}\right) \quad \text{and} \quad -\frac{m\ddot{y}_a}{\mathrm{EI}}\frac{a}{l}\left(\frac{a^2}{6} + \frac{l^2}{3}\right).$$

Hence

$$y = \frac{m\ddot{y}_a}{6\mathrm{EI}}\frac{b}{l}(x^3 - 2alx + a^2x),$$

and

$$y' = \frac{m\ddot{y}_a}{6\mathrm{EI}}\frac{l-x}{l}(a^3 - 2alx + ax^2).$$

In either case,

$$-\frac{\ddot{y}}{y} = \frac{3\mathrm{EI}l}{ma^2b^2}.$$

§ 9. *Clamped-Clamped Bar.*—With the same notation as in the preceding case.

If $x<a$,

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{M}_0+(\mathrm{M}_1-\mathrm{M}_0)\frac{x}{l}-m\ddot{y}_a\frac{bx}{l};$$

and if $x>a$ we must add $m\ddot{y}_a(x-a)$ to the above.

By integration we find

$$\mathrm{M}_0=m\ddot{y}_a\frac{ab^2}{l^2} \quad \text{and} \quad \mathrm{M}_1=m\ddot{y}_a\frac{a^2b}{l^2},$$

$$-\mathrm{EI}y=\frac{m\ddot{y}_a b^2x^2}{6l^3}[3a(l-x)-bx]$$

and

$$-\mathrm{EI}y'=\frac{m\ddot{y}_a a^2}{6l^3}[(a+3b)x^3+3(a-2b)lx^2+3l^2x-al^3].$$

In either case, $x=a$ gives

$$-\frac{\ddot{y}_a}{y_a}=\frac{3\mathrm{EI}}{m}\left(\frac{l}{ab}\right)^3.$$

This result and those of §§ 7 and 8 are identical with the corresponding results obtained, otherwise, by Chree.

§ 10. *Supported Bar with two symmetrically placed Concentrated Loads.*—When there are two equal masses, placed at equal distances a from each end of a bar whose mass is negligible, an exact solution can be obtained in a precisely similar manner. Thus, for a bar supported at each end,

For $x<a$,

$$\mathrm{EI}\frac{d^2y}{dx^2}=my_a x.$$

$$\mathrm{EI}\,y=m\ddot{y}_a x\left(\frac{x^2}{6}-\frac{la}{2}+\frac{a^2}{2}\right).$$

For $x=a$ to $(l-a)$,

$$\mathrm{EI}\frac{d^2y'}{dx^2}=m\ddot{y}_a a,$$

$$\mathrm{EI}\,y'=m\ddot{y}_a a\left(\frac{a^2}{b}-\frac{lx}{2}+\frac{x^2}{2}\right).$$

The curve between the two masses is a circular arc of radius $\mathrm{EI}/m\ddot{y}_a a$, and the inclination of the extreme ends is $m\ddot{y}_a\frac{a}{2}(a-l)/\mathrm{EI}$.

$$\frac{\ddot{y}_a}{y_a}=\frac{6\mathrm{EI}}{ma^2(4a-3l)}.$$

§ 11. *Clamped-Clamped Bar with two Concentrated Loads.*—In the previous case, if the ends are clamped, we must subtract M_0 (*i.e.* the couple at each end required to fix the directions) from the expression for $\frac{d^2y}{dx^2}$.

We find

$$M_0 = m\ddot{y}_a\frac{a}{l}(l-a),$$

$$EIy = m\ddot{y}_a\frac{x^2}{2l}\left(a^2-la+\frac{lx}{3}\right),$$

$$EIy' = m\ddot{y}_a\frac{a^2}{2l}\left(x^2-lx+\frac{la}{3}\right),$$

and

$$-\frac{\ddot{y}_a}{y_a}=\frac{6EIl}{ma^3(2l-3a)}.$$

§ 12. *Dynamical Method.*—Other methods for the solution of some of the problems of this section have been used by Dr. Chree, but they depend on the assumption of a curve of vibration.

A correct type having been assumed, expressions are obtained for the Kinetic and Potential Energies of the system. Employing these in Lagrange's Equations of Motion the frequency is readily obtained.

Taking, for example, a massless bar carrying a load m, as in § 8. Assume

$$y = \eta bx(l^2-b^2-x^2),$$
$$y' = \eta ax'(l^2-a^2-x'^2),$$

measuring x and x' from opposite ends of the bar.

The Kinetic Energy

$$T=\tfrac{1}{2}m\dot{y}_a^{\,2}-2ma^4b^4\dot{\eta}^2.$$

The Potential Energy of Bending, V,

$$=\tfrac{1}{2}EI\left\{\int_0^a\left(\frac{d^2y}{dx^2}\right)^2dx+\int_0^b\left(\frac{d^2y'}{dx'^2}\right)^2dx'\right\}$$
$$=6EIa^2b^2(a+b)\eta^2.$$

The Lagrangian Equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\eta}}\right)-\frac{\partial T}{\partial\eta}+\frac{\partial V}{\partial\eta}=0$$

becomes

$$4ma^4b^4\ddot{\eta}+12\mathrm{EI}\,a^2b^2(a+b)\eta=0\,;$$

$$\therefore\quad \frac{\ddot{\eta}}{\eta}=-\frac{3\mathrm{EI}\,l}{ma^2b^2}.$$

Section IV. *Loaded Massive Bars.*

§ 13. When the mass of the bar, in addition to that of the concentrated load, is taken into account, the expressions for the elastic curve and the frequency are more complicated. If, however, the position of the load and the ratio of the masses are given, the solutions are simple.

§ 14. *Clamped-Free Massive Bar, Load at End.*—In this case, if we assume

$$y=y_1\left(1-\frac{4}{3}\frac{x}{l}+\frac{1}{3}\frac{x^4}{l^4}\right),$$

the equation

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\rho\omega\int_0^x \ddot{y}_z(x-z)dz+mx\ddot{y}_1$$

gives

$$-\mathrm{EI}y=\rho\omega\ddot{y}_1\left(\cdot0819\dot{4}l^4-\cdot1126984\dot{1}l^3x+\cdot041\dot{6}x^4-\cdot0\dot{1}\frac{x^5}{l}\right.$$
$$\left.+\cdot00019841\frac{x^8}{l^4}\right)+m\ddot{y}_1(\cdot1\dot{6}x^3-\cdot5l^2x+\cdot\dot{3}l^3)$$

$$-\frac{\ddot{y}_1}{y_1}=\frac{\mathrm{EI}}{\cdot0819\dot{4}\rho\omega l^4+\cdot\dot{3}ml^3}.$$

The next approximation gives

$$-\mathrm{EI}y=\frac{\rho\omega\ddot{y}_1 10^{-3}}{\cdot0819\dot{4}\rho\omega l^4+\cdot\dot{3}ml^3}\left\{\rho\omega\left(6\cdot63084l^8-9\cdot12718l^7x\right.\right.$$
$$+3\cdot41435l^4x^4-\cdot93915l^3x^5+\cdot02480x^8-\cdot00367\frac{x^9}{l}$$
$$\left.+\cdot00002\frac{x^{12}}{l^4}\right)+m(26\cdot1905l^7-36\cdot\dot{1}l^6x+13\cdot\dot{8}l^3x^4$$
$$\left.-4\cdot1\dot{6}l^2x^5+\cdot1984x^7)\right\}+m\ddot{y}_1(\cdot\dot{3}l^3-\cdot5l^2x+\cdot1\dot{6}x^3).$$

Whence

$$-\frac{\ddot{y}_1}{y_1}=\mathrm{EI}\left\{\rho\omega l^4\frac{\cdot00663084\rho\omega l+\cdot0261905m}{\cdot0819\dot{4}\rho\omega l+\cdot\dot{3}m}+\cdot\dot{3}ml^3\right\}^{-1}$$

§ 15. *Supported Massive Bar, Load in any position.*—Let the load divide the bar into segments a and b. At each end

$$y=\frac{d^2y}{dx^2}=0\,;$$

and when $x=a$, $y=y_a$.

Hence, for the first approximation

$$y=\frac{y_a}{\mathrm{A}}\,(l^3x-2lx^3+x^4),$$

where, for brevity,

$$\mathrm{A}=l^3a-2la^3+a^4.$$

For $x<a$,

$$\mathrm{EI}\frac{d^2y}{dx^2}=\frac{b}{l}\,m\ddot{y}_a x+\frac{\rho\omega}{2}\frac{\ddot{y}_a}{y_a}x\int_0^l y_z dz-\rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z dz.$$

When $x>a$ we must add $-m\ddot{y}_a(x-a)$ to the above expression. Determining the constants of integration by equating the values for $\frac{dy}{dx}$ and y when $x=a$, we get for $x<a$,

$$y=\frac{\ddot{y}_a}{\mathrm{EI}}\left\{\frac{m}{6l}(lx^3-ax^3-a^3x-2al^2x+3a^2lx)+\frac{\rho\omega}{\mathrm{A}}(-\cdot01011905l^7x\right.$$
$$\left.+\cdot01\dot{6}l^5x^3-\cdot008\dot{3}l^3x^5+\cdot00238095lx^7-\cdot00059524x^8)\right\}.$$

Whilst for $x>a$ the first term in the brackets is

$$\frac{m}{6l}(la^3-a^3x-ax^3-2al^2x+3alx^2).$$

In either case

$$\frac{\ddot{y}_a}{y_a}=\mathrm{EI}\,.\,\mathrm{P}\,;$$

where

$$\mathrm{P}^{-1}=\frac{ma^2b^2}{3l}+\frac{\rho\omega}{\mathrm{A}}(-\cdot01011905l^7a+\cdot01\dot{6}l^5a^3-\cdot008\dot{3}l^3a^5$$
$$+\cdot00238095la^7-\cdot00059524a^8).$$

§ 16. In the second approximation, for $x<a$

$$\mathrm{EI}\frac{d^2y}{dx^2}=\frac{b}{l}\,m\ddot{y}_a x-\rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z dz$$
$$+\rho\omega\frac{\ddot{y}_a}{y_a}\frac{x}{l}\left[\int_0^a y\,(l-x)dx+\int_a^l y'\,(l-x)dx\right].$$

If $x>a$, the second expression on the right hand must be written

$$-\rho\omega\frac{\ddot{y}_a}{y_a}\left[\int_0^a (x-z)y_z dz+\int_a^x (x-z)y'_z dz\right],$$

and the term $-m\ddot{y}_a(x-a)$ must be added.

By the method already indicated we find the inclinations of the ends to be

at $x=0$:

$$\frac{dy}{dx}=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{m}{l}\left(\frac{a^2l}{2}-\frac{a^3}{6}-\frac{al^2}{3}\right)+\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\cdot 00010386\rho\omega l^{11}\right.\right.$$
$$\left.\left.+\frac{m\mathrm{A}}{6l}\left(-\frac{17a^7}{420}+\frac{la}{30}+\frac{a^5l^2}{60}-\frac{a^3l^4}{45}+\frac{4al^6}{315}\right)\right\}\right];$$

at $x=l$:

$$\frac{dy'}{dx}=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{m}{l}\left(-\frac{a^3}{6}-\frac{al^2}{3}\right)+\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\cdot 00010386\rho\omega l^{11}\right.\right.$$
$$\left.\left.+\frac{m\mathrm{A}}{6l}\left(-\frac{17a^7}{420}+\frac{a^5l^2}{60}-\frac{a^3l^4}{45}+\frac{4al^6}{315}\right)\right\}\right]$$

and the equations of the central line

$$y=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{mbx}{6l}(x^2-2la+a^2)-\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\rho\omega 10^{-4}(-1\cdot 0386l^{11}x+1\cdot 7085^9x^3\right.\right.$$
$$-\cdot 8433l^7x^5+\cdot 1984l^5x^7-\cdot 0276l^3x^9+\cdot 0030lx^{11}-\cdot 0005x^{12})$$
$$+\frac{m\mathrm{A}}{6l}\left[(l-a)\frac{x^7}{840}\cdot\left(\frac{a^2}{6}-\frac{al}{2}+\frac{l^2}{3}\right)\frac{ax^5}{20}-\left(\frac{a^4}{20}-\frac{a^3l}{4}+\frac{a^2l^2}{3}-\frac{2l^4}{15}\right)\frac{ax^3}{6}\right.$$
$$\left.\left.\left.-\left(\frac{la^6}{30}-\frac{17a^7}{420}+\frac{a^5l^2}{60}-\frac{a^3l^4}{45}+\frac{4al^6}{315}\right)x\right]\right\}\right].$$

$$y'=\frac{\ddot{y}_a}{\mathrm{EI}}\left[\frac{ma}{6l}(l-x)(a^2-2lx+x^2)-\frac{\rho\omega\mathrm{P}}{\mathrm{A}}\left\{\rho\omega 10^{-4}(-1\cdot 0386l^{11}x+1\cdot 7085l^9x^3\right.\right.$$
$$-\cdot 8433l^7x^5+\cdot 1984l^5x^7-\cdot 0276l^3x^9+\cdot 0030lx^{11}-\cdot 0005x^{12})$$
$$+\frac{m\mathrm{A}}{6l}\left[-\frac{ax^7}{840}+\frac{alx^6}{120}-\left(\frac{a^2}{6}+\frac{l^2}{3}\right)\frac{ax^5}{20}+\frac{a^3lx^4}{24}-\left(\frac{a^4}{20}+\frac{a^2l^2}{3}-\frac{2l^4}{15}\right)\frac{ax^3}{6}\right.$$
$$\left.\left.\left.+\frac{a^5lx^2}{40}+\left(\frac{17}{420}a^7-\frac{a^5l^2}{60}+\frac{a^3l^4}{45}-\frac{4al^6}{315}\right)x-\frac{17}{420}a^7l\right]\right\}\right].$$

Putting $x=a$ in either of these, we get the following relation between y_a and $\ddot{y}_a$:

$$-\left(\frac{\ddot{y}_a}{y_a}\right)^{-1}=\frac{1}{\mathrm{EI}}\left[\frac{ma^2b^2}{3l}+\rho\omega\mathrm{P}\left\{(-1{\cdot}0386l^{11}a+1{\cdot}7085l^9a^3-{\cdot}8433l^7a^5\right.\right.$$
$$+{\cdot}1984l^5a^7-{\cdot}0276l^3a^9+{\cdot}0030la^{11}-{\cdot}0005a^{12})\frac{\rho\omega10^{-4}}{\mathrm{A}}$$
$$\left.\left.+\frac{ma^2}{90l}({\cdot}3392857a^6+{\cdot}5178572a^5l-1{\cdot}\dot{3}a^4l^2+{\cdot}\dot{6}a^2l^4-{\cdot}1904762l^6)\right\}\right].$$

§ 17. In particular, if $a=\frac{l}{2}$,

$$\mathrm{A}=\frac{5}{16}l^4,\quad\text{and}\quad \mathrm{P}^{-1}=-\frac{ml^3}{48}-{\cdot}0103051\rho\omega l^4;$$

hence

$$-\frac{\ddot{y}_a}{y_a}=\mathrm{EI}\left[\frac{ml^3}{48}+\frac{3{\cdot}306\rho\omega l+7{\cdot}436m}{322{\cdot}038\rho\omega l+651{\cdot}042m}\rho\omega l^4\right]^{-1};$$

and further, if the masses of the load and bar be equal,

$$-\frac{\ddot{y}_a}{y_a}=\frac{31{\cdot}375\,\mathrm{EI}}{ml^3}.$$

§ 18. *Clamped-Clamped Massive Bar, Load in any position.*

The end conditions lead to the preliminary assumption

$$y=y_a\left(\frac{l}{ab}\right)^2\left(x^2-\frac{2x^3}{l}+\frac{x^4}{l^2}\right).$$

For $x<a$,

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{M}_0+(\mathrm{M}_1-\mathrm{M}_0)\frac{x}{l}+\rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x(x-z)y_z dz$$
$$-\frac{\rho\omega x}{2}\frac{\ddot{y}_a}{y_a}\int_0^l y_x\,dx-m\ddot{y}_a\frac{bx}{l}.$$

If $x>a$, add $m\ddot{y}_a(x-a)$.

By putting $x=a$, as in previous cases, we find

$$\mathrm{M}_0=m\ddot{y}_a\frac{ab^2}{l^2}+{\cdot}00357143\frac{\rho\omega\ddot{y}_a l^6}{a^2b^2},$$
$$\mathrm{M}_1=m\ddot{y}_a\frac{a^2b}{l^2}+{\cdot}00357143\frac{\rho\omega\ddot{y}_a l^6}{a^2b^2};$$

and hence for $x<a$,

$$-\mathrm{EI}y=\rho\omega\ddot{y}_a\left(\frac{l}{ab}\right)^2\left(2{\cdot}\dot{7}x^6-2{\cdot}38095\frac{x^7}{l}+{\cdot}59524\frac{x^8}{l^2}-2{\cdot}\dot{7}l^3x^3\right.$$
$$\left.+1{\cdot}78572\,l^4x^2\right)10^{-3}+m\ddot{y}_a\frac{bx^2}{6l^3}(3abl-l^2x-abx+a^2x);$$

and for $x > a$ the latter portion of the expression is changed to

$$m\ddot{y}_a \frac{a^2}{6l^3}(3lx^3 - 2ax^3 - 6l^2x^2 + 3alx^2 + 3l^3x - al^3).$$

When $x = a$, either of the above gives

$$-\frac{\ddot{y}_a}{y_a} = \mathrm{EI}\,.\,\mathrm{P},$$

where

$$\mathrm{P}^{-1} \equiv \frac{\rho\omega}{b^2}\,10^{-3}(1\cdot78572\,l^6 - 2\cdot\dot{7}\,al^5 + 2\cdot\dot{7}\,a^4l^2 - 2\cdot38095\,a^5l + \cdot59524\,a^6) + \frac{m}{3}\left(\frac{ab}{l}\right)^3.$$

§ 19. In the next approximation, for $x < a$:

$$-\mathrm{EI}\frac{d^2y}{dx^2} = \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\frac{\ddot{y}_a}{y_a}\int_0^x (x-z)y_z dz - \frac{\rho\omega}{l}\frac{\ddot{y}_a}{y_a}x\int_0^a y_x(l-x)dx - \frac{\rho\omega}{l}\frac{\ddot{y}_a}{y_a}x\int_0^l y'_x(l-x)dx - \frac{m\ddot{y}_a v x}{l};$$

$$= \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\mathrm{P}\ddot{y}_a\Bigg[\frac{\rho\omega l^2 10^{-4}}{a^2b^2}\left(-\cdot33068l^7x + 1\cdot48809l^4x^4 - 1\cdot3\dot{8}l^3x^5 + \cdot49603x^8 - \cdot33068\frac{x^9}{l} + \cdot06614\frac{x^{10}}{l^2}\right)$$
$$+ \frac{m}{6l^3}\left\{\left(\frac{3}{20}a^2l - \frac{l^3}{20} - \frac{a^3}{10}\right)x^5 + \left(\frac{al^3}{4} - \frac{a^2l^2}{2} + \frac{a^3l}{4}\right)x^4 - \left(\frac{3}{20}a^2l^5 - \frac{7}{20}a^3l^4 + \frac{a^4l^3}{4} - \frac{a^5l^2}{20}\right)x\right\}\Bigg] - m\ddot{y}_a\frac{bx}{l}.$$

Similarly for $x > a$:

$$-\mathrm{EI}\frac{d^2y'}{dx^2} = \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\frac{\ddot{y}_a}{y_a}\left[\int_0^a (x-z)y_z dz + \int_a^x (x-z)y_z' dz\right] - \rho\omega\frac{\ddot{y}_a}{y_a}\frac{x}{l}\left[\int_0^a y_x(l-x)dx + \int_a^l y'_x(l-x)dx\right] - m\ddot{y}_a\left(\frac{bx}{l} - x + a\right);$$

$$= \mathrm{M}_0 + (\mathrm{M}_1 - \mathrm{M}_0)\frac{x}{l} + \rho\omega\mathrm{P}\ddot{y}_a\Bigg[\frac{\rho\omega l^2}{a^2b^2}10^{-4}\left(-\cdot33068l^7x + 1\cdot48809l^4x^4 - 1\cdot3\dot{8}l^3x^5 + \cdot49603x^8 - \cdot33068\frac{x^9}{l} + \cdot06614\frac{x^{10}}{l^2}\right)$$
$$+ \frac{m}{6l^3}\left\{\left(\frac{3}{20}a^2l - \frac{a^3}{10}\right)x^5 + \left(\frac{a^3l}{4} - \frac{a^2l^2}{2}\right)x^4 + \frac{a^2l^3x^3}{2} - \frac{a^3l^3x^2}{2} - \left(\frac{3}{20}l^3 - \frac{7}{20}al^2 - \frac{a^3}{20}\right)a^2l^2x - \frac{a^5l^3}{20}\right\}\Bigg] - m\ddot{y}_a\left(\frac{bx}{l} - x + a\right).$$

Integrating and equating when $x=a$, we find

$$M_0=\rho\omega P\ddot{y}_a\left[\cdot 0000071145\frac{\rho\omega l^{10}}{a^2b^2}+\frac{m}{6l^3}\left(\frac{a^7l}{140}-\frac{a^6l^2}{30}+\frac{a^5l^3}{20}-\frac{11}{210}a^3l^5+\frac{a^2l^6}{35}\right)\right]+m\ddot{y}_a\frac{ab^2}{l^2};$$

$$M_1=\rho\omega P\ddot{y}_a\left[\cdot 0000071145\frac{\rho\omega l^{10}}{a^2b^2}+\frac{m}{6l^3}\left(-\frac{a^7l}{140}+\frac{a^6l^2}{60}-\frac{13}{420}a^3l^5+\frac{3}{140}a^2l^6\right)\right]+m\ddot{y}_a\frac{a^2b}{l^2}.$$

Whence the form of the centre-line is

$$-\mathrm{EI}y=\rho\omega P\ddot{y}_a\left[\rho\omega\left(\frac{l}{ab}\right)^2 10^{-6}\left(3\cdot 5573l^8x^2-5\cdot 5113l^7x^3+4\cdot 9603l^4x^6-3\cdot 3069l^3x^7+\cdot 5511x^{10}-\cdot 3006\frac{x^{11}}{l}+\cdot 0501\frac{x^{12}}{l^2}\right)\right.$$
$$+\frac{m}{6l^3}\left\{(3a^2l-l^3-2a^3)\frac{x^7}{840}+(al^3-2a^2l^2+a^3l)\frac{x^6}{120}+(7a^6l-2a^7-35a^4l^3+52a^3l^4-22a^2l^5)\frac{x^3}{840}+(3a^7l-14a^6l^2+21a^5l^3-22a^3l^5\right.$$
$$\left.\left.+12a^2l^6)\frac{x^2}{840}\right\}\right]+m\ddot{y}_a\left\{\frac{ab^2}{2}\frac{x^2}{l^2}-(2a+l)\frac{b^2}{6}\frac{x^3}{l^3}\right\}$$

and

$$-\mathrm{EI}y'=\rho\omega P\ddot{y}_a\left[\rho\omega\left(\frac{l}{ab}\right)^2 10^{-6}\left(3\cdot 5573l^8x^2+5\cdot 5113l^7x^3+4\cdot 9603l^4x^6-3\cdot 3069l^3x^7+\cdot 5511x^{10}-\cdot 3006\frac{x^{11}}{l}+\cdot 0501\frac{x^{12}}{l^2}\right.\right.+\frac{m}{6l^3}\left\{(3a^2l-2a^3)\frac{x^7}{840}\right.$$
$$+(a^3l-2a^2l^2)\frac{x^6}{120}+\frac{a^2l^3x^5}{40}-\frac{a^3l^3x^4}{24}+(52a^3l^4-22a^2l^5-2a^7+7a^6l)\frac{x^3}{840}$$
$$\left.\left.+(3a^7l-14a^6l^2-22a^3l^5+12a^2l^6)\frac{x^2}{840}+\frac{a^6l^3x}{120}-\frac{a^7l^3}{840}\right\}\right]$$
$$+\frac{m\ddot{y}_a}{6l^3}\{(3a^2l-2a^3)x^3+(3a^3l-6a^2l^2)x^2+3a^2l^3x-a^3l^3\}.$$

When $x=a$ these give

$$-\left(\frac{\ddot{y}_a}{y_a}\right)^{-1}=\frac{\rho\omega P}{\mathrm{EI}}\left[\rho\omega\frac{l^2}{b^2}10^{-6}\left(3\cdot 5573l^8-5\cdot 5113l^7a+4\cdot 9603l^4a^4-3\cdot 3069l^3a^5+\cdot 5511a^8-\cdot 3006\frac{a^9}{l}+\cdot 0501\frac{a^{10}}{l^2}\right)\right.$$
$$\left.+\frac{ma^4}{1260l^3}(3l^6-11l^5a+13l^4a^2-2l^3a^3-7l^2a^4+5la^5-a^6)\right]$$
$$+\frac{a^3b^3}{3l^3}\frac{m}{\mathrm{EI}}.$$

§ 20. As a particular case, if the load be at the centre

$$P^{-1} = \cdot 0020213\rho\omega l^4 + \cdot 005208\dot{3} m l^3$$

and

$$-\frac{\ddot{y}_a}{y_a} = \mathrm{EI}\left[\frac{ml^3}{192} + \frac{4\cdot 0398\rho\omega l + 10\cdot 076m}{2021\cdot 3\rho\omega l + 5208\cdot 3m}\rho\omega l^4\right]^{-1}.$$

If the masses of load and bar be equal we have a further simplification to

$$-\frac{\ddot{y}_a}{y_a} = \frac{139\cdot 65\,\mathrm{EI}}{ml^3}.$$

Section V. *Practical Formulæ for Loaded Bars.*

§ 21. The formulæ obtained in the last section for the frequency of the lateral vibrations of loaded bars of appreciable mass are too cumbersome for general use. In every case, however, given the position and mass of the concentrated load and the mass of the bar (assumed uniform throughout its length), the frequency is given by a formula of the type

$$\mathrm{N} = \frac{1}{2\pi}\sqrt{-\frac{\ddot{y}_a}{y_a}},$$

and the ratio $\ddot{y}_a/y_a$ is given by

$$-\frac{\ddot{y}_a}{y_a} = \frac{\beta \mathrm{EI}}{\rho\omega l^4}.$$

In the following paragraphs the values of β are given for all positions of the load and for ratios of mass of load to mass of bar varying from zero to 1·0.

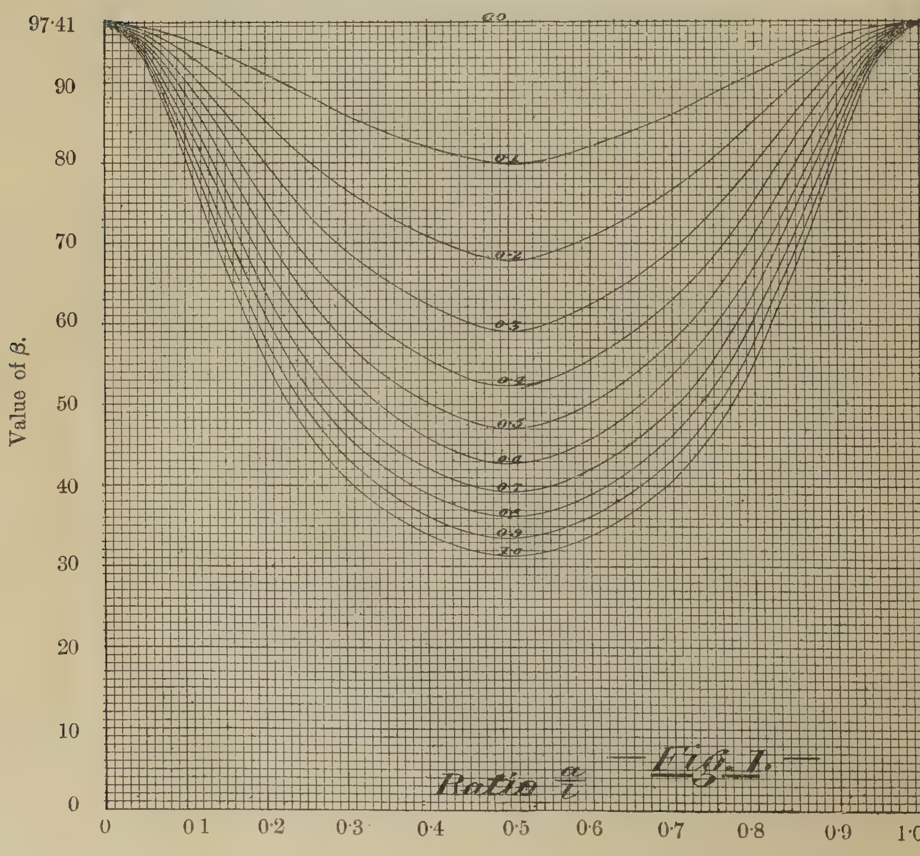
§ 22. *Bar supported at both ends.*—In this case the values of β have been obtained from the last equation of § 16. They are given in Table I. and represented graphically in figure 1. If the ratio of mass of load to that of bar cannot be expressed exactly in tenths, further interpolation is required for accuracy. This may easily be obtained graphically by plotting the values of β given by the curves for the required position of the load.

In the tables l is the effective length of the bar, and a the distance of the centre of gravity of the load from either end. The numbers on the curves represent the ratio of mass of load to that of effective length of bar.

TABLE I.

Values of β for Bars Supported at Each End.

Ratio. Mass of Load. / Mass of Bar.	Values of β for different positions of Load.				
	$a/l=\cdot1$.	$a/l=\cdot2$.	$a/l=\cdot3$.	$a/l=\cdot4$.	$a/l=\cdot5$.
0·1	95·56	90·98	85·91	82·03	7[illegible]
0·2	93·74	85·20	76·71	70·85	6[illegible]·73
0·3	91·95	80·02	69·22	62·35	58·99
0·4	90·21	75·36	63·01	55·68	52·31
0·5	88·69	71·17	57·81	50·30	47·03
0·6	86·86	67·39	53·91	45·86	42·74
0·7	85·24	63·96	49·56	42·15	39·18
0·8	83·67	60·84	46·25	38·99	36·17
0·9	82·15	57·99	43·35	36·27	33·60
1·0	80·66	55·39	40·78	33·91	31·38



— Fig. I. —

§ 23. *Bar Clamped at Both Ends.*—The values of β are here found from the last equation of § 19. They are given in Table II. and fig. 2.

TABLE II.

Values of β for Clamped-Clamped Bar.

Ratio. $\frac{\text{Mass of Load}}{\text{Mass of Bar.}}$	Values of β for different positions of Load.				
	$a/l=\cdot1$.	$a/l=\cdot2$.	$a/l=\cdot3$.	$a/l=\cdot4$.	$a/l=\cdot5$.
0·1	499·9	480·9	443·9	411·2	399·0
0·2	497·9	460·8	396·7	347·9	331·3
0 3	495·9	441·3	357·4	301·1	283·0
0 4	493 9	422·6	324·6	265·2	247·0
0·5	491·8	404·7	296 9	236·7	219 0
0·6	489·6	387·8	273·3	213·8	196·7
0·7	487·3	371·9	253·0	194·8	178·5
0·8	484·9	357·0	235·4	178·9	163 3
0·9	482·6	342·9	217·8	165·4	150·6
1·0	480·1	329·8	206·4	153·7	139·7

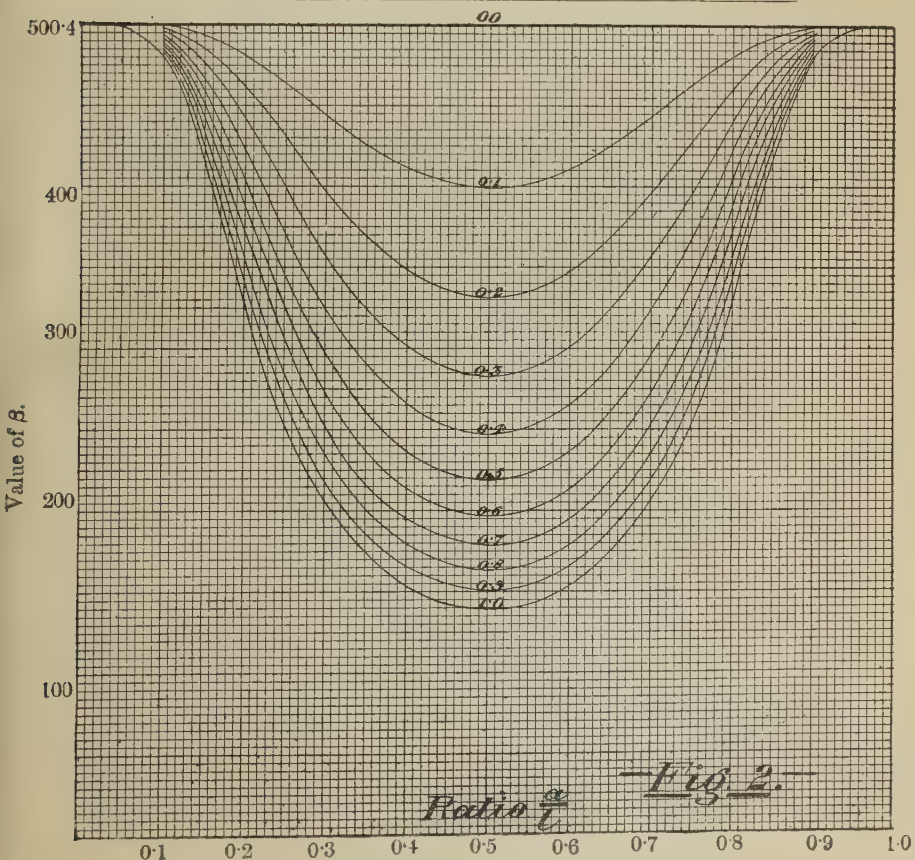


Fig. 2.

Section VI. *Correction for Rotatory Inertia.*

§ 24. Hitherto the terms depending on the angular motion, both of the concentrated masses and of the sections of the bars themselves, have been neglected. This is equivalent to supposing the inertia of each element to be concentrated at its centre.

The effect of the angular acceleration of the bar is sufficiently small to be neglected in practice, but in the case of a concentrated mass the additional terms may be important.

There are various methods of procedure. An important approximate one being to assume in the first place that the type of vibration is unaltered by rotatory inertia and then to find the correction due to this cause.

A general method has been given by Lord Rayleigh. It is applied to a special case in the next paragraph, in which

I' = Moment of Inertia of concentrated mass about an axis perpendicular to the plane of bending ($=mk'^2$).

T and V are the Kinetic and Potential Energies of the system.

§ 25. For a Fixed-Free Massless Bar (*cf.* Rayleigh's 'Sound,' Art. 183). Origin at fixed end and mass m at free end, the equation of the bar is

$$y=(3y_1-l\theta)\left(\frac{x}{l}\right)^2+(l\theta-2y_1)\left(\frac{x}{l}\right)^3,$$

where y_1 and θ are the values of y and $\frac{dy}{dx}$ at the point where the load is attached. This equation is deduced from

$$\mathrm{EI}\frac{d^2y}{dx^2}=m\ddot{y}_1(l-x)+\mathrm{I}'\ddot{\theta}.$$

Integrating, and determining $\ddot{y}_1$ and $\ddot{\theta}$ from

$$\left.\begin{aligned}\mathrm{EI}y_1&=m\ddot{y}_1\frac{l^3}{3}+\mathrm{I}'\ddot{\theta}\frac{l^2}{2}\\ \mathrm{EI}\theta&=m\ddot{y}_1\frac{l^2}{2}+\mathrm{I}'\ddot{\theta}l\end{aligned}\right\}$$

we get the desired result.

Following Rayleigh's solution, the equations of motion are

$$\left.\begin{aligned} m\ddot{y}_1+\frac{2\mathrm{EI}}{l^3}(6y_1-3l\theta)=0 \\ \mathrm{I}'\ddot{\theta}+\frac{2\mathrm{EI}}{l^3}(-3ly_1+2l^2\theta)=0 \end{aligned}\right\}$$

whence

$$-\frac{\ddot{y}}{y_1}=\frac{2\mathrm{EI}}{l\mathrm{I}'}\left\{1+\frac{3k'^2}{l^2}\pm\sqrt{1+\frac{3k'^2}{l^2}+\frac{9k'^4}{l^4}}\right\},$$

answering to the two different periods.

§ 26. The solution can be simplified if we assume to start with that the effect of I′ is small. This is usually the case in practice, and the method has been fully investigated by Dr. Chree.

Adopting this simpler method (*cf.* Chree, *l. c.* p. 511, § 9) the vibration curve in § 7 can be written

$$y=y_1\left(\frac{x^3}{2l^3}-\frac{3x}{l}+1\right),$$

and we can take

$$\theta=\left[\frac{dy}{dx}\right]_{x=0}=-\frac{3y_1}{2l}.$$

Then, since

$$\mathrm{T}=\tfrac{1}{2}m\dot{y}_1^{\,2}+\tfrac{1}{2}\mathrm{I}'\dot{\theta}^2=\tfrac{1}{2}m\dot{y}_1^{\,2}+\frac{9}{8}\frac{\mathrm{I}'}{l^2}\dot{y}_1^{\,2},$$

and

$$\mathrm{V}=\tfrac{1}{2}\mathrm{EI}\int_0^l\left(\frac{d^2y}{dx^2}\right)^2dx=\frac{3}{2}\frac{\mathrm{EI}}{l^3}y_1^{\,2},$$

the equation of motion

$$\frac{d}{dt}\frac{\partial\mathrm{T}}{\partial\dot{y}_1}-\frac{\partial\mathrm{T}}{\partial y_1}+\frac{\partial\mathrm{V}}{\partial y_1}=0$$

becomes

$$m\ddot{y}_1+\frac{9}{4}\frac{\mathrm{I}'}{l^2}\ddot{y}_1+\frac{3\mathrm{EI}}{l^3}y_1=0\,;$$

$$\therefore\quad -\frac{\ddot{y}_1}{y_1}=\frac{3\mathrm{EI}}{ml^3\left(1+\frac{9}{4}\frac{k'^2}{l}\right)}.$$

§ 27. The same problems worked by the methods of this

paper do not involve Lagrange's Equations and are as follows :—

Taking the origin at the free end,

$$-\mathrm{EI}\frac{d^2y}{dx^2}=m\ddot{y}_1x-\mathrm{I}'\ddot{\theta}.$$

Whence

$$\left.\begin{aligned}-\mathrm{EI}y_1&=m\ddot{y}_1\frac{l^3}{3}-\mathrm{I}'\ddot{\theta}\frac{l^2}{2}\\ -\mathrm{EI}\theta&=-m\ddot{y}_1\frac{l^2}{2}+\mathrm{I}'\ddot{\theta}l\end{aligned}\right\}.$$

Putting $\frac{\ddot{\theta}}{\theta}=\frac{\ddot{y}_1}{y_1}$ we get

$$\frac{ml^2}{6}\left(\frac{\ddot{y}_1}{y_1}\right)^2+2\left(\frac{1}{l}+\frac{ml}{3\mathrm{I}'}\right)\mathrm{EI}\frac{\ddot{y}_1}{y_1}=-\frac{2}{\mathrm{I}'}\left(\frac{\mathrm{EI}}{l}\right)^2;$$

$$\therefore \qquad -\frac{\ddot{y}_1}{y_1}=\frac{2\mathrm{EI}}{\mathrm{I}'l}\left(1+\frac{3\mathrm{I}'}{ml^2}\pm\sqrt{1+\frac{3\mathrm{I}'}{ml^2}+\frac{9\mathrm{I}'^2}{m^2l^4}}\right)$$

Or, for an approximate solution, starting with

$$-\mathrm{EI}\frac{d^2y}{dx^2}=m\ddot{y}\,x-\mathrm{I}'\ddot{\theta},$$

we may put

$$\theta=-\frac{3}{2}\frac{y_1}{l}.$$

Whence

$$-\mathrm{EI}y=\ddot{y}_1(x-l)^2\left\{\frac{m}{6}(x+2l)+\frac{3\mathrm{I}'}{4l}\right\},$$

and

$$-\frac{\ddot{y}_1}{y_1}=\frac{3\mathrm{EI}}{ml^3\left(1+\frac{9}{4}\frac{k'^2}{l^2}\right)}.$$

§ 28. When the mass of the bar is taken into account, and Rayleigh's method is used, the kinetic energy is

$$\mathrm{T}=\frac{1}{2}\left\{m\dot{y}_1{}^2+\mathrm{I}'\left[\frac{d^2y}{dx\,dt}\right]^2+\int\rho\omega\dot{y}^2dx+\int\rho\omega k^2\left(\frac{d^2y}{dx\,dt}\right)^2dx\right\},$$

where the part in square brackets is to be taken for the position of the load, and those under the integral signs are between the limits. The potential energy of bending is, as before,

$$\mathrm{V}=\frac{1}{2}\int\mathrm{EI}\left(\frac{d^2y}{dx^2}\right)dx.$$

In general the evaluation of these, though perfectly simple, is tedious. The values of y and $\frac{dy}{dx}$ are taken from earlier parts of this paper, and the energy expressions then used in the Lagrangian Equation just as in the example already given.

§ 29. It is, however, simpler merely to add the terms due to Rotatory Inertia to the previous solutions. A general investigation of these terms would occupy too much space, but in a numerical example the work is not difficult.

Referring to § 14 for the case of a loaded massive bar, the term to be added to the value of $-\mathrm{EI}\frac{d^2y}{dx^2}$ for the rotatory inertia of m is

$$-\mathrm{I}\,\frac{d^2}{dt^2}\left(\frac{dy}{dx}\right)_{x=0};$$

and for an approximate solution we can write

$$\left(\frac{dy}{dx}\right)_{x=0}=-\frac{4}{3}\frac{y_1}{l}.$$

Hence the term to be added to $-\mathrm{EI}y$ is

$$\frac{4}{3}\frac{\mathrm{I}'}{l}\left(\frac{x^2}{2}-lx+\frac{l^2}{2}\right);$$

and for a first approximation,

$$-\frac{\ddot{y}_1}{y_1}=\frac{\mathrm{EI}}{\cdot 0819\dot{4}\,\rho\omega l^4+\cdot\dot{3}\,ml^3+\cdot\dot{6}\,\mathrm{I}'l}.$$

This example is sufficient to indicate the procedure to be adopted in all such cases.

University College, Bristol,
October 1905.

VII. *On the Electrical Conductivity of Flames containing Salt Vapours for Rapidly Alternating Currents.* *By* H. A. Wilson, *M.A., D.Sc., M.Sc., Professor of Physics, King's College, London, Fellow of Trinity College, Cambridge, and* E. Gold, *B.A., Hutchinson Student, St. John's College, Cambridge* *.

The following paper contains an account of a series of experiments on the electrical conductivity of a Bunsen flame containing various alkali salt vapours, the currents used being alternating ones with frequencies varying from $7{\cdot}14 \times 10^4$ to $6{\cdot}2 \times 10^6$ per second.

The conductivity was measured between two platinum electrodes immersed in the flame, and the variation of the conductivity with the amount of salt present in the flame and with the nature of the salt was investigated. The variation of the conductivity with the frequency of alternation, the maximum electromotive force, and the distance between the electrodes was also examined. The results obtained enable a comparison to be made between the conductivities of the various alkali salt vapours for alternating currents and their conductivities for steady currents as previously determined †.

It appears that the relative conductivities for rapidly alternating currents are nearly proportional to the square roots of the corresponding conductivities for steady currents.

The flame is found to behave for very rapidly alternating currents more like a dielectric of high specific inductive capacity than like a conducting medium; and it is shown that this result is in accordance with the ionic theory.

The rest of the paper is divided into the following sections:—

(1) Description of apparatus used.

(2) Variation of the conductivity with the concentration and nature of the salt vapour.

* Read November 24, 1905.

† "The Electrical Conductivity and Luminosity of Flames containing Vaporised Salts," by A. Smithells, H. M. Dawson, and H. A. Wilson, Phil. Trans. A 193. 1899. "On the Electrical Conductivity of Flames containing Salt Vapours," by H. A. Wilson, Phil. Trans. A 192. 1899.

(3) Variation of the conductivity with the maximum P.D., the frequency, and the distance between the electrodes.

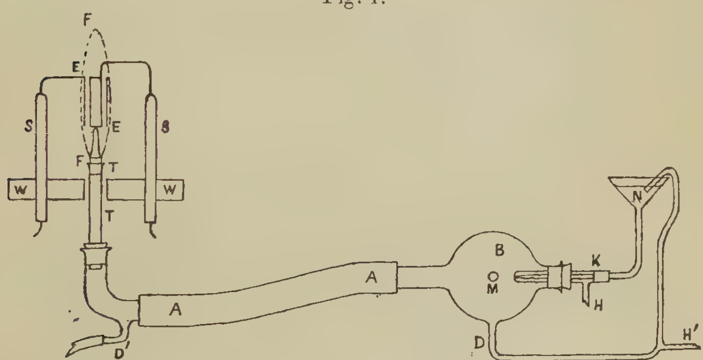
(4) Theory of the conductivity for rapidly alternating currents.

(5) Summary of results.

(1) *Description of Apparatus used.*

To produce a steady flame containing a definite amount of salt vapour, an apparatus similar to those described in the two papers referred to above was used. The principal parts of the apparatus are shown in fig. 1.

Fig. 1.



FF. Flame.	B. Sprayer-bulb.
EE. Electrodes.	K. Sprayer.
SS. Glass tubes.	N. Solution-tube and funnel.
WW. Wood block.	HH′. Compressed-air tubes.
TT. Flame-tube.	M. Gas tube.
AA. Wide indiarubber tube.	DD′. Solution-overflow tubes.

A mixture of coal-gas and air containing spray of a salt solution was burnt from a glass tube TT tipped with a short thin copper tube 1 cm. in diameter.

The mixture was formed in a glass bulb B, from which it passed through a wide tube AA to the burner. In the bulb and AA the coarser spray settled, and was allowed to escape through the tubes D and D′. The spray was produced by a Gouy sprayer K, worked by compressed air, which was

supplied by a rotary blower worked by a $\frac{1}{2}$-H.P. motor. The compressed air entered at H, and the salt solution at N. The pressure of the air-supply was indicated by a water-manometer, and was always kept at the same value, about 80 cms. of water. The salt solution was contained in a large funnel, and its level always kept a constant height above the sprayer-nozzle. The amount of gas entering the bulb was measured by a water-meter, and its pressure kept constant by means of a regulator; it did not vary appreciably.

The gas and air supplies were so adjusted that a "non-luminous" flame having a sharply-defined inner cone was obtained. This flame was very steady in appearance, and could be maintained constant for any length of time required.

The electrodes used in most of the experiments consisted of two concentric cylinders made of thin platinum, of the following dimensions:—

Diameter of outside cylinder . .	2·4 cms.
Height " " . .	5·0 "
Diameter of inside cylinder . .	1·2 "
Height " " . .	5·2 "

They were supported symmetrically about the axis of the flame at such a height that the inner cone of the flame just reached up to the level of the lower ends of the cylinders.

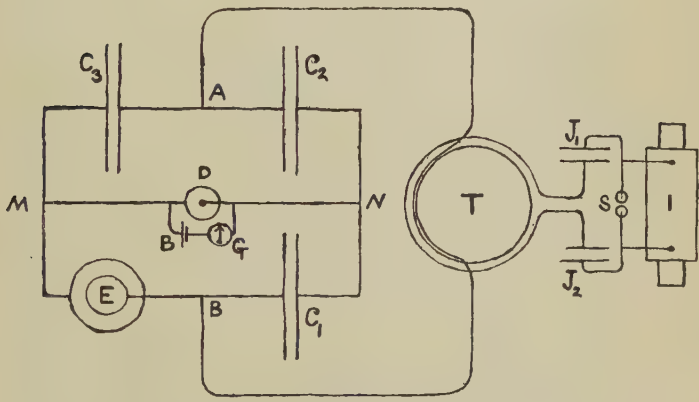
The conductivity between the electrodes was determined by means of a Wheatstone-bridge arrangement, of which the electrodes formed one arm, and the other three arms consisted of small air-condensers, the capacity of one of which was adjustable. The arrangement is shown in fig. 2.

An induction-coil, I, charged up two Leyden-jars, J_1 J_2, and these discharged at a spark-gap S. The outside coatings of the jars were connected through the primary of a Tesla coil, T. The primary of this coil consisted of 33 turns wound into a spiral, 29 cms. long and 19·2 cms. in diameter, on a large glass cylinder. The secondary coil had three turns, and was placed inside the glass cylinder, halfway up it. It was connected to the bridge arrangement at A and B, as shown.

An "electrolytic detector," D, was connected to the points M and N of the bridge. This detector consisted of two

platinum electrodes dipping into 20 per cent. sulphuric acid. One electrode was a platinum cylinder 3 cms. in diameter and 4 cms. high, while the other was a platinum wire $\frac{1}{1000}$ inch in diameter sealed into a glass tube and cut off close to the surface of the glass. The large electrode was contained in a test-tube, which it just fitted, and the glass tube carrying the small electrode was supported by a cork

Fig. 2.



E. Flame-Electrodes.
C_1 C_2. Air-condensers.
C_3. Adjustable Air-condenser.
D. Detector.
B. Cell.
G. Galvanometer.
T. Tesla Coil.
J_1 J_2. Leyden jars.
S. Spark-gap.
I. Induction-coil.

fitting the test-tube. The small electrode was just below the surface of the acid at the middle of the tube. The two electrodes were connected to a silver-chloride cell B and a moving-coil galvanometer G. The cell B gave about one volt and served to polarize the electrodes. When an alternating P.D. was produced between M and N, the detector was depolarized and a current passed through the galvanometer. The deflexion of the galvanometer-coil was read by means of an incandescent lamp and a scale, a current of 10^{-9} ampere giving a deflexion of one scale-division.

The condensers C_1, C_2 each consisted of two parallel circular disks 10 cms. in diameter, supported on ebonite rods. The distance between the disks of C_1 was 0·15 cm.

and of C_2 0·75 cm., in most of the experiments. The condenser C_3 consisted of two brass disks 10 cms. in diameter, whose distance apart could be adjusted and measured by means of an accurate micrometer-screw.

To determine the conductivity between the flame-electrodes, the rapidly alternating P.D. produced by the Tesla coil was applied at A and B (fig. 2), and the condenser C_3 was adjusted until the deflexion of the galvanometer was a minimum. On starting the alternating current, the galvanometer deflexion increased to a maximum value and then fell off slowly. The alternating current was always kept on for 15 secs. and the maximum deflexion noted, and then after an interval of 15 secs. the current was turned on again and so on. During the intervals the condenser was adjusted, and the deflexions corresponding to a series of positions of the adjustable condenser-disk were thus obtained. A curve was then drawn on squared paper showing the relation between the galvanometer deflexion and the condenser-screw reading, and so the screw reading for which the deflexion was a minimum was obtained. The observations at each position of the condenser-disk were repeated several times and the mean taken ; the series of observations was also repeated first in one direction and then in the other, while various intermediate positions were also tried so as to make as certain as possible that the correct relation between the deflexion and the condenser distance was obtained. These precautions were very necessary because the apparatus was sometimes irregular in its action. No great difficulty was experienced in keeping the flame sufficiently constant during the experiments, but it was not easy to keep the alternating current constant. This was due partly to variations taking place at the spark-gap and partly to irregularity in the action of the induction-coil interrupter. The spark-gap finally adopted consisted of two platinum spheres kept in an atmosphere of hydrogen. In most of the experiments the length of the gap was about 0·2 cm.

Several different kinds of interrupters were tried, but finally the ordinary App's platinum contact-breaker was used. The platinum contacts were always carefully filed smooth before starting an experiment and a 10-inch coil was

used with a four-volt battery, the contact-breaker being adjusted so that the coil gave only a short spark when not connected to the Leyden jars. In this way the apparatus was made to work sufficiently steadily to obtain fairly satisfactory observations. The distance between the condenser-plates corresponding to the minimum galvanometer-deflexion could be obtained within about 5 per cent. of its value. It was only after a long series of attempts extending over nearly a year that the apparatus was got to work well enough to obtain reliable results, and numerous modifications were tried before the form above described was finally adopted.

To compare the conductivities due to different salts, the increase in the apparent capacity of the flame-electrodes consequent on introducing each salt was calculated in terms of the capacities of the three condensers.

With a bridge arrangement each arm of which is a capacity with negligible self-induction, the condition for a balance is $C_1 C_3 = C_2 C_4$, a condition independent of the frequency. It was found that when the ratio $\frac{C_2}{C_1}$ was altered, then C_3 changed approximately proportionally, so that it appeared justifiable to apply the equation $C_1 C_3 = C_2 C_4$. Let d_1 be the distance between the plates of the adjustable condenser at the minimum for the flame free from salt, and d_2 for the flame containing salt. The capacity of the adjustable condenser in the first case is $\frac{A}{4\pi d_1} + D$, where D is a quantity nearly independent of d_1 and A the area of each condenser-plate [Clerk Maxwell, 'Electricity and Magnetism,' art. 202].

We have therefore

$$C_1 \left[\frac{A}{4\pi d_1} + D\right] = C_2 C_4,$$

where C_4 is the apparent capacity of the flame-electrodes with the flame free from salt. Also if C_4' is the apparent capacity with salt in the flame we have

$$C_1 \left(\frac{A}{4\pi d_2} + D\right) = C_2 C_4'.$$

Hence
$$\frac{C_1 A}{4\pi}\left(\frac{1}{d_2} - \frac{1}{d_1}\right) = C_2 \left(C_4' - C_4\right);$$

so that the change, in the apparent capacity, due to the introduction of the salt is given by the equation

$$C_4' - C_4 = \frac{C_1}{C_2}\frac{A}{4\pi}\left(\frac{1}{d_2} - \frac{1}{d_1}\right).$$

In the experiments we have made, the self-inductions of the arms of the bridge could be neglected in comparison with the capacities without appreciable error.

It was found possible with the bridge arrangement described to obtain an approximate balance. That is, the minimum galvanometer-current was always small compared with the currents when the adjustable condenser-plate was far from the position which gave the minimum deflexion. This showed that the arm of the bridge containing the flame behaved like a capacity simply, or like a capacity and self-induction in series. If the flame had behaved like a capacity and resistance in parallel, then a balance could not have been obtained. The current through the flame with a given maximum P.D. and frequency is proportional to the apparent capacity, so that it is reasonable to regard the apparent capacity as a measure of the conductivity of the flame for the rapidly alternating currents employed.

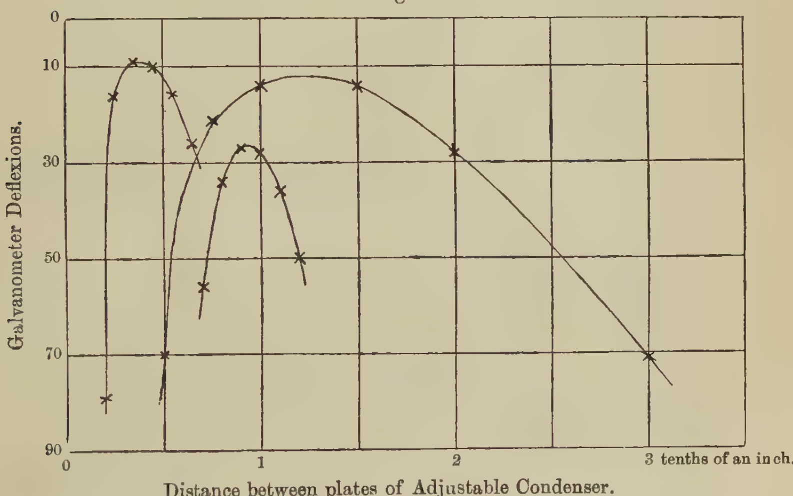
To obtain relative values of the apparent capacities, it is therefore sufficient to calculate $\frac{1}{d_2} - \frac{1}{d_1}$ for each salt solution sprayed. Since the absolute amount of salt in the flame could be only roughly estimated, it was useless to attempt to obtain exact values for the absolute apparent capacities; so that it was unnecessary to know the ratio $\frac{C_1}{C_2}$ exactly.

The value of d_1, the minimum position for the flame free from salt, was about half the value obtained with no flame. Consequently $\frac{1}{d_1}$ corresponds to twice the capacity of the flame-electrodes with air as dielectric. The capacity in this case was 3·6 cms., and $\frac{C_1}{C_2}$ was very nearly $\frac{1}{5}$ in most of the experiments described below.

(2) *Variations of the apparent Capacity with the Concentration and Nature of the Salt Vapour.*

The salt solutions sprayed were made up with distilled water and pure salts carefully dried. We shall first give a few examples of the curves obtained, showing the relation between the distance apart of the condenser-plates and the galvanometer deflexion, from which curves the position of the minima were deduced. Fig. 3 shows several such curves. The sharpness of the minimum was usually greater

Fig. 3.

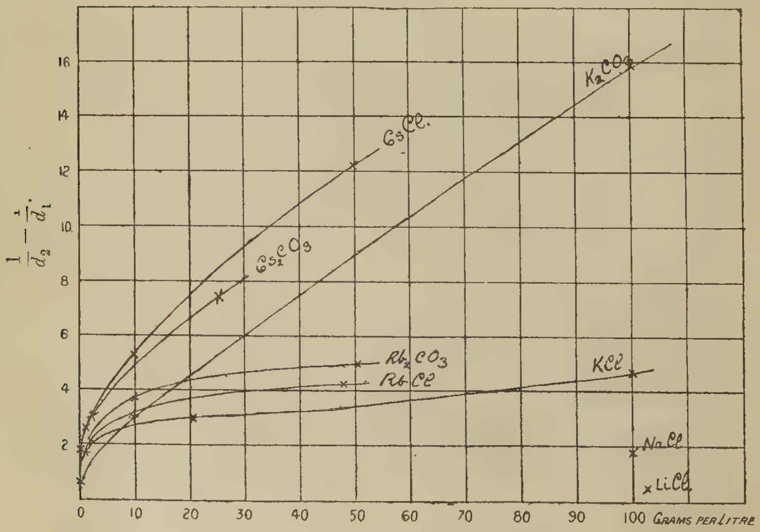


when the conductivity was large and the distance therefore small, than when the converse was the case. The accuracy with which the minimum could be found was consequently nearly the same for large as for small conductivities.

During the course of the experiments the ratio of the two condensers, C_1, C_2, was changed on one or two occasions. When this had been done, some of the observations made before the change were repeated, and the factors required to reduce all the results to the same standard so determined. The following table contains the results obtained, the number of alternations per second being $3{\cdot}2 \times 10^5$, and $d_1 = 3{\cdot}33$ tenths of an inch.

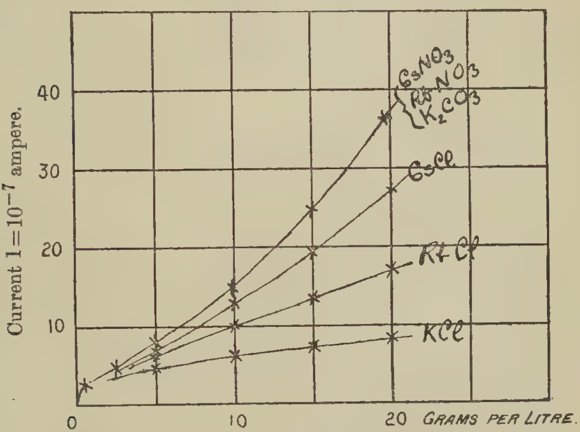
Salt.	Grams per litre.	d_2 (in tenths of an inch).	$\frac{1}{d_2} - \frac{1}{d_1}$.
CsCl	50	0·08	12·2
,, ...	10	0·18	5·26
,, ...	1	0·34	2·64
,, ...	0·333	0·48	1·78
,, ...	0·1	1·10	0·61
Cs_2CO_3	25·5	0·13	7·40
,, ...	2·55	0·30	3·03
,, ...	0·26	0·59	1·40
,, ...	0·026	1·90	0·23
RbCl	48·1	0·22	4·24
,, ...	9·6	0·29	3·15
,, ...	1·92	0·41	2·14
Rb_2CO_3	50·4	0·19	4·96
,, ...	10	0·25	3·70
,, ...	1	0·50	1·70
K_2CO_3	100	0·062	15·8
,, ...	10	0·297	3·07
KCl	100	0·20	4·70
,, ...	20·8	0·30	3·03
,, ...	1	0·50	1·70
NaCl	100	0·47	1·83
LiCl	102·5	1·3	0·47

Fig. 4.



In fig. 4 the above values of $\frac{1}{d_2} - \frac{1}{d_1}$, and the corresponding strengths of the solutions, are shown graphically for each salt. Fig. 5 shows the steady currents due to an E.M.F. of 0·227 volt, taken from the papers referred to above.

Fig. 5.



The amount of salt entering the flame in the present experiments was nearly the same as in the older experiments.

The following table gives the values of $\frac{1}{d_2} - \frac{1}{d_1}$ for decinormal solutions obtained from the curves in fig. 4, and also the steady currents due to 0·227 volt.

Salt.	$\frac{1}{d_2} - \frac{1}{d_1}$ (k)	Salt.	Steady Current ($1 = 10^{-7}$ ampere). (k')	Ratios.	$\frac{\sqrt{k'}}{k}$.
CsCl	6·7	CsCl.........	22·2	3·3	0·70
$\frac{1}{2}Cs_2CO_3$ ...	5·9	$CsNO_3$......	36·6	6·2	1·03
$\frac{1}{2}Rb_2CO_3$ ...	3·7	$RbNO_3$......	25·9	7	1·37
RbCl	3·2	RbCl	11·3	3·5	1·05
$\frac{1}{2}K_2CO_3$......	2·9	$\frac{1}{2}K_2CO_3$......	11·2	3·9	1·15
KCl	2·6	KCl	5·75	2·2	0·92

The fifth column contains the ratios of the numbers

expressing the conductivities for steady currents to the values of $\frac{1}{d_2}-\frac{1}{d_1}$. In the previous work the conductivities of cæsium and rubidium carbonates were not measured; so the values for nitrates are given, since the conductivities of all oxysalts of the same metal were found to be nearly equal for steady currents. It will be seen that, roughly speaking, the relative conductivity for steady currents varies in a similar way to the conductivity for rapidly alternating currents as represented by $\frac{1}{d_2}-\frac{1}{d_1}$.

The last column contains the square root of the conductivity for steady currents divided by $\frac{1}{d_2}-\frac{1}{d_1}$. The numbers in this column do not vary much, showing that the conductivity for rapidly alternating currents varies, roughly speaking, as the square root of the conductivity for steady currents.

The conductivities of KCl and RbCl were found to vary nearly as the square root of the concentration in the case of steady currents; so that we should expect them to vary as the fourth root of the concentration for rapidly alternating currents. The following table shows that this is nearly the case.

Salt.	Grams per litre. (C)	$\frac{1}{d_2}-\frac{1}{d_1}$. ($k$)	$k/\sqrt[4]{C}$.
RbCl.........	48·1	4·24	1·6
"	9·6	3·15	1·79
"	1·92	2·14	1·81
KCl	100	4·70	1·49
"	20·8	3·03	1·42
"	1·0	1·70	1·70

Considering the large variation in C the values of $k/\sqrt[4]{C}$ are surprisingly constant for these two salts.

The following table shows the relative variation of the conductivity CsCl for steady and alternating currents. The numbers are taken from figs. 4 and 5.

	Grams per litre.	$\frac{1}{d_2}-\frac{1}{d_1}$. (k)	Steady Current. (k')	$\sqrt{k'}/k$.
CsCl	20	7·3	26	0·70
"	15	6·4	19·5	0·69
"	10	5·3	13	0·68
"	5	4·1	7·4	0·66
"	1	2·6	3·0	0·67

It is clear from these results that $\frac{1}{d_2}-\frac{1}{d_1}$ varies nearly as the square root of the steady current due to a small P.D.

(3) *The Variation of the Conductivity with the Potential-Difference and Number of Alternations per Second.*

The variation of the conductivity of the flame as measured by $\frac{1}{d_2}-\frac{1}{d_1}$, with the maximum P.D. applied to the bridge arrangement, was effected by varying the length of the spark-gap in the primary circuit of the Tesla coil. The spark passed between two platinum spheres in air at the ordinary pressure. The following table gives the results obtained when spraying a solution of CsCl containing one gram per litre :—

Spark-length.	P.D. (E.S. Units at gap.)	d_2.	$\frac{1}{d_2}-\frac{1}{d_1}$. (k).	$k\sqrt{\text{P.D.}}$
0·0055 cms.	2·60	0·23	4·05	6·55
0·011 "	3·6	0·42	2·08	3·95
0·017 "	4·7	0·43	2·03	4·4
0·028 "	6·4	0·48	1·78	4·5
0·044 "	8·8	0·50	1·70	5·0
0·10 "	16	0·53	1·59	6·3
0·20 "	27·8	0·60	1·37	7·2

The last column contains the products of $\frac{1}{d_2}-\frac{1}{d_1}$ and the square root of the corresponding potential-difference as estimated from the spark-length in the primary circuit of

the Tesla coil. The numbers in this column vary between 4 and 7, while the P.D. varies by a factor of 11. Taking into account the roughness of the method used to estimate the P.D. and the large change made in it, we may conclude that $\frac{1}{d_2}-\frac{1}{d_1}$ probably varies approximately inversely as the square root of the maximum P.D. applied.

The effect of varying the number of alternations per second was tried by altering the capacity in the primary circuit of the Tesla coil. A solution containing one gram of CsCl per litre was sprayed. The following table gives the results obtained :—

Capacity.	Self-Induction.	Alternations per second.	d_2.	$\frac{1}{d_2}-\frac{1}{d_1}$.
33000 cms.	136500 cms.	$7{\cdot}14\times10^4$	0·4	2·1
6400 „	„	$1{\cdot}62\times10^5$	0·4	2·1
1600 „	„	$3{\cdot}24\times10^5$	0·34	2·6
150 „	„	$10{\cdot}8\times10^5$	0·30	3·0
150 „	4100 cms.	$6{\cdot}2\times10^6$	0·30	3·0

It will be seen that the value of $\frac{1}{d_2}-\frac{1}{d_1}$ varies as the number of alternations per second is changed. Unfortunately it was not found possible to obtain the variation of d_2 with the frequency very exactly; and all that can be said is that changing the frequency from $7{\cdot}14\times10^4$ to $6{\cdot}2\times10^6$ per second does not change $\frac{1}{d_2}-\frac{1}{d_1}$ by more than 25 per cent. of its mean value.

If we suppose that $\frac{1}{d_2}-\frac{1}{d_1}$ varies as n^x, where n is the frequency, then the results just given show that x lies between $+0{\cdot}05$ and $-0{\cdot}05$.

For a pure capacity $x=0$, and for a pure self-induction $x=-1$; so that it appears that the flame behaves nearly like a pure capacity. That it does not include much conductivity in parallel with the capacity is shown by the

fact that it could be nearly balanced by three condensers. In the section below, on the theory of conductivity, it is shown that these conclusions from the experiments might have been anticipated.

To obtain the apparent specific inductive capacity of a salt vapour, it is only necessary to multiply the numbers for $\frac{1}{d_2}-\frac{1}{d_1}$ given above by $2d_1=6{\cdot}66$ and add on unity. The apparent specific inductive capacities so obtained vary from about 100 for the strongest solution of K_2CO_3 sprayed to about 4 for the solution of LiCl *.

Some experiments were made to find out how the conductivity varied with the distance between the electrodes.

Two parallel vertical platinum disks each 1·5 cms. in diameter were used, and it was found that $\frac{1}{d_2}-\frac{1}{d_1}$ was independent of the distance between them when this was less than 3 or 4 millimetres.

When they were 10 millimetres apart $\frac{1}{d_2}-\frac{1}{d_1}$ was about double its value between 0 and 4 mms. This increase in the apparent capacity is no doubt due to the fact that the cross-section of flame acted on by the P.D. is greater when the distance between the electrodes is comparable with their diameter.

We may therefore conclude that $\frac{1}{d_2}-\frac{1}{d_1}$ would be independent of the distance between the electrodes if they were very large. This is in agreement with the results for constant P.D.'s, for which the current is independent of the distance between the electrodes when they are near together.

(4) *Theory of the Conductivity of Ionized Gases for rapidly alternating Currents.*

It will be convenient now to describe an approximate theory of the conductivity of ionized gases for rapidly

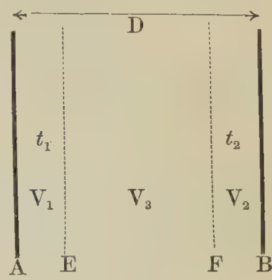
* These values of the apparent specific inductive capacity have of course no relation to the true specific inductive capacity, which must be nearly unity.

alternating currents—a theory which affords an explanation of the experimental results obtained in this investigation.

Suppose a large parallel plate-condenser filled with a uniformly ionized gas, and let the distance between the plates be D cms. Let the potential-difference between the plates be given by the formula $V = V_0 \sin pt$, and let the number of positive or negative ions per c.c. be n, each ion carrying a charge $\pm e$. In a Bunsen flame the velocity of the negative ions is about $1000 \frac{\text{cms.}}{\text{sec.}}$ for one volt per cm., while the velocity of the positive ions under the same potential gradient is only about $60 \frac{\text{cms.}}{\text{sec.}}$ Moreover, the mass of the positive ions is probably very large compared with that of the negative ions. Consequently, in a rapidly alternating electric field the amplitude of vibration of the negative ions will be large compared with that of the positive ions. As a first approximation, therefore, we shall assume that the positive ions do not move at all, so that all the current is carried by the negative ions. We shall also suppose that all the negative ions move in the same way with the same velocity, so that the number of negative per c.c. remains n except within a distance d of each electrode, d being twice the amplitude of vibration of the negative ions. It is easy to see that on these assumptions the negative ions will practically all be contained in a slab of thickness $D-d$, which will vibrate between the plates so as just not to touch either of them. For if a new negative ion is formed outside the slab, it will almost immediately strike the electrode near it; whereas a new negative ion formed in the slab cannot reach either electrode except by diffusion, which we shall neglect. Thus we may regard the whole space between the plates as filled with positive electricity of density $+ne$, and the vibrating slab of thickness $D-d$ as containing also negative electricity of density $-ne$. Thus inside the slab the total density is zero and outside is $+ne$.

Let X denote the electric intensity between the plates at a distance x from one of them. Then, inside the slab $\frac{dX}{dx} = 0$, and outside $\frac{dX}{dx} = 4\pi ne$.

Let A and B be the two plates, and let the slab be represented by the space between the two dotted lines E, F. Let $AB=D$, $AE=t_1$, $FB=t_2$, and suppose the potential of A kept zero while that of $B=V$. Let the rise of potential in AE be V_1, in EF be V_3, and in FB be V_2. In EF $\frac{dX}{dx}=0$ so that X is constant. Let its value be X_0. Then



$$\therefore \qquad V_3=-X_0(D-d),$$

where $d=t_1+t_2$. In AE we have

$$\frac{d^2V}{dx^2}=-4\pi\rho,$$

where $\rho=ne$. From this we get

$$\frac{dV}{dx}=-4\pi\rho x+C \quad \text{and} \quad V=-2\pi\rho x^2+Cx+D,$$

where C and D are constants to be determined. When $x=0$

$$V=D=0,$$

and when $x=t_1$

$$\frac{dV}{dx}=-4\pi\rho t_1+C=X_0.$$

Hence $$V_1=2\pi\rho t_1^2-X_0t_1.$$

In the same way in FB we have

$$V=-2\pi\rho x^2+C'x+D'.$$

When $x=D-t_2$

$$\frac{dV}{dx}=-X_0 \quad \text{and} \quad V=V_1+V_3\,;$$

so that we get for V at $x=D$,

$$V=-X_0D+2\pi\rho(t_1^2-t_2^2).$$

Now $t_1+t_2=d$, so that

$$V=-X_0D+2\pi\rho d(2t_1-d). \quad . \quad . \quad . \quad . \quad (1)$$

The force acting on a negative ion is $-X_0e-A\frac{dt_1}{dt}$, where A is a constant representing the viscous resistance to

motion with unit velocity. Let m be the mass of a negative ion; then its equation of motion is

$$-X_0 e = m\frac{d^2t_1}{dt^2} + A\frac{dt_1}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The current-density inside the slab is given by the equation

$$i = -\rho\frac{dt_1}{dt} + \frac{K}{4\pi}\frac{dX_0}{dt},$$

where K is the specific inductive capacity of the medium between the plates in the absence of ions. Thus K is unity, and

$$i = -\rho\frac{dt_1}{dt} + \frac{1}{4\pi}\frac{dX_0}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now in a flame containing a salt vapour the fall of potential nearly all takes place near the electrodes, so that X_0 is probably very small, even when rapidly alternating currents are used. Consequently, since ρ is large, $\frac{1}{4\pi}\frac{dX_0}{dt}$ may be neglected in comparison with $-\rho\frac{dt_1}{dt}$. Hence (3) becomes $i = -\rho\frac{dt_1}{dt}$ approximately. Substituting in (1) the value of X_0 got from (2) we get

$$V = V_0 \sin pt = D\left(\frac{m}{e}\frac{d^2t_1}{dt^2} + \frac{A}{e}\frac{dt_1}{dt}\right) + 2\pi\rho d(2t_1 - d).$$

This gives

$$V_0 p \cos pt = \frac{mD}{e}\frac{d^3t_1}{dt^3} + \frac{AD}{e}\frac{d^2t_1}{dt^2} + 4\pi\rho d\frac{dt_1}{dt}.$$

But $\frac{dt_1}{dt} = -\frac{i}{\rho}$. Hence

$$-V_0 p \cos pt = \frac{mD}{e\rho}\frac{d^2i}{dt^2} + \frac{AD}{e}\frac{di}{dt} + 4\pi di. \quad . \quad . \quad (4)$$

The solution of this equation is

$$i = \frac{-\frac{e\rho}{mD}V_0 \sin(pt-\alpha)}{\left\{\left(1-\frac{4\pi de\rho}{p^2mD}\right)^2 p^2 + \frac{A^2}{m^2}\right\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (5)$$

where

$$\tan\alpha = \frac{-\left(1-\frac{4\pi de\rho}{p^2mD}\right)pm}{A}.$$

If a P.D. $V = V_0 \sin pt$ is applied to a condenser of capacity C, the current is given by the equation $i = CV_0 p \cos pt$. For the flame, if A and m are both negligible (5) becomes

$$i = -\frac{V_0 p \cos pt}{4\pi d},$$

so that the apparent capacity is $\frac{1}{4\pi d}$ per unit area. Now $\frac{d}{2}$ is the amplitude of vibration of the negative ions so that $d\rho$ must be the amount of electricity flowing during a half-vibration.

Let $\qquad i = -\frac{dQ}{dt} \qquad$ so that $\quad Q = \frac{V_0}{4\pi d} \sin pt.$

Then we have, integrating from 0 to π,

$$d\rho = \frac{V_0}{2\pi d} \quad \text{or} \quad d = \sqrt{\frac{V_0}{2\pi\rho}},$$

so that the apparent capacity per unit area is

$$\frac{1}{4\pi d} = \sqrt{\frac{\rho}{8\pi V_0}}.$$

If $\rho = 0$ this makes the capacity zero, whereas it should be $\frac{1}{4\pi D}$. This is due to the omission of $\frac{1}{4\pi}\frac{dX}{dt}$, which would not be negligible if ρ were very small. If, however, we take $\sqrt{\rho/8\pi V_0}$ to be not the apparent capacity but the increase in the apparent capacity due to the presence of the ions, then no error will be made even if ρ be small. The quantity $\frac{1}{d_2} - \frac{1}{d_1}$ which has been determined is proportional, as we have seen, to the increase in the apparent capacity when salt is added to the flame. In what follows we shall speak of $\frac{1}{d_2} - \frac{1}{d_1}$ as the apparent capacity.

Thus we should expect the apparent capacity per unit area to vary as the square root of the number of ions per c.c., and inversely as the square root of the maximum P.D. applied, but to be independent of the distance between the electrodes. The experimental results are in surprisingly good agreement with these conclusions. Thus we have seen that the apparent

capacity is nearly independent of the number of alternations per sec., and varies as the square root of the steady current conductivity. For small E.M.F.'s the conductivity for steady currents is proportional to the number of ions per c.c. and to the velocity of the ions. But it has been shown * that all alkali-salts in flames give ions having the same velocity, so that the conductivity for steady currents should vary nearly as the number of ions per c.c. present. Hence the observed apparent capacity varies nearly as the square root of the number of ions present. The apparent capacity was also independent of the distance between the electrodes, provided this was small compared with their diameters. Further, we have seen that the apparent capacity varies roughly inversely as the square root of the maximum P.D., measured by the length of the spark-gap in the primary coil of the Tesla transformer. It appears, therefore, that the expression $\sqrt{\rho/8\pi V_0}$ does represent approximately the observed variations of the apparent capacity of the electrodes in the flame containing salt vapour.

This expression has been obtained by neglecting the mass of the ions and the resistance to their motion through the flame-gases, so that it appears that the amount of alternating current through the flame is determined merely by the density of the layer of positive charge left in the gas near the electrodes when the negative ions move under the action of the applied field.

If a steady P.D. V is applied to two electrodes immersed in an ionized gas, and if the positive ions cannot move, it is easy to see that a current will only pass for the short time required for the accumulation of positive charge near the negative electrode to become sufficient to make the electric force near the positive electrode zero. Thus the two electrodes will behave like a condenser when the P.D. is applied. When a rapidly alternating P.D. is applied it is easy to see that even if the positive ions can move, provided their velocity is small compared with that of the negative ions, the arrangement will behave like a condenser if the number of ions per c.c. is very large and the mass of the negative ions very small.

* H. A. Wilson, Phil. Trans. A. 1899.

Denoting the apparent capacity per unit area by C, we have

$$C = \sqrt{\frac{\rho}{8\pi V_0}} = \frac{K}{4\pi D},$$

where K is the apparent specific inductive capacity. Consequently $\rho = \frac{V_0 K^2}{2\pi D^2}$. The length of the spark-gap used in the experiments on the variation of the apparent capacity with the concentration and nature of the salt was about 2 mms. in hydrogen gas at atmospheric pressure. The Tesla coil had 33 turns in its primary coil and 3 turns in its secondary, so that V_0 was about 400 volts or 1·2 E.S. units. D was 0·6 cms., so that for the strongest K_2CO_3 solution sprayed, for which K = 100 (p. 496), we have

$$\rho = \frac{1 \cdot 2 \times 10000}{2 \times 3 \cdot 1 \times 0 \cdot 6^2} = 5400 \text{ E.S. units.}$$

The charge on one ion is 3×10^{-10} E.S. units, so that the number of ions per c.c. was

$$\frac{5400}{3 \times 10^{-10}} = 18 \times 10^{12}.$$

The amount of salt entering the flame was determined by finding the loss of weight of a bead of sodium-chloride placed in an equal Bunsen flame, so that the light emitted by this flame was equal to that emitted by the flame when a solution of NaCl containing 10 grams per litre was sprayed. In this way it was found that 0·53 milligram entered the flame per minute. Consequently the amount of K_2CO_3 entering the flame per minute with the strongest solution was 5·3 milligrams. The velocity of the flame-gases was about 200 cms. per second, and the diameter of the flame about 3 cms., so that the amount of salt per c.c. in the flame was about

$$\frac{5 \cdot 3}{200 \times \pi(1 \cdot 5)^2 \times 60} = 7 \times 10^{-5} \text{ milligram.}$$

Hence, taking the mass of an atom of hydrogen as 10^{-24} gram, the number of salt molecules per c.c. was about

$$\frac{7 \times 10^{-8}}{138 \times 10^{-24}} = 5 \times 10^{14}.$$

It thus appears that about one molecule in 30 molecules of K_2CO_3 was ionized in the flame. For most of the other salt

solutions sprayed the proportion of the molecules ionized in the flame comes out less than one in 30.

In the paper on the conductivity of flames for steady currents, referred to above, numbers are given which are proportional to the molecular conductivity of all the salts tried, so that it is not necessary here to discuss further the variation of the conductivity with the nature and concentration of the salt vapour.

Let q be the number of ions produced per c.c. per sec. in the flame and n the number present per c.c. Then $q=\alpha n^2$, where α is a constant which has been shown by Langevin to be equal to $f \,.\, 4\pi e(k_1+k_2)$, where f is a proper fraction, and k_1, k_2 are the ionic velocities due to unit electric intensity. For the flame we have $k_2=1000\,\frac{\text{cms.}}{\text{sec.}}$ for one volt per cm., or $3\times10^5\,\frac{\text{cms.}}{\text{sec.}}$ for one E.S. unit of electric intensity. Hence

$$\alpha=4\pi \,.\, f \,.\, 3\times10^{-10}\times3\times10^5=1{\cdot}1\times10^{-3}f.$$

Now for the strongest K_2CO_3 solution sprayed $n=1{\cdot}8\times10^{13}$, hence

$$q=f\,1{\cdot}1\times10^{-3}\times1{\cdot}8^2\times10^{26}=4\times10^{23}f.$$

The number of salt molecules per c.c. is about 5×10^{14}; so that it appears that each salt molecule is ionized and recombines about $10^9 f$ times per second. The value of f under ordinary conditions is about 0·2. Probably in the flame it is less, say 0·1. Taking the temperature of the flame-gases to be 2000° C., we get for the number of collisions made by a salt molecule per second in the flame rather less than 5×10^8. So it appears that the K_2CO_3 molecules are ionized once for every 5 collisions with another molecule. It is therefore probable that the cause of the ionization of salt vapours in flames is the shock of molecular collision.

The current which could be carried by the 5×10^{14} ions produced per c.c. is about 60 amperes, which is enormously greater than the observed currents per c.c. of flame between the electrodes. This result agrees with the conclusion * that the observed steady currents through flames containing salt vapours are very far from the saturation value. Since each salt molecule is ionized many times per second, the salt would

* H. A. Wilson, Phil. Mag. October 1905.

all be carried to the electrodes as ions if the current were sufficiently great. If we suppose that the electrodes absorb the ions which reach them, then the maximum possible current would be equal to that required to electrolyse the same amount of salt in a solution. This has been previously found to be the case for alkali salts vaporised in a current of air *.

Equation (5) may be written

$$i=\frac{-pe\rho V_0 \sin(pt-\alpha)}{\{(mDp^2-4\pi de\rho)^2+A^2D^2\}^{\frac{1}{2}}}.$$

It is easy to show that the term A^2D^2 is negligible compared with $(mDp^2-4\pi de\rho)^2$. We have $D=0{\cdot}6$ cm. and $Xe=Av$, where X = electric intensity and v = velocity of negative ions due to X. If $X=1$ E.S. unit, $v=3\times10^5\frac{\text{cms.}}{\text{sec.}}$; so that $D^2.A^2$ is equal to

$$\frac{0{\cdot}6\times9\times10^{-20}}{9\times10^{10}}=3\times10^{-29}.$$

In $(mDp^2-4\pi de\rho)^2$ the term $4\pi de\rho$ is about 6×10^{-9} for $e=3\times10^{-10}$, and $4\pi d\rho$ is about 20. Also when p is say 10^6, mDp^2 must be small compared with $4\pi de\rho$, because, as we have seen, changing p does not much affect the apparent capacity. Hence $(mDp^2-4\pi de\rho)^2$ is of the order 10^{-18}.

The expression for i becomes therefore on putting $\alpha=-90°$

$$i=\frac{pe\rho V_0 \cos pt}{4\pi de\rho-mDp^2}.$$

Hence the apparent capacity per unit area is

$$C=\frac{e\rho}{4\pi de\rho-mDp^2}.$$

Therefore if C_1 and C_2 are values of C corresponding to values p_1, p_2 of p, we have

$$4\pi d\rho-\frac{m}{e}Dp_1^2=\frac{\rho}{C_1},$$

$$4\pi d\rho-\frac{m}{e}Dp_2^2=\frac{\rho}{C_2};$$

$$\therefore\ D\,.\,\frac{m}{e}\,.\,(p_2^2-p_1^2)=\rho\left(\frac{1}{C_1}-\frac{1}{C_2}\right).$$

* H. A. Wilson, Phil. Mag. August 1902.

Also approximately $\rho = 8\pi V_0 C^2$ when p is small enough for mDp^2 to be small compared with $4\pi de\rho$, so that

$$\frac{e}{m} = \frac{D(p_2^2 - p_1^2)}{8\pi V_0\left(\frac{1}{C_2} - \frac{1}{C_1}\right) \times C^2}.$$

If C_1 nearly $= C_2$ we can put $C^2 = C_1C_2$, and get

$$\frac{e}{m} = \frac{D(p_2^2 - p_1^2)}{8\pi V_0(C_2 - C_1)}.$$

Thus if the variation of C with p were known with sufficien, accuracy, $\frac{e}{m}$ for the negative ions could be calculated. It is hoped that further experiments will enable this to be done for the negative ions of different salts.

Summary of Results.

(1) For rapidly alternating currents a flame containing an alkali-salt vapour behaves like an insulating medium having a high specific inductive capacity.

(2) The conductivity of different alkali-salt vapours in a flame for rapidly alternating currents as measured by the apparent capacity of platinum electrodes immersed in the flame varies as the square root of the conductivity of the same salt vapours for steady currents. This result confirms the view that the negative ions from all salts have the same velocity.

(3) The apparent capacity varies nearly inversely as the square root of the maximum applied P.D.

(4) The apparent capacity is nearly independent of the number of alternations per second.

(5) The apparent capacity is nearly independent of the distance between the electrodes.

(6) The results (1) to (5) are in agreement with the ionic theory of the conductivity of the flame for rapidly alternating currents when the velocity of the positive ions and the inertia and viscous resistance to the motion of the negative ions are neglected in comparison with the effects due to the number of ions per c.c.

(7) The apparent capacity per sq. cm. area of the electrodes is equal, according to the theory just mentioned, to $\sqrt{ne/8\pi V_0}$,

where n is the number of positive ions per c.c., e the charge on one ion, and V_0 the maximum applied P.D.

(8) Not more than one molecule in 30 salt molecules is ionized at any instant in the flame, but each molecule is probably ionized and recombines several million times per second.

(9) The steady currents observed through salt vapours in flames are very far from the maximum possible currents corresponding to the number of ions produced per second.

Discussion.

Mr. W. Duddell expressed his interest in the method of measurement, and referred to the fact that although the paper was entitled "The Electrical Conductivity of Flames," the Authors had not measured conductivities, but a complex quantity which was equivalent to a resistance shunted with a condenser. He drew attention to one of the tables given in the paper in which numbers referred to as constant varied by over 50 per cent., and to a figure in which the minimum galvanometer deflexion was shown as a maximum.

Dr. Wilson and Mr. Gold, in reply, said that the quantity which they had measured was the apparent capacity of the electrodes in the flame. The current through the flame for a particular frequency was proportional to the apparent capacity, so that it was considered proper to take this quantity as a measure of the conductivity of the flame. The experiments showed that the flame electrodes did not behave like a condenser shunted with a resistance, but very nearly like a condenser simply, and the theory which was worked out in the paper explained this fact in a satisfactory manner. The table of results referred to by Mr. Dnddell showed a variation of 50 per cent. in the numbers given, but these numbers were not referred to as constant. The figure which Mr. Duddell said showed a maximum, did not show a maximum but a minimum because the galvanometer deflexions were measured downwards.

VIII. *The Isothermal Distillation of Nitrogen and Oxygen and of Argon and Oxygen.* By J. K. H. Inglis *.

Many investigations have been made in order to find the relation connecting the composition of the vapour with the composition of the liquid, when a mixture of two liquids is distilled isothermally (see Young's 'Fractional Distillation,' Macmillan & Co., where a full summary of the literature on this subject may be found); and it has been shown that in some cases the relation takes the simple form $r_v = k \cdot r_l$, where r_v is the ratio of the two substances in the vapour, r_l the corresponding ratio in the liquid, and k a constant. In most cases, however, k is not an absolute constant but varies slightly with the molecular composition of the liquid; and we thus get mixtures (*a*) which have a maximum vapour-pressure, (*b*) which have a minimum pressure, and (*c*) which although having neither maximum or minimum vapour-pressure do not satisfy the above relation. These different cases have been very fully investigated by Zawidzki and others at ordinary temperatures; but the only paper dealing with distillation at low temperatures is one by Mr. E. C. C. Baly (Phil. Mag. vol. xlix. p. 517, 1900), who carried out a series of distillations of liquid air under a pressure of one atmosphere. In isobaric distillations, however, the conditions are not so simple as they are when the temperature is constant; so at Mr. Baly's suggestion I decided to complete his work by making a series of isothermal distillations at low temperatures, and I received considerable help from him in the early experiments when we hoped to make the research a joint one.

Many forms of apparatus have been devised for carrying out distillations isothermally, but none of them were suitable for work at low temperatures. Moreover, most of the forms used are open to the objection that care is not taken to ensure that the vapour is in complete equilibrium with the liquid; and in addition considerable error may be introduced by back condensation, &c. The apparatus used by Zawidzki (*Zeit. f. phys. Chemie*, vol. xxxv. p. 129) avoids most of these

* Read January 26, 1906.

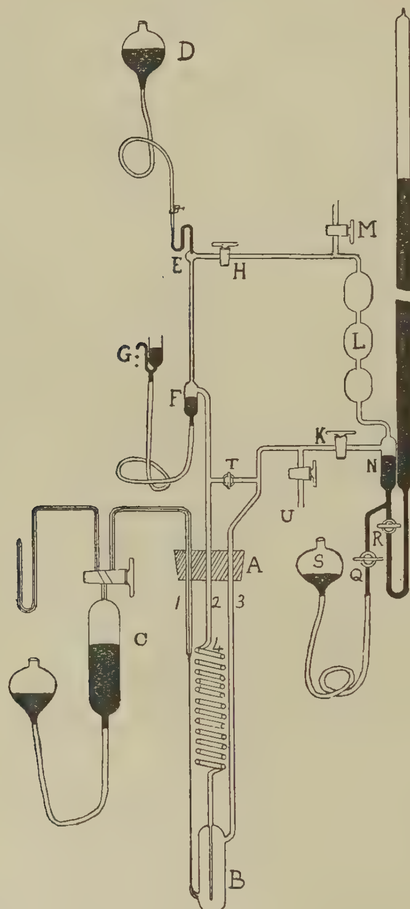
sources of error; but even in this case, since the sample of distillate is obtained by distilling over a small quantity of the liquid, the composition of the liquid may change during this operation, and so introduce error. Such an error would be of considerably greater importance in dealing with liquefied gases; for in this case one cannot work with such large quantities of liquid as Zawidzki used. But this error and many others may be eliminated by circulating the vapour through and through the liquid until no further change in either takes place; and then collecting and analysing a portion of the vapour which has in this way been brought into equilibrium with the liquid.

With such an arrangement it was necessary to keep the temperature of the distillation-bulb very constant for a considerable time, and an accurate means of measuring the temperature was also required. The distillation-bulb was therefore immersed in a long cylindrical vacuum-vessel containing about 600 c.c. of liquid air, and the pressure under which this air was boiling was varied by means of a Fleuss pump so as to keep the temperature constant. To measure the temperature, a bulb containing liquid oxygen and connected to a manometer was immersed in the liquid air so that the manometer registered the vapour-pressure of pure oxygen at the temperature of the distillation-bulb. With this arrangement the Fleuss pump was worked at such a rate that the manometer showed a constant pressure. In the different experiments this pressure amounted to either 100, 200, or 300 mm., and the variation of pressure seldom exceeded 0·5 mm. This variation corresponds to a temperature difference of 0°·03 C., 0°·02 C., and 0°·01 C. in the three different cases. These pressures, according to Travers (Phil. Trans. 200. p. 105) correspond to 74°·7, 79°·07, and 82°·09 Absolute measured on the hydrogen scale.

The apparatus used for carrying out the distillations is shown diagrammatically in fig. 1, the liquid oxygen bulb and the corresponding manometer being omitted. The rubber cork A, which fits the mouth of the vacuum vessel containing the liquid air, has passing through it the liquid oxygen bulb and several tubes of which three, 1, 2, 3, are joined to the distillation-bulb P, and another, not shown in

the figure, leads to the Fleuss pump. In an experiment, B is half filled with the mixture to be distilled, and the vapour, after being thoroughly cooled by the spiral 4, is blown through the liquid in B, the tube 3 then carrying the vapour on into

Fig. 1.



the circuit. The tube 1 is joined to the bulb B quite close to the bottom, and for a considerable part of its length consists of an extremely fine capillary drawn out of an ordinary piece of capillary tubing. This tube is also connected to a gas-holder C, so that when the reservoir of C is lowered liquid

is sucked up the fine capillary to a warm part above the liquid air, and there boils as a whole and is collected in the gasholder. The circulation of the vapour is carried on by means of the mercury circulator DEFG of the pattern described by Collie (Journ. Chem. Soc. 1889, p. 110). Mercury flows from the reservoir D through a rubber tube furnished with a screw-clip and then falls in drops down the tube EF. As each drop falls it drives on the gas contained in the tube EF and pumps gas in the direction KLHE. In this way the circulator pumps the vapour up the tube 3 and blows it down 2, and thus sends it round and round the circuit. The mercury after it falls down the circulator is collected at the exit G, the height of which above or below F is adjusted so as to keep the surface of the mercury just below the side tube leading from F.

The sample of vapour is collected in the bulbs L through which the vapour is circulated; by closing the taps HK this sample of vapour is shut off from communication with the liquid in P, and can be collected by opening the tap M which leads to a Töpler pump. The pressure under which the liquid is boiling can be measured in two different ways. It can either be calculated from the difference in height of the mercury surfaces in F and G and the height of the barometer; or it may be measured directly by means of the closed manometer NP. The shorter open limb N of this manometer can be shut off from the closed limb P by means of the tap R. By means of the taps QR and the reservoir S the amount of mercury in the manometer can be so regulated that when R is open the mercury surface in N is close below the side tube coming from the tap H. In this way dead space is as far as possible avoided. This manometer could be used for pressures up to 770 mm., and the manometer GF was used only for pressures greater than this. Readings of the manometer NP were made during the circulation, so that one could see when the reading became constant; but the final readings were taken by means of a telescope after the circulator had been stopped and after the tap T connecting the tubes 2, 3 had been opened. It was found convenient to close the tap R a moment after T had been opened and before the readings were taken.

The gases were stored over water in ordinary glass gasholders—the argon alone being kept over mercury; and after passing through soda-lime and phosphorus-pentoxide tubes, were admitted to the apparatus by means of the tap U. As regards the gases themselves, the oxygen was in all cases obtained by the decomposition of potassium permanganate. The nitrogen was prepared by heating a mixture of solutions of ammonium sulphate and potassium nitrite, and was fractionated by means of liquid air before being passed into the gasholders. A small quantity of oxygen was then added and the gas left standing over water. In this way, any traces of nitric oxide were turned into nitrous or nitric acids which dissolved in the water, and the small quantity of oxygen (0·5 per cent.) remaining did not matter, as mixtures of nitrogen and oxygen were to be used. The argon used, which was kindly supplied to me by Sir Wm. Ramsay, was purified by means of a hot mixture of quicklime and magnesium, and was fractionated with liquid air to remove traces of helium and neon. Its spectrum showed that it was extremely pure.

The analyses in each case were carried out by measuring off 10–13 c.c. of the gas, and then removing the oxygen by means of a pellet of yellow phosphorus. A measurement of the volume of the residual gas then gave the molecular composition. Tests showed that analyses carried out in this way did not have a greater error than 0·1 per cent., which was sufficiently accurate.

Two parts of the apparatus needed exhaustive testing before one could be sure of the results. The method of taking the sample of the liquid was based on the assumption that by taking a fine enough capillary, the liquid would evaporate as a whole and would not fractionate itself. The first capillaries employed were found to give very variable, and therefore untrustworthy samples; but by inserting a fine drawn-out capillary, as drawn in fig. 1, concordant results were obtained. These tests were carried out as follows:—About 5 litres of dry air were condensed in the bulb B, and samples of the liquid, which half-filled the bulb, were then taken and analysed. The percentages of nitrogen found were 78·77, 78·52, 78·74, 78·66, and 78·84, which

results, omitting the second one, give 78·75 per cent. as a mean, the deviation of the second one from this mean being 0·2 per cent. But all the results are lower than the true percentage of nitrogen in the air, which was found to be 79·06, 78·99 per cent. in two consecutive experiments. This difference can be explained as follows :—When the air was condensed, the spiral 4 and the part of the bulb B above the liquid were filled with vapour which was in equilibrium with the liquid, and which therefore did not have the same composition as the air. The vapour, in fact, contained an excess of nitrogen, so that the liquid contained too high a proportion of oxygen. The vapour also being at a temperature of less than 80° Abs., the dead space contained nearly 4 times as much gas as the same space would at ordinary temperatures. Thus 25 c.c. of dead space represented 100 c.c. of vapour measured at ordinary temperatures. This vapour contained about 5·5 per cent. of oxygen, so that the dead space contained $94{\cdot}5-4\times5{\cdot}5=72{\cdot}5$ c.c. of nitrogen above the proper amount for the oxygen. This amount of nitrogen is to be deducted from 5 litres of air ; and this brings down the true percentage in the liquid to 78·69 per cent., which is sufficiently near the mean value 78·75 per cent. found by experiment. It is certainly surprising that the amount of dead space can make so great a difference in the composition of the liquid ; but it is of course due to the temperature of the dead space being so low and to the great difference in the vapour-pressures of nitrogen and oxygen. These tests show therefore that the sample of the liquid can be taken pretty accurately, the error being certainly not more than 0·3 per cent. and usually not more than 0·1 per cent.

The sampling of the vapour offered no possibility of error, but it was necessary to ascertain how long the circulation must be carried on in order to obtain true equilibrium between the vapour and the liquid. About 5 litres of a mixture of nitrogen and oxygen containing about 75 per cent. of nitrogen were condensed in the bulb B, and after ten minutes circulation (the circulator pumped the gas at the rate of about 50 c.c. a minute) samples of the liquid and vapour were taken. Circulation was then carried on for a further fifteen minutes and fresh samples taken, and again for a further

twenty minutes and a third pair of samples taken. The tubes K, L, H were then pumped out, and while some more of the mixture was being taken into B through the tap U the tap K was opened, thus filling the bulbs L with the mixture; a second series of experiments was then made exactly as at first. The results are given in Table I.

TABLE I.

Time of circulation	10 min.	15 min.	20 min.	10 min.	15 min.	20 min.
Per cent. of Nitrogen in liquid	74·9	75·0	74·4	74·8	74·3	74·4
Per cent. of Nitrogen in vapour	93·1	93·0	92·7	93·0	92·8	93·0

These experiments show that the liquid sampling was not very reliable; but the experiments were made before the method of liquid sampling had been perfected. The vapour

TABLE II.

$P_{N_2}/P_{O_2}=5·31.$ $C=6·60-0·028\ m.$

Temp. 74°·7 Abs. Pure oxygen has vapour-press. = 100 mm.

No.	Molec. percent. N_2 in liquid. Obs.	Molecular percentage Nitrogen in Vapour: obs.	calc.	diff.	Total Pressure: mm. obs.	smoothed	diff.	Part. Pressure Nitrogen.	Part. Pressure Oxygen.
							%	mm.	mm.
1...	0·0	0·0	0	...	100·0	100·0	...	0·0	100·0
2...	5·3	25·5	26·5	−1·0	130·6	130·0	+·5	34·5	95·5
3...	7·3	34·3	33·6	+ ·7	140·7	141·0	−·2	47·5	93·5
4...	11·4	44·5	44·7	− ·2	162·9	162·7	+·1	72·7	90 0
5...	16·6	55·5	55·0	+ ·5	190·2	190·0	+·1	104·5	85·5
6...	21·0	61·4	61·5	− ·1	211·2	210·5	+·3	129·5	81·0
7...	25·6	66·5	66·9	− ·4	232·2	232·7	+·1	155·7	77·0
8...	30·5	71·2	71·6	− ·4	254·5	255·0	−·2	182·5	72·5
9...	33·4	73·6	73·9	− ·3	267·1	267·8	−·2	197·9	69·9
10...	37·5	76·8	76·7	+ ·1	284·8	285·0	−·1	218·6	66·4
11...	41·8	80·0	79·7	+ ·3	304·2	303·7	+·2	242·0	61·7
12...	44·5	81·5	81·1	+ ·4	315·6	315·0	+·2	255·4	59·6
13...	48·4	83·0	83·1	− ·1	330·1	330·5	−·1	274·6	55·9
14...	52·2	84·6	84·6	0	346·4	346·7	−·1	293·3	53 4
15...	56·4	86·4	86·6	− ·2	361·1	363·0	−·5	314·3	48·7
16...	57·3	87·2	87·0	+ ·2	366·0	366·5	−·1	318·8	47·7
17...	63·1	89·2	89·2	0	388·7	390·0	−·3	347·9	42·1
18 ..	68·6	91·0	91·1	− ·1	408·5	411·2	−·7	374·6	36·6
19...	74·8	93·0	93·0	0	432·5	435·5	−·7	405·0	30·5
20...	79·2	94·2	94·3	− ·1	453·3	453 0	+·1	427·2	25·8
21...	84·0	95·9	95·7	− ·2	467·5	471·0	−·8	450·7	20·3
22...	88·9	97·0	97·1	− ·1	489·7	490·0	−·1	475·8	14·2
23...	93·9	98·4	98·4	0	509·8	509·0	+·2	500 8	8·2
24...	99·4	99·9	99·9	0	530·0	529·0	+·2	528·5	0·5
25...	(100·0)	(100·0)	(100)	...	(531·0)	(531·0)	...	(531)	

samples show slight variations, and equilibrium is apparently reached after ten minutes circulation; so a *minimum* of twelve minutes was fixed upon as being sufficient to ensure equilibrium. It should be remembered that the percentage of nitrogen in the liquid sample must steadily fall; for the vapour is richer in nitrogen than the liquid, and some of the vapour is removed in taking each sample of the vapour. This to a certain extent explains the lower value obtained for the third sample, *i. e.* after twenty minutes circulation in each case.

The apparatus having thus shown itself accurate and convenient for working, two series of distillations were undertaken. In the first of these, the results of which are given in Table II., twenty-three mixtures of nitrogen and oxygen were distilled at that temperature (74°·7 Abs.) at which the vapour-pressure of pure oxygen is 100 mm.; and the total pressure, the composition of the vapour, and the composition of the liquid were determined. In the second series, which was carried out at 79°·07 Abs., where the vapour-pressure of pure oxygen is 200 mm., only eleven mixtures were taken; the results are given in Table III.

TABLE III.

$P_{N_2}/P_{O_2}=4{\cdot}655$. $C=5{\cdot}48-0{\cdot}0207\,m$.

Temp. 79°·07 Abs. Pure oxygen has vapour-press. = 200 mm.

No.	Molec. percent. N_2 in liquid. Obs.	Molecular percentage Nitrogen in Vapour: obs.	calc.	diff.	Total Pressure: mm. obs.	smoothed	diff.	Part. Pressure Nitrogen.	Part. Pressure Oxygen.
							%	mm.	mm.
1...	0	0	0	...	200	200	...	0	200
2...	3·4	16·7	16·0	+·7	233·2	231·5	+ ·7	37·0	194·5
3...	10·7	39·2	38·7	+·5	297·8	295·5	+ ·8	114·3	181·2
4...	19·7	55·5	55·4	+·1	371·8	371·5	+ ·1	205·8	165·7
5...	27·8	65·4	65·4	0	435·7	435·5	+ ·0	284·8	150·7
6...	40·5	75·8	75·9	−·1	529·7	530·0	− ·0	402·2	127·8
7...	51·0	82·1	82·1	0	(599)*	604·0	− ·8	495·5	108·5
8...	60·4	86·4	86·6	−·2	660·4	667·0	−1·0	577·6	89·4
9...	72·8	91·5	91·3	+·2	746·0	750·0	− ·5	684·7	65·3
10...	82·8	94·8	94·8	0	815·5	816·0	− ·0	773·6	42·4
11...	90·8	97·6	97·2	+·4	872·5	870·0	+ ·3	845·6	24·4
12...	99·7	99·7	99·95	−·15	931·0	928·5	+ ·2	928·0	·0·5
13...	(100·0)	(100·0)	...	...	...	931·0	...	931·0	0

* In the case of Exp. 7, the total pressure was by a slip read only during circulation and is therefore certainly too low.

Discussion of Results.

If the ratio of nitrogen to oxygen in the vapour be compared with the same ratio for the liquid, it is found that the quotient of these two ratios is a linear function of the molecular composition of the liquid. Thus taking the results of Table II. we find that the experimental values, *i.e.* those values given in the second and third columns, approximately satisfy the relation

$$\frac{\text{Ratio } N_2 : O_2 \text{ in vapour}}{\text{Ratio } N_2 : O_2 \text{ in liquid}} = 6{\cdot}60 - 0{\cdot}028\, m = C,$$

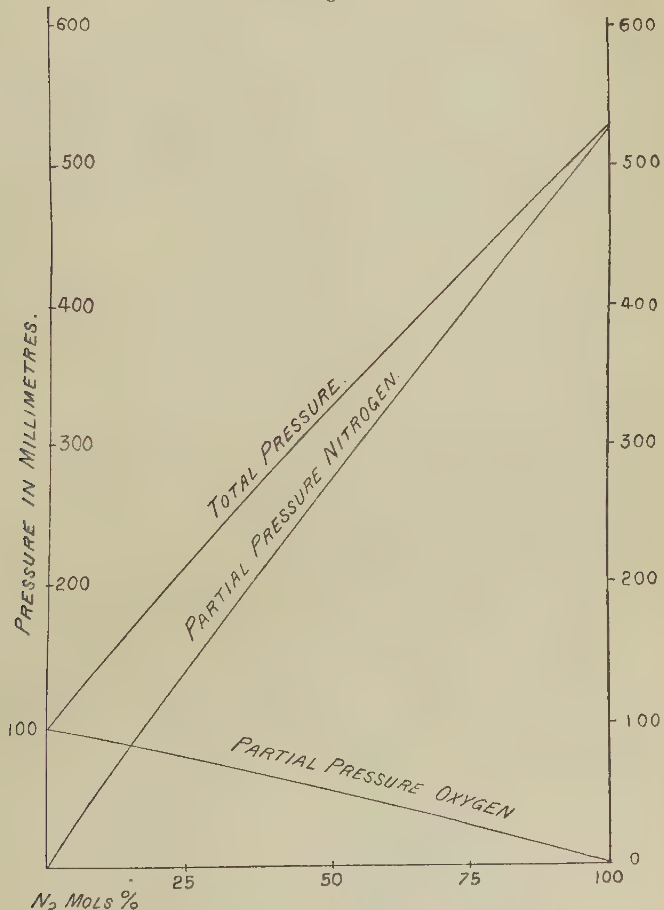
where m is the molecular percentage of nitrogen in the liquid. Similarly in Table III. the value of this same quotient is $5{\cdot}48 - 0{\cdot}0207\, m$. Now by means of these formulæ we can calculate the composition of the vapour from the composition of the liquid, and these results are given in the fourth column of the tables under the heading "nitrogen in vapour calculated." It will be seen that the differences from the observed values are slight except when the value of m is small. Now in this case an error of 0·1 per cent. in the liquid corresponds to an error of 0·5 per cent. in the calculation of the percentage of the vapour; so that the apparently large differences between the calculated and observed values correspond to only small errors in the analysis of the liquid. The formulæ given can therefore be fairly used to smooth the experimental results.

As regards the total pressure, since this could only be smoothed graphically, the results were plotted on squared paper on which 2 mm. corresponded to 1 per cent. in the liquid, the pressures being plotted full size. A steady curve was then drawn through the experimental points and the "smoothed" values of the total pressure taken from this curve. Comparison of the "smoothed" and "observed" values shows a fairly good agreement considering how rapidly the vapour-pressure changes with the temperature. From these "smoothed" values of the total pressure and the composition of the vapour, the partial pressures of the nitrogen and the oxygen may be calculated, and these results are given in the last columns of the tables.

When now these partial pressures and the total pressure

are plotted in the usual way against the molecular composition of the liquid (see fig. 2 for the results of Table II.), it is found that the curves obtained, though they have only slight curvature, are certainly not straight lines—a result

Fig. 2.



Vapour-pressures at 74°·70 Abs

which was to be expected from the fact that the relation $r_v = kr_l$ did not hold good. This relation is the mathematical expression of the property "the ratio of the concentrations in the vapour is proportional to the ratio of the concentrations

in the liquid." Hence we might expect to find the concentration of either substance in the liquid proportional to its partial pressure as vapour, this being the relation known as Henry's law. Now the vapour is usually plotted against the molecular percentage, and a straight line for the vapour-pressure would therefore indicate a proportionality between the vapour-pressure and molecular percentage. But since the volume of a gramme-molecule of a liquid is a quantity which is not the same for different liquids, the molecular percentages are not true concentrations. Hence a straight line in the usual method of plotting would indicate a deviation from Henry's law; and it is therefore evident that the correct method is to plot the partial pressures against the true concentration.

To obtain the true concentration one needs to know the density of each mixture used. But a close approximation to the actual value of the density may be obtained by calculating the density of the mixture from the densities of the pure components, *assuming that no contraction takes place.* Any contraction would of course make this calculated value wrong; but the error thus introduced would be of a higher order than the differences in density of different mixtures of the same two liquids. Measurements of the density of liquid nitrogen and oxygen have been made by Baly and Donnan (Trans. Chem. Soc. 1902, p. 907); and they found the values 0·8225 and 1·2160 respectively at that temperature at which Baly's curve for the vapour-pressure of oxygen gives the vapour as 100 mm.; and the values 0·8022 and 1·1947 respectively at the temperature corresponding to an oxygen vapour-pressure of 200 mm. According to Travers (*loc. cit.*), Baly's measurements of the temperature are erroneous; but the thermometer used being the same in the two cases the densities given are probably correct.

Hence for the results of Table II. the volume of one hundred gramme-molecules of a mixture containing m molecules per cent. of nitrogen will be

$$\frac{28m}{0{\cdot}8225}+\frac{32(100-m)}{1{\cdot}2160}\text{ c.c.},$$

and the molecular concentrations will be obtained from the

molecular percentages by dividing each molecular percentage by the corresponding value of this expression. Similarly for the results given in Table III. the factor is

$$\frac{28m}{0\cdot8022}+\frac{32(100-m)}{1\cdot1947} \text{ c.c.}$$

For simplicity, however, the concentrations can be better stated as grams of the corresponding substance per 100 c.c. of the mixture; and the results of Tables II. and III. are given in this way in Tables IV. and V. (p. 164). If Henry's law holds, the quotient obtained by dividing the concentration by the partial pressure should be a constant, so the values of this quotient are also given in the Tables.

These two tables show that the solubility of nitrogen in the oxygen obeys Henry's law quite rigidly up to a molecular percentage of nearly 70 per cent., but that oxygen does not obey this law. The value of the quotient $\frac{\text{concentration}}{\text{pressure}}$ for oxygen varies in such a way as to point to association of oxygen molecules when dissolved in nitrogen. Now the surface-tensions of liquid oxygen, nitrogen, argon, and carbon monoxide were determined by Baly and Donnan (*loc. cit.*), and they concluded that the pure liquids showed no association. But looking at their results more closely, we see that the value of the temperature coefficient of the molecular surface-energy of oxygen, viz. 1·917, is not the same as that found for the other three gases, which all have a coefficient nearly equal to 2·003. Baly and Donnan concluded that the probable value of this coefficient at low temperatures was too uncertain for any conclusion to be drawn as to association; but the fact that argon, nitrogen, and carbon monoxide all have the same coefficient points to the value of that coefficient being the normal one, so that oxygen must be slightly associated. The association calculated from these figures would be $\left(\frac{2\cdot003}{1\cdot917}\right)^{\frac{3}{2}}=1\cdot068$. Now if oxygen is associated so that n molecules in the vapour become *one* molecule in the liquid, we should have, according to Henry's law, concentration = const. × (pressure)^n.

TABLE IV.—Temperature 74°·7 Abs.

No.	Concentration of Nitrogen.	Partial Pressure of Nitrogen.	$\frac{\text{Concen.}}{\text{Press.}}$	Concentration of Oxygen.	Partial Pressure of Oxygen.	$\frac{\text{Concen.}}{\text{Press.}}$	$\frac{\text{Conc. } O_2}{(\text{Press. } O_2)^{1\cdot09}}$
		mm.			mm.		
1 ...	0·0	0·0	...	121·6 (pure)	100·0	1·216	
2 ...	5·6	34·5	·1624	113·4	95·5	1·187	
3 ...	7·6	47·5	·1600	110·4	93·5	1·180	
4 ...	11·7	72·7	·1602	104·2	90·0	1·158	
5 ...	16·8	104·5	·1608	96·7	85·5	1·131	
6 ...	21·0	129·5	·1622	90·5	81·0	1·117	
7 ...	25·3	155·7	·1625	84·1	77·0	1·093	
8 ...	29·8	182·5	·1633	77·6	72·5	1·070	
9 ...	32·3	197·9	·1630	73·7	69·9	1·055	
10 ...	36·0	218·6	·1647	68·4	66·4	1·031	
11 ...	39·6	242·0	·1637	63·0	61·7	1·022	
12 ...	41·9	255·4	·1641	59·7	59·6	1·001	
13 ...	45·1	274·6	·1642	54·9	55·9	·982	
14 ...	48·2	293·3	·1643	50·4	53·4	·944	
15 ...	51·5	314·3	·1639	45·5	48·7	·934	
16 ...	52·2	318·8	·1637	44·4	47·7	·932	
17 ...	56·6	347·9	·1626	37·8	42·1	·899	
18 ...	60·8	374·6	·1623	31·8	36·6	·868	·630
19 ...	65·2	405·0	·1610	25·1	30·5	·824	·607
20 ...	68·3	427·2	·1599	20·5	25·8	·795	·581
21 ...	71·7	450·7	·1591	15·6	20·3	·769	·586
22 ...	75·0	475·8	·1576	10·7	14·2	·753	·593
23 ...	78·3	500·8	·1563	5·8	8·2	·709	·586
24 ...	81·9	528·5	·1550	(0·5)	(0·5)	(1·0)	
25 ...	82·3 (pure)	(531·0)	·1550	0	0	...	

TABLE V.—Temperature 79·07 Abs.

No.	Concentration of Nitrogen.	Partial Pressure of Nitrogen.	$\frac{\text{Concen.}}{\text{Press.}}$	Concentration of Oxygen.	Partial Pressure of Oxygen.	$\frac{\text{Concen.}}{\text{Press.}}$	$\frac{\text{Concen.}}{(\text{Press.})^{1\cdot15}}$
		mm.			mm.		
1 ...	0	0	...	119·5	200	·597	
2 ...	3·5	37·0	·0951	114·2	194·5	·587	
3 ...	10·8	114·3	·0948	103·4	181·2	·571	
4 ...	19·4	205·8	·0944	90·5	165·7	·546	
5 ...	26·8	284·8	·0941	79·5	150·7	·528	
6 ...	37·7	402·2	·0938	63·3	127·8	·496	
7 ...	46·2	495·5	·0931	50·7	108·5	·467	
8 ...	53·4	577·6	·0924	40·0	89·4	·447	·228
9 ...	62·3	684·7	·0910	26·6	65·3	·408	·218
10 ...	69·2	773·6	·0894	16·4	42·4	·387	·221
11 ...	74·4	845·6	·0880	8·6	24·4	·353	·219
12 ...	80·0	928·0	·0863	(0·3)	(0·5)		
13 ...	80·2	931	·0861	0	0		

But this equation takes no account of any relation between the association and the concentration, and therefore at its best can only approximately represent the facts when the concentration is small. In the equation, n and the constant are both unknown, but may be determined very easily by plotting the logarithm of the concentration against the logarithm of the pressure. If, now, this be done for experiments 23–20 of Table IV. (for all of which the concentration of the oxygen is low), it is found that the four points obtained lie close to a straight line the slope of which indicates that the pressure and concentration satisfy the relation, — concentration $=$ const. $\times$ (press.)$^{1\cdot 09}$, thus indicating an association factor $=1\cdot 09$. This is of the same order as Baly and Donnan's factor 1·068, so that the agreement is satisfactory. Similarly the results of Table V. point to an association factor 1·15, but a slight error in the composition of the vapour in experiment 11 in that table would be sufficient to explain the increase from 1·09 to 1·15. Values of the expressions $\dfrac{\text{concentration } O_2}{(\text{pressure})^{1\cdot 09}}$ and $\dfrac{\text{concentration } O_2}{(\text{pressure})^{1\cdot 15}}$ are given in Tables IV. and V. (for low concentrations of oxygen), and the values obtained show how the relation is satisfied. Since the size of the associated oxygen molecule is not known, one cannot introduce the corresponding modifications in the formula deduced from Henry's law, so that the formula used cannot be expected to give good results. In addition to this, Richardson (Phil. Mag. [6] vii. p. 266) has shown that one must consider separately the solubilities of the simple and associated molecules.

The results point, therefore, to nitrogen obeying Henry's law and to oxygen only obeying it when we allow for association. Now in the results of the many isothermal distillations which have been carried out at ordinary temperatures (see Zawidzki, *loc. cit.*) agreement with Henry's law has not been looked for, as investigators have only looked for a linear relation between the partial pressure and the molecular percentage. The same difficulty that we have experienced in the calculation of the densities of the mixtures arises here also in calculating the concentrations; but it may be overcome in the same way as before. A few of Zawidzki's

results have therefore been recalculated as concentrations, and the results showing the relations between concentration and partial pressure are given in the Tables VI. to X.

TABLE VI.

Propylene Bromide and Ethylene Bromide. 85°·05 C.

No.	Concentration of Propylene Bromide.	Pressure of Propylene Bromide.	Concen. / Press.	Concentration of Ethylene Bromide.	Pressure of Ethylene Bromide.	Concen. / Press.
1 ...	0	0	...	203·4	172·6	1·178
2 ...	4·4	3·2	1·38	198·6	167·8	1·183
3 ...	15·5	10·2	1·52	186·3	158·6	1·175
4 ...	31·3	19·9	1·57	169·0	145·1	1·165
5 ...	46·7	29·4	1·59	152·6	132·2	1·152
6 ...	60·7	37·3	1·63	136·9	121·4	1·128
7 ...	63·2	38·1	1·66	134·1	120·8	1·110
8 ...	82·8	52·9	1·56	112·7	101·7	1·108
9 ...	85·1	52·9	1·61	110·2	100·5	1·097
10 ...	105·1	67·7	1 55	88·2	81 9	1·077
11 ...	122·1	79 3	1·54	69·6	64·0	1·088
12 ...	139·5	92·5	1·51	50·5	48·0	1·052
13 ...	153·1	102·5	1·49	35·6	34·3	1·038
14 ...	163·1	110·4	1·48	24·7	23·5	1·051
15 ...	172·0	117·1	1·47	14·9	13·8	1·077
16 ...	175·1	120·1	1·46	11·3	10·1	1·123
17 ...	179·9	123·8	1·45	6·2	4·6	1·359
18 ...	182·7	126·5	1·45			
19 ...	185·6	127·2	1·45			

TABLE VII.

Benzene and Ethylene Chloride. 49°·99 C.

No.	Concentration of Ethylene Chloride.	Pressure.	Concen. / Press.	Concentration of Benzene.	Pressure.	Concen. / Press.
1 ...	0	0	...	84·5	268	·315
2 ...	16·3	32·0	·511	73·1	231·5	·316
3 ...	32·4	69·2	·469	61·7	189·8	·325
4 ...	33·0	70·5	·468	61·3	188·5	·325
5 ...	46·7	98·9	·472	51·7	156·0	·332
6 ...	59·3	123 6	·480	42·9	127·9	·335
7 ...	75·7	154·9	·489	31·3	92·5	·339
8 ...	88·1	178·1	·495	22·7	65·9	·344
9 ...	109·6	216·9	·505	7·5	21·7	·347
10 ...	120·4	236·2	·510	0	0	...

TABLE VIII.

Carbon Tetrachloride and Ethyl Iodide. 49°·99 C.

No.	Concentration of Iodide.	Pressure.	Concen./Press.	Concentration of Chloride.	Pressure.	Concen./Press.
1	0	0	...	153·4	306·3	·501
2	5·8	15·3	·381	148·6	295 6	·503
3	14·3	38·4	·373	141·7	280 8	·505
4	31·8	80·6	·394	127·3	249·4	·510
5	46·3	110 9	·417	115·4	227·2	·508
6	67 0	153·1	·437	97·4	193·0	·505
7	84·1	183·4	·459	84·3	167·5	·503
8	186·7	354·0	·527	0	0	...

TABLE IX.

Carbon Tetrachloride and Benzene. 49°·99 C.

No.	Concentration of Chloride.	Pressure.	Concen./Press.	Concentration of Benzene.	Pressure.	Concen./Press.
1 ...	0	0	...	84·5	268	·315
2 ...	8·4	18·5	·455	79·9	253·4	·315
3 ...	19·3	40·5	·477	73·9	237·1	·312
4 ...	28·9	59 7	·484	68·6	221·8	·309
5 ...	41·0	82·9	·494	62·0	202·5	·306
6 ...	48·0	97·0	·495	58·2	191·3	·304
7 ...	63·7	128·7	·495	49·5	165·8	·299
8 ...	89·0	176·4	·505	35·6	124·6	·286
9 ...	106·5	211·8	·503	26·0	93·4	·279
10 ...	119 8	238·5	·502	18·7	68·3	·273
11 ...	153·3	306·3	·501	0	0	...

In two cases, viz. propylene and ethylene bromide and benzene and ethylene chloride, Zawidzki found that the partial pressure of either component of the mixture was proportional to its molecular percentage. Now, as will be seen in Tables VI. and VII., the change from molecular percentage to concentration upsets this relation; but the deviation from Henry's law is not very great, and any association would be very slight. In the case of mixtures of carbon tetrachloride with ethyl iodide, Zawidzki obtained slightly curved partial pressure-lines; but it will be seen from Table VIII. that the carbon tetrachloride obeys Henry's law very closely, while the agreement for ethyl iodide is not so good. Similarly in Table IX., carbon tetrachloride and

benzene show a fair agreement with Henry's law. It must be remembered that in all these cases the concentration has

TABLE X.

Benzene and Acetic Acid. 49°·99 C.

No.	Concentration of Benzene.	Pressure.	Concen. / Press.	Molec. Weight of Acetic Acid in Vapour.	Concentration of Acetic Acid.	Pressure.	Concen. / Press.
1 ...	84·5	(268·9)	(·314)	...	0	0	
2 ...	83·3	262·3	·318	83·0	1·44	3·63	·397
3 ...	81·6	258·7	·316	89·1	3·51	6·53	·539
4 ...	81·0	257·2	·315	90·1	4·30	7·25	·593
5 ...	77·4	249·6	·310	94·7	8·56	11·5	·745
6 ...	74·6	244·8	·305	97·1	11·92	14·2	·839
7 ...	69·3	231·8	·299	99·3	18·3	18·4	·994
8 ...	65·9	224·7	·294	100·4	22·4	20·5	1·092
9 ...	57·9	211·2	·274	102·0	32·1	24·8	1·29
10 ...	53·3	200·9	·265	102·5	37·6	27·1	1·39
11 ...	51·4	195·6	·263	103·1	39·8	28·7	1·39
12 ...	33·0	153·2	·215	104·8	62·1	36·3	1·71
13 ...	30·6	147·2	·208	105·0	65·0	36·8	1·77
14 ...	26·6	135·1	·197	105·4	69·8	40·2	1·74
15 ...	11·8	75·3	·157	107·0	87·5	50·7	1·73
16 ...	1·5	13·3	·114	107·9	99·9	54·7	1·83
17 ...	0·5	3·5	·148	107·9	101·1	54·7	1·85
18 ...	0	0	...	...	101·7	(56·5)	(1·80)

been calculated from the densities of the two components *assuming that no contraction takes place on mixing*. Until the densities of these mixtures have been determined, a close relation between the calculated concentrations and the pressures cannot be looked for. It is possible that this explains the fact that in Tables IX. and X. the quotient for benzene rises with the concentration, whereas in Table VII. the quotient falls.

But the case of the distillation of acetic acid and benzene, the results of which are given in Table X., is particularly interesting from the point of view of agreement with Henry's law. Acetic-acid vapour has a molecular weight which shows that the vapour consists of a mixture of simple and double molecules. Now when acetic acid is dissolved in benzene, even in the most dilute solutions, the acetic acid consists entirely of double molecules. Hence, when acetic-acid

vapour dissolves in benzene, association takes place, and this ought to be shown if the partial pressure be plotted against the concentration. From Table X. it will be seen that neither the partial pressure of the acetic acid nor that of the benzene is proportional to the concentration, and that the association factor in the case of acetic acid is considerable. The association varies with the concentration, so that calculations from the partial-pressure curve are not very accurate. If the log of concentration be plotted against the log of the partial pressure as before, the slope of the curve at each point may be taken as a measure of the association for that concentration. The result obtained is an association factor 1·50, when the concentration of acetic acid is 3·51 gr. per 100 c.c. According to the molecular weight the factor should be 1·35. It will be seen, therefore, that again the agreement is not close; but since the method of calculation does not take account of the separate solubilities of the single and double molecules, as is really necessary, a better agreement can hardly be looked for.

Zawidzki's experiments, therefore, to a certain extent support the view that the relation between the partial pressure and the concentration can be obtained by means of Henry's law, so that it may be concluded that oxygen is associated when dissolved in nitrogen and also in the pure state.

Distillation of Argon and Oxygen.

In the separation of two gases from one another by means of fractionation at low temperatures, it often happens that at the temperature used the one substance is below its melting-point. In this case the relations which hold during distillation are modified by the fact that the total pressure of the saturated solution of the one substance in the other may be greater than the vapour-pressure of either pure substance. This happens in the case of argon and oxygen. The melting-point of argon is only a little below its boiling-point; and at the temperature of fairly fresh liquid air, argon is a solid with a vapour-pressure of over 400 mm. In order to see how a mixture of argon and oxygen behaved when distilled isothermally, a few experiments were made at

82·09 Abs. with the apparatus already described. As, however, the quantity of argon available was only 550 c.c., the greatest percentage of argon that could be used was 13·6 per cent. by volume. The results of these experiments are given in Table XI. The vapour-pressure of pure oxygen at 82·09 Abs. is 300·0 mm., and this value, together with the experiments 2, 3, 4, 5, gives five points on the vapour-pressure diagram. Experiment 6 was carried out in quite a

TABLE XI.

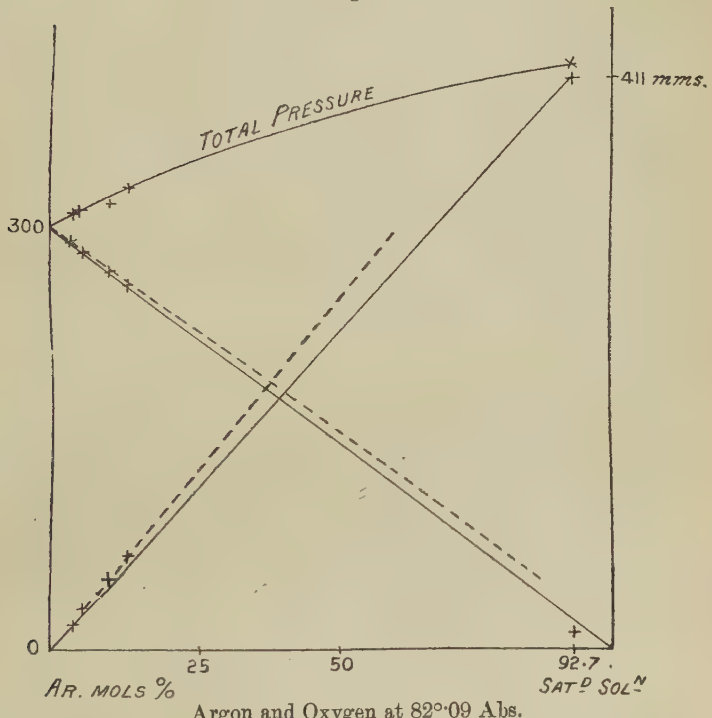
Argon and Oxygen. Temp. 82°·09 Abs.

No.	Molec. percent. Argon in Liquid. Obs.	Molec. percent. Argon in Vapour.		Total, obs.	Pressure, smoothed	Partial Pressure, Oxygen.	Partial Pressure, Argon.
		Obs.	Calc.			mm.	mm.
1......	0	0	0	300·0	300·0	300·0	0·0
2......	3·30	5·76	...	308·2	307·5	290·5	17·0
3	5·6	9·15	...	310·8	312·0	283·5	28·5
4......	10·2	16·0	...	320·1	321·2	269·8	51·4
5......	13·6	20·6	...	328·7	327·5	260·0	67·5
6......	92·7 Saturated	...	97·8	420·0	420·0	9·0	411·0

different way. A quantity of pure argon was prepared and its vapour-pressure at 82°·09 Abs. was found to be 411·0 mm. On adding a small quantity of oxygen the vapour-pressure rose to 420 mm., and it remained equal to this in spite of continued addition of oxygen so long as any of the argon remained solid. Hence, 420 mm. is the total pressure, above a saturated solution of argon in oxygen at 82°·09 Abs. Since the solution is saturated, the partial pressure of the argon must be equal to the vapour-pressure of solid argon at the same temperature. Hence the partial pressure of the oxygen is (420−411) mm. = 9·0 mm. and the percentage of argon in the vapour is $\frac{411}{420} \times 100 = 97{\cdot}8$ per cent. Several samples of the saturated solution were collected in the way used earlier for liquid samples, and it was found that the

composition of the liquid remained constant, however the amount of oxygen added might vary, so long as there was solid argon present. The analysis of the samples showed that the liquid contained 92·7 per cent. by vol. of argon. Hence we see that the liquid containing 92·7 per cent. of argon gave a vapour containing 97·8 per cent. of argon and exerted a vapour-pressure =420 mm.

Fig. 3.



Argon and Oxygen at 82°·09 Abs.

Since the densities of mixtures of argon and oxygen at 82°·09 Abs. are not known, and the density of liquid argon at that temperature cannot be measured, one cannot plot the pressures against the concentrations. In default of this, the pressures may be plotted against the molecular percentages, and this has been done in fig. 3. The broken lines in this figure are drawn through the points representing the partial pressures for low concentrations of argon; and

it will be noticed that they deviate considerably from the full lines which are drawn to show what the partial pressures would be if they were proportional to the molecular percentage. It will be noticed also that the partial pressure of oxygen above the saturated solution is much less than would be expected from analogy with the nitrogen-oxygen curve. This shows, therefore, that the separation of a solid from a liquid in which it is soluble is a much more complicated distillation-process than the separation of two liquids. A simple separation can be obtained, however, if one can lower the temperature so far that the vapour-pressure of the pure solid is negligible compared with the vapour-pressure of the other substance. In this case a single distillation effects a complete separation, and therefore in practice one always endeavours to obtain this condition of affairs.

In conclusion, I wish to express my thanks to Sir William Ramsay for the kind interest he took in this research.

University College,
London, W.C.

Discussion.

Dr. J. A. Harker said the paper was interesting at the present time, as in the 'Journal de Physique' for January a new process was described for obtaining liquid oxygen commercially by fractional distillation of liquid air with very small expenditure of power. In this proeess advantage was taken of the difference in composition of the liquid and the vapour in the distillation of liquid air.

IX. *A Note on Talbot's Lines.*

By JAMES WALKER, *M.A., Oxford* *.

IT is sometimes felt that Stokes's explanation of the "polarity" of Talbot's lines is mathematical rather than physical, so that those who are unable to follow the analysis still require an adequate reason for the fact that the retarding plate must be inserted on the one side of the aperture rather than on the other.

Prof. Schuster† has indeed supplied this want by considering the source of light to be due to a succession of impulsive velocities; but there is perhaps still room for an elementary explanation of the phenomenon on the old familiar lines of regarding the light as resolved into a congeries of monochromatic constituents.

1. When a stream of monochromatic light, coming from a distant slit, falls upon the object-glass of a telescope that is limited by a rectangular aperture with its sides parallel to the luminous line, it is easily shown that the diffraction-pattern in the focal plane of the telescope is characterised by a series of dark lines, arranged at equal distances on either side of the geometrical image of the slit. When half the aperture is covered by a retarding plate, the minima of an even order retain their former positions, but those of an odd order are displaced towards the side on which the plate is placed by an amount depending upon the retardation introduced by the plate.

Let us now suppose that the primitive light is white and that its monochromatic constituents have been separated before reaching the aperture by a prism or a grating, so that the different colours occupy different angular positions in the field of view.

The minima of an even order will now disappear: for on account of the dispersion there will be a gradual shift of the

* Read February 23, 1906.

† An Introduction to the Theory of Optics, p. 329; Phil. Mag. [6] vol. vii. p. 1 (1904).

centre of the system and consequently of these minima on passing from one wave-length to the next*.

The case of the minima of an odd order is, however, different; for as the wave-length alters, there is not only a shift due to the dispersion of the light, but also a dispersion due to the varying displacement caused by the change in the retardation introduced by the plate, and if these balance one another, the bands corresponding to different wave-lengths will be superposed and will therefore be rendered visible.

It is clear then that the plate must be so placed that the dispersion of the bands that it produces must oppose the dispersion of the light introduced by the prism or the grating. Thus the retardation of phase caused by the plate increasing with decreasing wave-length, the plate must be placed over the right or the left half of the aperture, according as the dispersion in the focal plane of the telescope from red to blue is from right to left or from left to right.

2. The following analysis of what takes place may perhaps tend to make the foregoing explanation somewhat clearer.

Let us suppose that the light, falling normally upon the aperture, is nearly monochromatic, as is the case when a small portion of the spectrum is considered, and that the slit of a spectroscope is placed in the focal plane of the telescope in a direction perpendicular to the bands of the diffraction pattern.

Then the light will be drawn out into a short spectrum, and the aperture being uncovered, this will be traversed by bands running along it, that are nearly straight and parallel to the sides of the spectrum (fig. 1).

When half of the aperture is covered by the retarding plate, the bands (a) of an even order will retain their positions, but those (b) of an odd order will be shifted, and as the amount of the displacement varies with the wave length, will be made to slant across the spectrum. Thus the right half being that which is covered, the appearance will be that represented diagrammatically in figure (2).

* The slight change in the distance between the minima as the wave-length alters, may be left out of account when a small region of the spectrum is considered.

Now the effect of a dispersion of the light will be to move each horizontal line of the spectrum in its own direction by an amount depending upon its distance from the end—in fact to give the figure a kind of shear. If this motion be to the right, the result will be that both sets of bands will slant

Fig. 1.

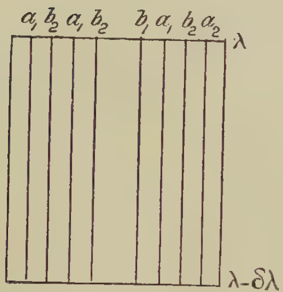
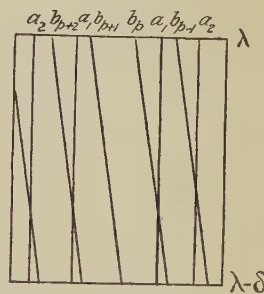


Fig. 2.



across the spectrum, but if it be to the left, the bands (a) will be made to slant, while the others (b) will be placed more nearly along the spectrum and, the shift being properly adjusted, will become exactly parallel to the sides.

Hence in the former case no bands will be seen when the focal plane is viewed directly with an eyepiece, but in the latter case the bands (b) will become visible.

3. The best thickness of the retarding plate for a given part of the spectrum may now be determined.

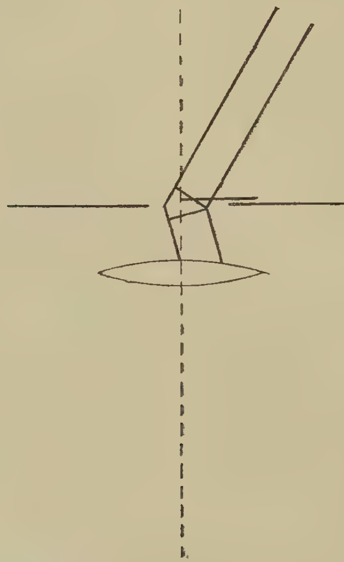
It is obvious that the resultant disturbances in any direction from the two halves of the aperture have the same amplitude, and that their phases are those of the secondary waves emanating from the central elements. Since the source of light is a line parallel to the sides of the aperture, directions perpendicular to these sides alone need be considered, and if ϕ be the angle of incidence, the disturbance in a direction θ may be represented by

$$A \cos\left(\frac{2\pi}{\lambda} vt - \Delta\right) + A \cos \frac{2\pi}{\lambda} \{vt - h(\sin\theta + \sin\phi)\}$$

$$= 2A \cos\left\{\frac{\pi}{\lambda} h\,(\sin\theta + \sin\phi) - \frac{\Delta}{2}\right\}$$

$$\times \cos\left[\frac{2\pi}{\lambda}\left\{vt - \frac{h(\sin\theta + \sin\phi)}{2}\right\} - \frac{\Delta}{2}\right],$$

$2h$ being the width of the aperture, Δ the retardation of phase introduced by the plate, and θ, ϕ being regarded as positive when measured on the side of the normal on which the plate is inserted.



Hence the intensity is

$$4A^2 \cos^2 \left\{ \frac{\pi}{\lambda} h(\sin\theta + \sin\phi) - \frac{\Delta}{2} \right\},$$

and there will be a primary series of minima given by $A=0$, being those due to a rectangular aperture of width h, and a secondary series of minima in directions given by

$$h(\sin\theta + \sin\phi) = \left\{ \frac{\Delta}{\pi} \pm (2n-1) \right\} \frac{\lambda}{2}, \quad n = 1, 2, \ldots$$

Let the telescope be directed so that light of wave-length λ_0 is incident normally upon the aperture, and let p be the order of the secondary minimum next to the focal point. Then θ being very small,

$$h\theta = \left\{ \frac{\Delta}{\pi} - (2p-1) \right\} \frac{\lambda_0}{2},$$

and for light of wave-length $\lambda_0+\delta\lambda_0$

$$h(\theta+\delta\theta+\delta\phi)=\left\{\frac{\Delta}{\pi}-(2p-1)\right\}\frac{\lambda_0+\delta\lambda}{2}+\frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda$$

$$\doteq\left\{\frac{\Delta}{\pi}-(2p-1)\right\}\frac{\lambda_0}{2}+\frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda,$$

whence

$$h(\delta\theta+\delta\phi)=\frac{\lambda_0}{2\pi}\frac{d\Delta}{d\lambda_0}\delta\lambda.$$

Hence for coincidence of the bands due to the monochromatic constituents λ_0 and $\lambda_0+\delta\lambda$, we must have

$$\frac{d\Delta}{d\lambda_0}=\frac{2\pi h}{\lambda_0}\frac{d\phi}{d\lambda_0},$$

which gives the best thickness of the plate.

Also $\frac{d\Delta}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ must have the same sign: whence $\frac{d\Delta}{d\lambda}$ being negative, the angle of incidence must increase on the positive side of the normal with decreasing wave-length.

X. *The Construction and Use of Oscillation Valves for Rectifying High-Frequency Electric Currents.* By J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London*.*

Attention was directed by the author in 1890 to the fact that if two carbon filaments are sealed into a single vacuous glass bulb so as to make an incandescent lamp with two separate carbon loops, the resistance between these filaments, though infinite when the carbon is cold, becomes quite small as soon as the loops are made incandescent †. Moreover, if a metal plate is sealed into an incandescent lamp it was shown that the space between the metal plate and the incandescent carbon filament possesses a unilateral

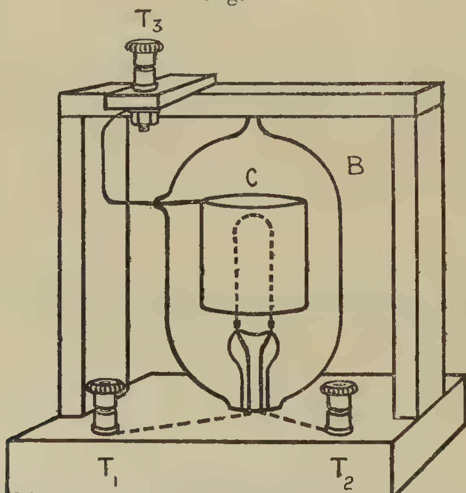
* Read March 23, 1906.

† See J. A. Fleming, "On Electric Discharge between Electrodes at Different Temperatures in Air and High Vacua," Proc. Roy. Soc. Lond. vol. xlvii. p. 122 (1890); also "Problems on the Physics of an Electric Lamp," Proc. Royal Institution, vol. xiii. part 34, p. 45 (1890).

conductivity, negative electricity being able to pass freely from the hot carbon to the plate, but not in the opposite direction *. More recently the author discovered that such an arrangement may be used as a valve to permit the passage of one constituent current only of a high-frequency current or to rectify an electric oscillation †. The reason for this action is now recognized to be the copious emission of negative ions or electrons from the incandescent carbon. This operation has been studied quantitatively by the present writer and many other observers.

For the purpose of rectifying electrical oscillations and thus be able to detect them by an ordinary galvanometer,

Fig. 1.



B, Exhausted glass bulb. C, Nickel cylinder. T_1 T_2, Carbon filament terminals. T_3, Insulated cylinder terminal.

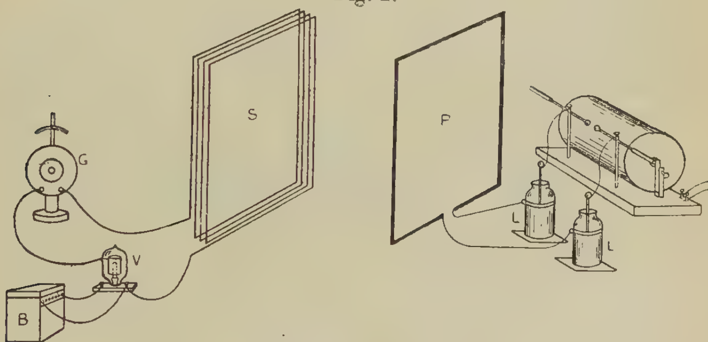
these *oscillation valves* are now made as follows :—A carbon-filament glow-lamp is constructed, the carbon loop of which is upheld in the centre of a highly exhausted glass bulb (see fig. 1). Around the loop is fixed a small cylinder

* See J. A. Fleming, "On a Further Examination of the Edison Effect in Glow-Lamps," Phil. Mag. July 1896.

† See also Proc. Roy. Soc. Lond. vol. lxxiv. p. 476, 1905, "On the Conversion of Electrical Oscillations into Continuous Currents by means of a Vacuum Valve."

of nickel, C, which is connected to a platinum wire sealed through the side of the bulb. The valve is used as follows:—The carbon loop is made incandescent by a suitable battery of secondary cells, a sliding rheostat being added to adjust the voltage on the terminals of the lamp. The circuit in which oscillations are to be detected is joined in series with a dead-beat mirror-galvanometer, and the valve connected with the circuit by wires joined respectively to the terminal of the nickel cylinder and the negative terminal of the carbon loop. The oscillation valve is most conveniently mounted for this purpose on a special form of stand (see fig. 1). In using the valve the carbon filament must be brought to bright incandescence, about equal to that which in a carbon glow-lamp would correspond to a so-called "efficiency" of 3 watts per candle. So used, the valve enables us to employ a sensitive

Fig. 2.



P, Primary oscillation circuit. S, Secondary oscillation circuit. G, Galvanometer. V, Valve. B, 12-volt battery for incandescing filament of valve.

mirror-galvanometer of the ordinary type to detect the presence of electric oscillations in a circuit and to institute comparative measurements.

Thus, for instance, we form an oscillatory circuit (see fig. 2) by connecting a Leyden jar in series with a square coil of wire of a few turns P, and join the condenser and inductance across a spark-ball discharger connected to the secondary terminals of an induction-coil. At a certain distance we place another square coil of wire S in series with

a galvanometer G and oscillation valve V. We then find that when oscillations are set up in the primary circuit, we obtain a steady deflexion of the galvanometer indicating that its coils are being traversed by a series of discharges in the same direction, all those in the opposite direction being practically stopped.

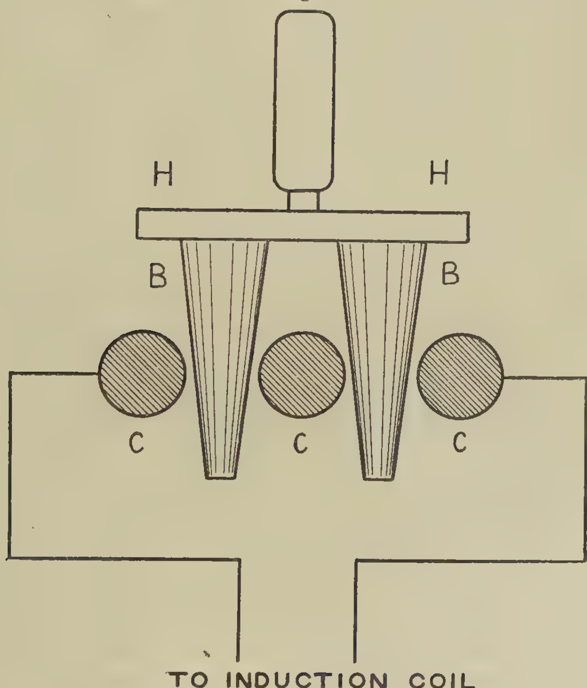
The author has already described the methods by which the amount of rectification produced by the valve can be ascertained (see Proc. Roy. Soc. vol. lxxiv. p. 484, 1905). Perfect rectification does not exist, but, as shown, the number expressing the percentage which the actual unilateral electric flow is of that which would flow if the unilateral conductivity were perfect, can be ascertained by sending the current which passes through the vacuous space of the valve through a calibrated galvanometer and electrodynamometer placed in series with each other. In valves as described this rectification may amount to 90 per cent.

We may use the above arrangements to investigate the effect of different kinds of discharge-balls and different lengths of spark. If we employ a fairly long spark in the primary condenser-circuit we may find that we obtain a very small effect on the galvanometer in the secondary circuit, but if we shorten the spark-gap until the spark at the balls is hardly visible, the galvanometer deflexion is generally increased. The reason for this is partly because the oscillations are damped out much more by the long spark than by the short one, and partly because with a short spark the condenser discharges occur more frequently. Hence, although in the latter case the condenser is charged to a less voltage owing to the lower discharge potential, the decreased damping and greater charge frequency causes the galvanometer to be traversed by a larger quantity of electricity per second, and therefore to give a greater deflexion.

In the same manner, we can exhibit the difference in the damping due to variations in the material of the spark-balls. Thus, using iron, brass, and zinc spark-balls of the same diameter and a spark-length of 0·1 mm., the galvanometer deflexions in one case were respectively 40, 57, and 70 scale-divisions, thus showing the smaller damping of zinc spark-balls. The writer has found by this means that carbon in

the state used for arc-lamp carbons presents many advantages as a discharge surface. All who have experimented much with Hertzian oscillators know how the state of the polish of the surface of the metal balls (generally brass) affects the electric wave-producing power. It can be shown by the use of an oscillation valve that for quantitative work a discharger made of carbon rods, as follows, presents many advantages:—A row of arc-lamp carbons C, C, C (see fig. 3) are fixed like posts in a piece of ebonite and another row of slightly conical

Fig. 3.



carbon rods B, B are inserted transversely between them, the distances between the rods being fixed so that very small air-gaps are left between carbon and carbon. We thus construct a multiple spark-gap of carbon surfaces which has small damping and great constancy. By enclosing the rods in a non-oxidizing atmosphere we can prevent the rods burning away. Another advantage of the arrangement is the ease with which new surfaces can be brought into use.

We can also investigate by the same means whether the use of spark-balls immersed in oil presents any advantages. Also the same arrangements may be used to exhibit the screening action of conductors for high frequency magnetic fields. For if we interpose between the primary and secondary circuits a sheet of tinfoil or zinc, we see a notable decrease in the galvanometer deflexion, thus making the screening action of the metal very evident. Employed in this manner, it enables us to show strikingly the rapid rate at which the field due to a current in a square or circular circuit decreases with distance from the circuit, and therefore to illustrate one of the disadvantages under which wireless telegraphy by electromagnetic induction labours when compared with space telegraphy by electric waves. When using the valve to detect the oscillations in an antenna produced by the impact of Hertzian waves, an oscillation transformer is inserted in the circuit of the receiving antenna, and its secondary circuit is connected through a valve with a dead-beat mirror-galvanometer. We are thus able to receive signals over short distances by the direct effect of the rectified oscillations themselves on the galvanometer.

The action of other substances besides incandescent carbon as a cathode in a vacuum-value has also been studied. It has been found by G. Owen* and by A. Wehnelt† that glowing metallic oxides, including the rare oxides employed in the manufacture of the Nernst lamp-glowers, copiously emit negative ions when incandescent both at atmospheric and at reduced pressures. Wehnelt found that the incandescent oxides of calcium, barium, and strontium produce an abnormally powerful electronic discharge, and, following the recommendations of the author, he has proposed to employ vacuum-tubes with one electrode covered with such oxides and heated, as rectifying valves for alternating currents.

As far back as 1890 the writer showed in a lecture experiment at the Royal Institution that the so-called Edison effect, that is the passage of negative electricity across space

* See G. Owen, Phil. Mag. vol. viii. p. 230 (1904). "On the Discharge of Electricity from a Nernst Filament."

† See A. Wehnelt, Phil. Mag. vol. x. p. 80 (1905). "On the Discharge of Negative Ions by Glowing Metallic Oxides and Allied Phenomena."

from an incandescent carbon filament to a metallic plate near it, could take place at atmospheric pressure if the plate was very near the filament. It is easy to show a similar experiment with a Nernst electric glow-lamp. If a Nernst lamp is supported with the bare glower horizontal and placed within a few millimetres of a vertical insulated metal tube kept cold by being filled with water, it is found that negative electricity will pass freely across the glower to the cold metal, but not in the opposite direction. Hence if the glower and metal tube are inserted as a gap in an electric circuit containing a sensitive galvanometer, and if secondary oscillations are created by induction in this circuit, we find that the galvanometer gives a steady deflexion showing the passage of a continuous current through it, and therefore of the unilateral conductivity of the space between the glower and the metal tube.

The distance over which this transference of electricity can take place depends very much upon the temperature of the glower, and the amount of rectification of the alternating current obtained upon success in keeping down the temperature of the metallic electrode. This is best achieved by circulating water through it.

It follows as a consequence from the above facts, that there is a considerable emission of negative ions or electrons from the incandescent portion of the lime cylinder used with the oxy-coal gas-burner to produce the lime-light, and that the space near the incandescent portion of the lime cylinder as well as the space near the Nernst lamp-glower is highly conductive by reason of the presence there of negative ions emitted from the oxide surface.

In the practical construction of oscillation valves, the advantage of placing the heated and non-heated electrodes in a vacuum is that the plate which acts as an anode can be placed at a greater distance from the incandescent surface and thereby kept cool, since the electrons ejected from the heated surface are projected to a much greater distance when the atmospheric pressure is reduced. Although platinum coated with calcium or barium oxides undoubtedly emits a much larger electronic current per square millimetre than carbon at the same temperature and under the same surrounding

conditions as to gas pressure, I find that for rectifying electric oscillations the carbon-filament oscillation-valve as I have designed it, affords more conveniently all that is required. There are some advantages in employing a thick carbon filament and constructing it to be worked at about 12 volts and take a fairly large current of 2 or 3 amperes. For one thing, the filament is much less likely to be destroyed by overheating in working, and hence the valve lasts longer. In some cases an advantage may ensue from working valves in parallel, that is joining up a number of such carbon-filament valves with their carbon filaments in parallel on the same heating battery, connecting together the insulated metal cylinders contained in each bulb together, and then using the multiple arrangement as if it were a single valve.

When used, however, to rectify such oscillations as are employed in the receiving circuits of wireless telegraph apparatus, a single valve will do all that is required because the quantity of electricity which has to be carried is small and the electronic emission from even a 4-volt 1-ampere carbon filament is amply sufficient to carry the negative component of the feeble oscillations used across the vacuous space.

It should be noted that such oscillation-valves as are here described have quite a different range of use from other rectifying arrangements such as the Cooper-Hewitt mercury-vapour tube, and the electrolytic aluminium-carbon valve of Nodon and others.

The electrolytic valves produce no rectifying effects with high-frequency alternating currents, because the time element enters into the formation of the aluminic hydroxide film on which their action depends. On the other hand, the mercury-vapour tubes which have been proposed for use with high-tension alternating currents will not operate below a certain minimum potential-difference between the electrodes. The vacuum-valve as here described, however, will pass current unilaterally with a fraction of a volt difference of potential between the incandescent and the cold electrode, and there is no minimum potential difference below which they will not act; hence their use is conditioned solely by the sensitiveness of the galvanometer employed with them.

By its simplicity and ease of use the carbon-filament vacuum-valve recommends itself as a useful addition to our resources for experimental work in connexion with electric oscillations and electric-wave telegraphy.

Discussion.

Dr. R. T. Glazebrook expressed his interest in the paper, and hoped that Dr. Fleming would be able in a further communication to give numerical data so that the sensitiveness of the arrangement described could be compared with those of other rectifying devices.

XI. *On the Use of the Cymometer for the Determination of Resonance-Curves.* *By* G. B. Dyke, *B.Sc.**

Dr. Fleming has shown in his recent Cantor Lectures before the Society of Arts †, that by the introduction of a hot-wire ammeter into the circuit of his direct-reading cymometer, the effective or root-mean-square value of the oscillation current set up in the cymometer circuit can be measured. This instrument was originally designed for the determination of the wave-lengths used in wireless telegraphy by the direct inspection of a scale, and also for the measurement of capacities and inductances; but it has been found that a small addition renders the instrument also available for the determination of resonance-curves, and therefore of the decrement of oscillation-trains, and of oscillatory spark-resistances.

A direct-reading cymometer was used constructed as described by Dr. Fleming in a paper read before this Society on March 24th, 1905 ‡. The instrument consists essentially of a closed circuit containing a condenser and an inductance,

* Read March 23, 1906.

† Cantor Lectures, 1905. Dr. J. A. Fleming on "The Measurement of High Frequency Currents and Electric Waves."

‡ "On the Application of the Cymometer to the Determination of the Coefficient of Coupling of Oscillation Transformers," by Dr. J. A. Fleming. Proc. Phys. Soc. Lond. vol. xix. p. 603; and Phil. Mag. June 1905

the distinctive feature being the fact that the capacity and inductance are so arranged as to be varied simultaneously and in the same proportion by one movement of a handle. A portion of the closed circuit consists of a straight copper rod, which is placed in the neighbourhood of, and parallel to, the circuit on which the measurements are being made. Then, as Dr. Fleming has shown in the paper referred to above, resonance will take place between the two circuits when the cymometer is so adjusted that its oscillation constant, that is the square-root of the product of the capacity and inductance, has the same value as that of the circuit under test. The position of resonance is detected by the illumination of a Neon vacuum-tube connected between the inner and outer coatings of the condenser. The Neon vacuum-tube, although an excellent detector of the position of resonance, gives but little indication of the relative value of the current in any other position of the cymometer, and is of the nature of an indicator rather than a measuring instrument.

For the determination of the logarithmic decrement, it is requisite to know the relative values of the current in the cymometer for points in the neighbourhood of resonance, and hence a quantitative current-meter must be employed. The instrument should fulfil the following conditions:—

(i.) Its capacity and inductance should be negligible compared with that of the cymometer, otherwise the disturbance of the scale-readings would lead to erroneous results.

(ii.) Its damping factor should be small and should be capable of calculation.

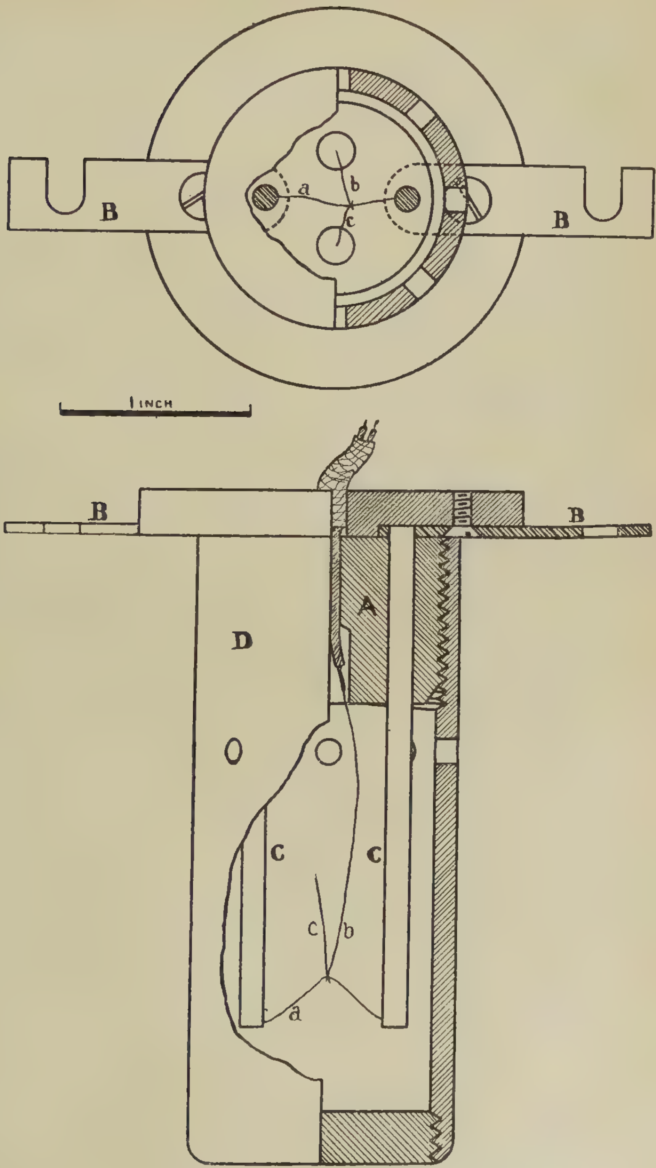
(iii.) It should be fairly rapid in its action.

The above requirements are fulfilled by an addition to the cymometer, made recently by Dr. Fleming, and shown in his Cantor Lectures referred to above.

The ammeter adopted is of the hot-wire type and is inserted into a gap made by cutting the bar of the cymometer; the current passing through the heated wire being measured by a thermojunction in contact with it.

The instrument consists of an ebonite block A (fig. 1), from the sides of which project lugs BB which are used for connexion to the cymometer. To these lugs are

Fig. 1.



Thermoelectric junction employed with Fleming Cymometer to determine mean-square value of the induced current.

soldered brass rods CC, each about 3 inches long, and between the extremities of these rods is stretched a fine platinoid wire *a* (diameter ·05 mm.) having a resistance of 3·5 ohms. To the middle of this wire the thermojunction is soldered. This junction consists of two very fine wires, *b*, *c*, one of pure iron (diameter ·20 mm.) and the other of bismuth (diameter ·17 mm.); these are attached to the platinoid with special solder of low fusing-point, the contact area being made as small as possible. The other ends of the wires are connected to the galvanometer by a flexible cord. The junction is shielded by the ebonite cap, D, which screws over the plug A. The electromotive force of the couple is observed by means of a low-resistance Paul single-pivot galvanometer, having a resistance of 4·88 ohms and a figure of merit of 19·5 microvolts per division.

The method of calibrating the junction is as follows:—For wires so fine as the platinoid used, the high-frequency resistance is the same as that for low-frequency or continuous current. Hence it is only necessary to connect up the hot-wire ammeter in series with an adjustable resistance and a secondary cell, and to pass currents of known strength through it, observing the corresponding deflexions of the galvanometer.

These observations enable a curve to be plotted from which the root-mean-square (or equiheating) value of the current in the cymometer for any deflexion can be readily read off. For the instrument described the calibration curve is such that the deflexion varies as the 1·9th power of the current.

An auxiliary resistance is also required and is constructed similarly to the hot-wire ammeter just described, except that the thermojunction is omitted and the resistance of the fine platinoid wire is 7·2 ohms. This resistance is arranged so that when required it can be put into a second gap cut in the cymometer bar, a short-circuiting strip being used to complete the circuit when it is not in use. It will be seen that if the cymometer bar is placed in the proximity of a circuit in which oscillations are taking place, the value of the R.M.S. current induced in it can be determined by means of the hot-wire ammeter described above for any position of the cymometer-handle; that is, for any oscillation constant or any frequency of the oscillations in the cymometer within the range of the instrument.

From observations of this R.M.S. current and frequency, it is possible to deduce the logarithmic decrements of both primary and secondary circuits. The logarithmic decrement of an oscillation per semiperiod is here defined to be the Naperian logarithm of the ratio of two successive maximum oscillations in *opposite* directions. In this connexion it is to be noted that most German physicists have defined the decrement as the logarithm of the ratio of two successive maximum oscillations in the *same* direction.

The problem of the oscillation transformer has been attacked more particularly by Oberbeck, Bjerknes, Drude and Wien; and Bjerknes and Drude have given solutions for obtaining the decrements, and although the proof is very long the final equations are simple and easily interpreted. For the complete proof we must refer to the original papers *; the essential parts, however, have been translated into English nomenclature by Dr. Fleming, and are given in his book on "The Principles of Electric Wave Telegraphy" (Longmans, Green & Co.). In this note we can do little more than state the final result of their work. We shall use the following symbols :—

δ_1 = logarithmic decrement per semi-period of the oscillation in the condenser circuit.

δ_2 = logarithmic decrement per semi-period in the secondary circuit inductively coupled with the primary.

n_1 = frequency of oscillation in condenser circuit.

n_2 = frequency in secondary circuit.

J = value of R.M.S. current in the secondary circuit corresponding with the frequency n_2.

J_r = maximum value of R.M.S. current in secondary circuit, *i. e.* the "resonance current."

Then Bjerknes shows that the following equation holds:

$$n_1\delta_1 + n_2\delta_2 = \pi(n_1 - n_2)\frac{J}{\sqrt{J_r^2 - J^2}};$$

or, if n_2 is nearly equal to n_1, that is the secondary is very

* V. Bjerknes: *Annalen der Physik*, vol. lv. (1895) p. 121; and vol. xliv. (1891) p. 74. P. Drude: *Annalen der Physik*, vol. xiii. (1904) p. 512.

nearly resonant to the primary, this becomes

$$\delta_1+\delta_2=\pi\left(1-\frac{n_2}{n_1}\right)\frac{J}{\sqrt{J_r^2-J^2}}.$$

Writing
$$x=1-\frac{n_2}{n_1},$$
$$y=\left(\frac{J}{J_r}\right)^2,$$
this becomes
$$\delta_1+\delta_2=\pi x\sqrt{\frac{y}{1-y}}.$$

This equation gives us the sum of the decrements in the two coupled circuits. In order to obtain the values of δ_1 and δ_2 separately we require some other relation between them. This relation has been given by Drude. He shows that the resonance current in the secondary circuit can be calculated from a formula equivalent to

$$J_r^2=V_1{}^2\frac{C_1C_2}{8}\,\frac{\pi^4 n_1 k^2}{\delta_1\delta_2(\delta_1+\delta_2)},$$

where V_1 = maximum value of primary condenser potential-difference.

C_1 and C_2 = capacities in primary and secondary circuits respectively.

k = coefficient of coupling of the two circuits

$$=\frac{M}{\sqrt{L_1L_2}},$$

where M = mutual inductance between the circuits ;

L_1 and L_2 = self-inductances of the primary and secondary circuits respectively.

Passing now to the delineation of the resonance curve, we proceed as follows :—Insert the hot-wire ammeter into the cymometer circuit and place the cymometer parallel to some straight portion of the primary circuit and at such a distance from it that when the cymometer is adjusted to resonance the current is not too large to be measured on the galvanometer. Then move the cymometer handle slowly from one end of the scale until a readable deflexion appears

on the galvanometer. From this point move the handle in small steps, noting at each step the current J in the cymometer as given by the calibration curve of the galvanometer, and the frequency n_2 as read on the cymometer-scale. Proceed in this way until the maximum deflexion is passed and the current has again fallen to a small value. Plot the results thus obtained as a curve having ordinates proportional to J, and abscissæ proportional to n_2. Take off from this curve the maximum value of J. This will be the resonance current J_r; and the corresponding frequency will be the resonance frequency, that is, the frequency n_1 of the primary circuit. Now repeat the observations inserting the auxiliary resistance into the cymometer circuit in addition to the ammeter, taking care not to alter the relative positions of the circuits in so doing. Plot the results as before and obtain the value of the resonance current J_r'. The resonance frequency should of course remain unaltered. The two curves should now be redrawn, taking the ratio $\frac{J}{J_r}\left(\text{or } \frac{J'}{J_r'}\right)$ as ordinate, and the ratio $\frac{n_2}{n_1}$ as abscissa.

We are now in the position to determine the decrements for the two circuits. Take out from the curve corresponding values of $\frac{J}{J_r}$ and $1-\frac{n_2}{n_1}$, taking the mean for the two sides of the curve and noting that $\frac{n_2}{n_1}$ lies between 0·95 and 1·05, or that $\left(1-\frac{n_2}{n_1}\right)$ is not greater than 0·05.

Then, using Bjerknes' formula

$$\delta_1+\delta_2=\pi x\sqrt{\frac{y}{1-y}},$$

obtain a series of values for $(\delta_1+\delta_2)$.

Let the mean value of $(\delta_1+\delta_2)=\mathrm{X}$.

In like manner, if δ_2' is the logarithmic decrement of the auxiliary resistance, obtain a series of values for $\delta_1+\delta_2+\delta_2'$ by taking out from the second curve corresponding values of $\left(\frac{J'}{J_r'}\right)$ and $\left(1-\frac{n_2'}{n_1}\right)$ and applying Bjerknes' formula.

Let the mean value of $(\delta_1+\delta_2+\delta_2')=\mathrm{X}'$.

On writing out Drude's formula for the two cases we get: when the ammeter only is in circuit

$$J_r^2 = V_1^2 \frac{C_1 C_2}{8} \frac{\pi^4 n_1 k^2}{\delta_1 \delta_2 (\delta_1 + \delta_2)};$$

and when the auxiliary resistance is inserted

$$J_r'^2 = V_1^2 \frac{C_1 C_2}{8} \frac{\pi^4 n_1 k^2}{\delta_1 (\delta_2 + \delta_2')(\delta_1 + \delta_2 + \delta_2')}.$$

Hence we have the relation

$$J_r^2 \delta_2 (\delta_1 + \delta_2) = J_r'^2 (\delta_2 + \delta_2')(\delta_1 + \delta_2 + \delta_2'),$$

or

$$J_r^2 \delta_2 X = J_r'^2 (\delta_2 + \delta_2') X'.$$

Hence

$$\delta_2 = \frac{\delta_2' X'}{\left(\frac{J_r}{J_r'}\right)^2 X - X'}.$$

The value of δ_2' may either be taken as equal to $(X' - X)$ or may be calculated from its resistance and the frequency and inductance of the circuit, for we have

$$\delta_2' = \frac{R \times 10^9}{4 n_1 L_1};$$

where R = resistance of auxiliary wire in ohms,

L_1 = inductance of cymometer in resonance position in cms.

n_1 = resonance frequency.

And as δ_2' is known from either of these equations, δ_2 becomes known, and $\delta_1 = X - \delta_2$;

therefore δ_1 is known.

Hence the decrements of the two circuits are determined.

The resistance of the primary spark can be deduced from the value of δ_1 in the following manner.

Let R' = high frequency resistance of the wire of the primary circuit in ohms.

r = resistance of spark in ohms.

L = inductance of primary circuit in cms.

n_1 = resonance frequency.

Then we know

$$\delta_1 = \frac{(R'+r)10^9}{4n_1 L}.$$

$$\therefore \qquad R'+r = \frac{4n_1 L\delta_1}{10^9}.$$

$$\therefore \qquad r = \frac{4n_1 L\delta_1}{10^9} - R'.$$

The following numerical example of the deduction of the decrements from the resonance curves may be useful as illustrating the method of arranging the work. The primary oscillation circuit consisted of a rectangle of wire (diameter ·162 cm.) inductance =5000 cms., a condenser of capacity= 5560 micro-microfarads, and a 2 mm. spark-gap between 1·25 inch iron balls. The oscillations were excited by a high tension transformer. The cymometer was set up parallel to one side of the rectangular inductance and about 6 inches away from it.

We will suppose that the resonance curves have been drawn as described above, and the result to be as shown in

Fig. 2.

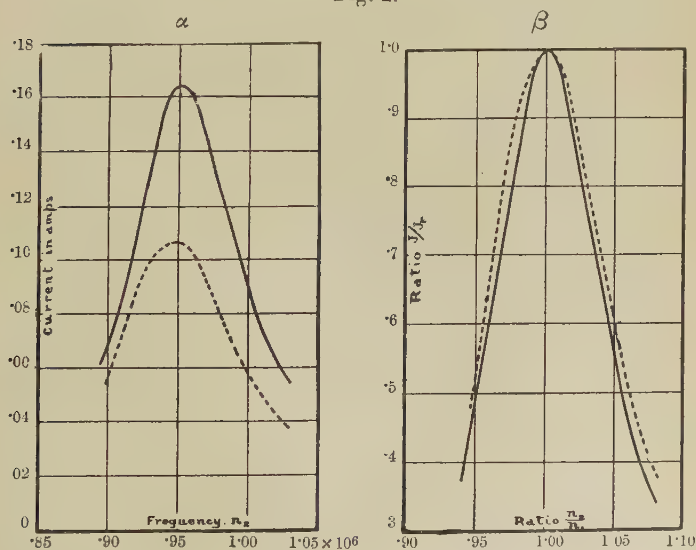


fig. 2 (α and β), where the full-line curve shows the result obtained with the ammeter (resistance 3·5 ϖ) only in circuit,

and the dotted curve the result with the extra resistance ($7{\cdot}2\ \varpi$) also connected.

We will now form a table showing the relative values of $\left(\frac{J}{J_r}\right)$ and $\left(1-\frac{n_2}{n_1}\right)$, and the calculated values of $(\delta_1+\delta_2)$ and $(\delta_1+\delta_2+\delta_2')$.

$\frac{J}{J_r}$.	$1-\frac{n_2}{n_1}$.	$\delta_1+\delta_2$.	$1-\frac{n_2'}{n_1}$	$\delta_1+\delta_2+\delta_2'$.
·95	·0120	·115	·0125	·120
·90	·0165	·112	·0210	·138
·85	·0205	·104	·0255	·130
·80	·0255	·107	·0300	·125
·75	·0293	·105	·0345	·124
·70	·0335	·103	·0385	·119

Taking out the means we find

$$X = {\cdot}108,$$

$$X' = {\cdot}126.$$

$$\therefore \qquad X'-X = {\cdot}018 = \delta_2'.$$

If δ_2' is calculated from the formula

$$\delta_2' = \frac{R' \times 10^9}{4 n_1 L_1},$$

$$R' = 7{\cdot}2\varpi, \quad L_1 = 768000 \text{ cms.}, \quad n_1 = {\cdot}95 \times 10^6,$$

we get $\qquad \delta_2' = {\cdot}025.$

Hence

$$\text{mean } \delta_2' = {\cdot}022.$$

From fig. 2 (α) we see

$$J_r = {\cdot}164 \text{ amps.}$$

$$J_r' = {\cdot}104 \text{ amps.}$$

$$\therefore \qquad \frac{J}{J_r'} = 1{\cdot}53.$$

Calculating δ_2 from the formula

$$\delta_2 = \frac{\delta_2' X'}{\left(\frac{J_r}{J_r'}\right)^2 X - X'}$$

we get $\delta_2 = \cdot 022.$

Hence

$$\text{mean value of } \delta_1 = \tfrac{1}{2}\{(X-\delta_2)+(X'-\delta_2-\delta_2')\}$$
$$= \cdot 077.$$

Now the decrement δ of the ammeter *per se* is given by the formula

$$\delta = \frac{R \times 10^9}{4 n_1 L_1},$$

where $R = 3 \cdot 5 \varpi$, $L_1 = 708000$ cms., $n_1 = \cdot 95 \times 10^6$.

$$\therefore \qquad \delta = \cdot 012.$$

Hence decrement of cymometer *per se*

$$= \delta_2 - \delta$$
$$= \cdot 022 - \cdot 012 = \cdot 010.$$

Passing to the primary circuit, we had the formula

$$r = \frac{4 n_1 L \delta_1}{10^9} - R',$$

for the spark resistance r; where

$$L = 5000 \text{ cms.}, \quad \delta_1 = \cdot 077, \quad n_1 = \cdot 95 \times 10^6.$$

According to Lord Rayleigh's formula

$$R' = R \frac{\pi d}{80} \sqrt{n},$$

where R = low frequency resistance, and d = diameter of wire in cms.

In our case $R = \cdot 0386$ ohms.,

$d = \cdot 162$ cms.,

$n = \cdot 95 \times 10^6.$

$$\therefore \qquad R' = \cdot 23\, \varpi.$$

Putting in these values we get

$$r = 1 \cdot 23\, \varpi.$$

The above example is chosen rather as as example of the ease and speed with which reasonably good results can be obtained, than as a criterion of the accuracy of the method, as all the observations necessary for drawing the resonance curves were taken in less than half an hour. If more time is taken over the observations much closer agreement between the calculated and observed values of the decrements can be obtained.

The example first worked out is a case of two rather loosely coupled oscillation circuits, and in this case we have seen that the resonance curve has a single peak. If, however, the coupling is at all tight, the resonance curve develops a double hump, the maxima becoming more and more separated as the coupling becomes tighter and tighter, until when the coupling is perfect (*i. e.* when the mutual inductance is the geometric mean of the two self-inductances), one of the maxima has gone off to infinity, and we are again left with a single-hump resonance curve.

Oberbeck has developed the theory of two coupled oscillation circuits, and has given formulæ by means of which the two resonance frequencies may be predetermined. For the general solution we must refer to the original paper *, but in one particular case the result deserves special mention on account of its importance in wireless telegraphy. If the primary and secondary circuits are tuned, that is, are so adjusted that when far apart they have the same oscillation constant and the same frequency n_0, then, when put near together so that the coupling coefficient has a value k, the following very simple relations hold between the two frequencies n_1 and n_2 induced in the secondary circuit and the natural frequency n_0:—

$$n_1 = \frac{n_0}{\sqrt{1+k}},$$

$$n_2 = \frac{n_0}{\sqrt{1-k}},$$

* A. Oberbeck, Wied. *Ann. der Physik*, 1895, vol. lv. p. 623. See also Dr. J. A. Fleming, "Principles of Electric Wave Telegraphy," chap. iii.

or $$\frac{n_1}{n_2}=\frac{\sqrt{1-k}}{\sqrt{1+k}}.$$

Now in the case of some oscillation transformers used in wireless telegraphy the coupling may be about 0·5.

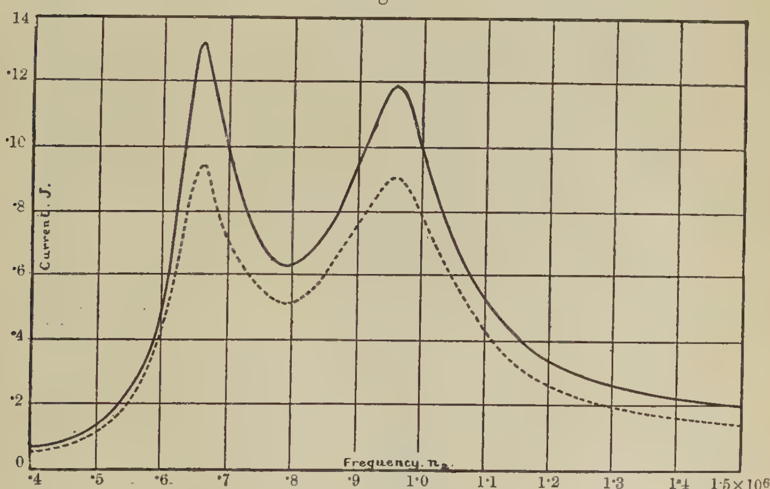
Hence we should then have

$$\frac{n_1}{n_2}=\sqrt{\frac{\cdot 5}{1\cdot 5}}=\frac{1}{1\cdot 732},$$

or, The frequency of one wave is about $1\frac{3}{4}$ times that of the other.

For a coupling of the order of 0·5 the method above described may be applied to each hump of the double-humped resonance-curve, and will enable a fairly accurate determination of the decrements of the two oscillations to be made; but, as shown by C. Fisher (see *Annalen der Physik*, vol. xix. p. 182, 1906), when the coupling is very loose we cannot consider the resultant resonance-curve to be identical with the sum of the curves due to each oscillation separately.

Fig. 3.



The curve shown in fig. 3 was taken from a closely coupled circuit of this type, and may be taken as indicative of the general type of result to be expected.

The accurate determination of the logarithmic decrement in

open oscillation circuits such as these is a matter of considerable practical importance, as most of the damping is due to the radiation of energy from the wire, and not to the resistance as is the case with closed or non-radiative circuits; and as the amount of this radiant energy is not easily estimated mathematically, the value of this important quantity must rest on experimental evidence alone.

The objection may be raised that the type of ammeter described greatly increases the decrement of the cymometer, as its decrement is almost equal to that of the cymometer *per se*. It must be remembered, however, that the part of the instrument in the cymometer circuit is a short piece of very fine wire whose decrement can be easily and accurately calculated, and secondly that the decrement of the cymometer itself does not matter, the primary circuit being generally the one whose decrement is required.

At first sight it would seem that a very obvious variation of the method could be employed which would not suffer from this defect.

Suppose that, instead of cutting the cymometer-bar, a few turns of well-insulated wire are wound round a section of the inductance-coil of the cymometer, and the ends of the coil connected to the fine wire of the ammeter described above, then currents will be induced in this circuit proportional to the currents in the cymometer. This method, however, has more serious drawbacks than the other, as:—

(1) The damping of the cymometer will be just as great if this method is employed, as the same amount of energy has to be supplied to heat the ammeter wire to a definite temperature.

(2) The actual current in the cymometer is not measured but only a current bearing some unknown ratio to it.

(3) The scale of the cymometer is slightly altered when the ammeter is in place, as the arrangement is of the nature of a transformer with a closed secondary circuit, part of the cymometer inductance spiral forming the primary; this will of course annul a portion of the inductance and so alter the scale.

The extent of this alteration of scale may be determined by measurements made at the point of resonance with the Neon tube, first with the secondary open, and second with the secondary closed through the ammeter wire.

Several curves have been obtained using the ammeter in this manner, but as the method shows no points of advantage over the one first described and has several inherent disadvantages, it was discarded in favour of what might be called the series arrangement.

In conclusion, the author wishes to thank Dr. J. A. Fleming, F.R.S., for his kindness in permitting him to publish the results of these experiments, which were carried out under his direction at the Pender Laboratory, University College, London.

Discussion.

Mr. A. Campbell asked the Author if he could give an idea of the sensitivity of the hot-wire ammeter. He pointed out that if the instrument was used in a vacuum, the sensitivity was much increased.

Mr. W. Duddell said he had tried thermojunctions in a vacuum and found their sensitivity increased five or ten times. He suggested that the Author should replace the bismuth in the junction by constantan.

Mr. Dyke, in reply to Mr. Campbell, said that a current with a maximum value of ·17 amps. gave a deflexion of 80 divisions.

XII. *Secondary Röntgen Radiation.* By CHARLES G. BARKLA, *D.Sc.* (*Liverpool*), *M.Sc.* (*Victoria*), *B.A.* (*King's College, Cambridge*); *Lecturer in Advanced Electricity, University of Liverpool* *.

Introduction.

SINCE the discovery by Röntgen of secondary radiation from substances exposed to X-rays, the subject has been investigated by a number of experimenters, including Perrin, Townsend, Dorn, Curie, Sagnac, Langevin, and Bumstead †.

The results of these investigations have shown that a substance upon which a beam of Röntgen radiation is incident emits two kinds of radiation: an easily absorbed radiation consisting of negatively charged corpuscles or electrons, and a heterogeneous beam of X-radiation differing from the primary in penetrating power. In almost all cases the secondary X-radiation has been found to be more easily absorbed than the primary producing it; and never has it been found to possess greater penetrating power. The absorbability of the radiation from a given substance varies with that of the primary; and probably as a consequence, the results obtained by different experimenters for the relative absorptions of the radiations from different substances do not agree. Recently Bumstead has made experiments from which he concludes that the absorption of a given X-ray beam and the secondary beam arising from it, results in the generation of about twice as much heat when the absorbing and radiating substance is lead than when it is zinc.

The writer ‡ investigated the radiation proceeding from gases and light substances when subject to X-rays, and found

* Read February 23, 1906.

† Perrin, *Annales de Chimie et de Physique*, [7] xi. p. 496 (1897). Townsend, Proc. Camb. Phil. Soc. x. p. 217 (1899). Dorn, *Abhand. d. naturf. Ges. zu Halle*, xxii. p. 39 (1900). Sagnac, *Annales de Chimie et de Physique*, [7] xxii. p. 493 (1901). Curie & Sagnac, *Journal de Physique*, [4] i. p. 13 (1902). Langevin, *Recherches sur les gaz ionisés.* Bumstead, Phil. Mag. xi. pp. 292–317, Feb. 1906.

‡ Barkla, Phil. Mag. v. pp. 685–698, June 1903; vii. pp. 543–560, May 1904.

that the radiation not absorbed by a few centimetres of air under ordinary atmospheric conditions differed exceedingly little from the primary. From certain gases and light solids the radiation was found by direct experiment not to differ appreciably in absorbability from the primary; but there was indirect evidence of greater absorbability in air. From these substances the intensity of radiation was found to be proportional to the quantity of matter passed through by the primary radiation of given intensity.

These laws were accounted for by the theory that the corpuscles or electrons constituting the atoms scattered the primary radiation. The intensity of radiation was experimentally investigated, and a close agreement found between that and the result of a calculation by Prof. J. J. Thomson * when applied to the electrons. The theory was further verified by the discovery of partial polarization in a primary beam proceeding from an X-ray tube, by a study of the intensity of secondary radiation in different directions, and later by the demonstration of much more complete polarization of secondary radiation proceeding from one of the substances (carbon) to which the theory was supposed applicable †.

These results furnished data and suggested methods for investigating the complex secondary radiations proceeding from metals subject to X-rays more completely than had previously been attempted.

The experiments described below were made on the more penetrating radiations, that is the radiations which had passed through several centimetres of air under ordinary atmospheric conditions and very thin aluminium leaf.

Density, Molecular Weight, Atomic Weight of Radiator.

Preliminary experiments on the absorbability of the secondary radiation proceeding from different substances when subject to X-rays were made in the manner described in previous papers. It was seen that in general light substances emitted radiations differing very little in this

* J. J. Thomson, 'Conduction of Electricity through Gases,' p. 270.

† Barkla, Phil. Trans. A, vol. 204 (1905) pp. 467–479; Proc. Roy. Soc. A. vol. lxxvii. pp. 247–255 (1906).

respect from the primary producing them, while from heavy substances the radiation was more easily absorbed. Whether the character depended on density, molecular weight, or atomic weight had to be investigated.

The gases hydrogen, air, sulphuretted hydrogen, carbon dioxide, and sulphur dioxide had all been found to emit radiations closely resembling the primary. Carbon, paper, wood, and even aluminium and sulphur, were found to emit radiations differing comparatively little from the radiation producing them; while calcium, iron, copper, zinc, tin, platinum, and lead emitted radiations of considerably less penetrating power than the primary producing them.

It should be observed that this is only a general way of classifying the radiations emitted by different substances; for upon close examination it was concluded that in all cases there was a difference between the primary and secondary rays. The difference was, however, so small in many cases—including those in which aluminium and sulphur were the radiators—in comparison with that when one of the second class of substances was experimented upon, that these substances may be conveniently spoken of as scattering and transforming the radiations *.

The compounds ammonium carbonate, lime, calcium carbonate, and copper sulphate were afterwards tested. Ammonium carbonate was found to emit a radiation closely resembling the primary, and the others radiations much more absorbable than the primary. Thus ammonium carbonate, a substance of greater molecular weight than calcium or lime, belonged to what may be called the scattering class, while calcium and lime were among the transforming substances. Aluminium also belonged to the former class, while calcium, a substance of less density, belonged to the latter.

On the other hand, all the elements in the former class had atomic weights lower than any in the latter class, and the radiation proceeding from a compound was such as would be obtained by a mixture of the radiations proceeding from the constituent elements.

These experiments indicated that the character of secondary

* Further reason for this classification is given later.

radiation set up by a given primary depends upon the atoms subject to that radiation, and not to any great extent, if at all, on their distances apart or on their combination with atoms of other substances.

Absence of Purely Scattered Radiation.

In further investigating the secondary rays from substances of higher atomic weight, experiments were made to ascertain if the radiation consisted of a radiation such as was found to proceed from substances of lower atomic weight superposed on a more easily absorbed radiation.

The radiation from tin when placed in the primary beam was studied by the method described in a previous paper. The absorbability of the radiation was measured by placing successive layers of thin aluminium in front of the electroscope through which the secondary beam passed. The ionization was initially large; but as sheet after sheet of aluminium was placed in the path of the secondary radiation, the reduction was so great that the resultant ionization produced in the secondary electroscope was found to be much less than what would have resulted if simple scattering had occurred in the tin such as was found in substances of low atomic weight. That is, taking account of the absorption of the primary beam in the plate of tin and of the secondary in the same plate and in the aluminium absorbing plates, if the same percentage had been scattered as was found from light atoms, a much greater ionization would have been produced than was actually measured. The absorptions of primary and secondary rays were determined by separate experiment. Thus the secondary radiation was found to consist almost entirely, if not entirely, of a completely transformed radiation.

Temperature, Electric Conductivity, Magnetic Permeability.

To determine if the character of secondary radiation was in any way connected with the temperature, electric conductivity, or magnetic property of the radiator, a grating was made of iron wire in the form of a bolometer-grating, and was exposed to primary X-rays just as sheets of metal had

been. The ionization produced in the electroscope was considerable on account of the great absorbability of the secondary rays. A current was then passed through the wire grating till it became almost "white hot." It was seen that though very flexible, it was no longer deflected under the action of a strong magnet placed near. With this great rise of temperature, consequent increase of resistance and disappearance of magnetic property, no change was observed in the ionization produced by the secondary beam, though a change in intensity or absorbability of the radiation by 2 or 3 per cent. would have been detected.

Thus there was no appreciable direct connexion between the character of the radiation and the temperature, conductivity, or magnetic permeability of the radiating substance.

The observations were taken in the order shown below, the experiments being made with the wire alternately hot and cold.

State of Radiator.	Readings and Deflexion of Electroscope per minute.	
Hot	40·4 66·5	26·1
Cold	41·7 68·7	27
Hot	41·5 69·5	28
Cold	40·2 70·1	29·9

Selective Absorption.

It has been stated by experimenters that the radiation proceeding from a heavy metal is more easily absorbed by that metal than would be expected by comparing the absorbabilities of other rays in different substances; that is, that the rays are specially absorbed by the metal from which they are emitted. If this were so, it would indicate a more or less definite period of vibration of corpuscles or electrons in a given substance producing a radiation having some of the properties of Röntgen radiation; also that this vibration is set up by the passage of X-rays through the substance.

Experiments were made to verify this important conclusion if possible, and to learn the extent of this special absorption. A comparison of the absorbability of the secondary rays from lead in aluminium and in tin was made. (The secondary rays studied passed through about 11 cm. of air at atmospheric pressure before passing through the absorbing plates into the electroscope. This distance was chosen because the rays previously experimented upon had travelled an equal distance from the radiator.) Then the absorptions of the secondary rays from tin in aluminium and tin were determined. Absorbing plates of thickness to diminish the ionization by approximately the same amount were used. It was found that the ratio of the two absorptions by aluminium and tin for the radiation from lead was, within a small possible error, the same as that ratio when the radiation from tin was absorbed.

No evidence then of special absorption for the radiation from tin by tin was obtained.

Variation in Intensity of Primary Radiation.

Experiments were made to determine the effect on the secondary radiation of variation in intensity of the primary. Calcium was chosen as the radiating substance, because it was the substance of lowest atomic weight experimented upon which emitted a radiation differing considerably from the primary, and consequently might be expected to be specially sensitive to such variations. The apparatus used in these experiments was that described in previous papers on polarized Röntgen radiation. The electroscope receiving the vertical secondary beam was used to standardize the intensity of the horizontal secondary beam, while plates of aluminium were placed in the path of the latter to the electroscope.

The ratio of the rates of deflexion of the two electroscopes appeared constant under definite controllable conditions, so no particular period of discharge was taken. Variation in intensity of the primary was obtained by varying the distance of the X-ray tube from the calcium radiator.

The results are given below :—

Conditions.	Distance of Anticathode of X-ray tube from Radiator.	Deflexion of Upper Electroscope (standard).	Deflexion of Lower Electroscope.	Ratio of Deflexions.
No absorbing plate	28 centims.	44·8 / 54·3 } 9·5	16·8 / 32·4 } 15·6	10 to 16·4
Aluminium plate (·01 cm. thick) before lower electroscope	28 "	55·4 / 70·0 } 14·6	34·8 / 42·4 } 7·6	10 to 5·2
" "	78 "	49·2 / 66·2 } 17	18·8 / 27·3 } 8·5	10 to 5
No absorbing plate	78 "	65·3 / 77·4 } 12·1	25·8 / 46·0 } 20·2	10 to 16·7

From these we see that the ionization produced by the secondary radiation from the intense primary beam was reduced by 68·3 per cent. of its original value, while that produced by the secondary radiation set up by the weak primary beam was diminished by 70 per cent.

The difference between these was within the limits of possible error of experiment. We thus see that the character of the secondary radiation from calcium did not depend to any appreciable extent on the intensity of the primary.

Similar experiments were made when iron was used as the radiating substance. The sheet of aluminium when placed in the path of the secondary beam diminished its ionizing effect in successive experiments by 90·6 per cent., 90·9 per cent., 90·9 per cent., and 90·5 per cent. In the first experiment the radiation emitted by the tube was very weak. Before the second the tube was heated, consequently it worked much more easily and produced a very much greater ionization. For the third it was removed to something like double the distance from the iron radiator, and for the fourth it was brought back. The variation in absorption as shown by the above numbers must have been exceedingly small, if it existed at all.

Experiments have been made at different times to determine if the intensity of secondary radiation was proportional to that of the primary. In all cases so far investigated, the proportionality has been demonstrated within two or three per cent.

Variation in Penetrating Power of the Primary Radiation.

It was of interest to determine in what way and to what extent the secondary radiation from a particular substance depended on the character of the primary radiation. From substances of low atomic weight it had been seen that the character was dependent solely, or almost solely, on that of the primary and to an inappreciable extent on the nature of the radiator.

It was found that as the difference in character between the secondary and primary rays became more marked by increasing the atomic weight of the radiator, the effect on the secondary of a change in character of the primary diminished.

From those substances which emitted a radiation varying in intensity in different directions when the primary beam was polarized, there were considerable variations in absorbability, the secondary becoming more penetrating as the primary became more penetrating. The radiation from those substances, however, which produced considerable transformation and which gave no evidence of polarization of the primary beam, was extremely little affected by considerable changes in the character of the primary, though in all cases it appeared slightly more penetrating when a penetrating primary beam was used. The change was remarkably small in some cases.

The slight change in the character of the secondary radiation from copper when the primary was changed is shown by the following results.

The absorption of the secondary beam from copper by a plate of aluminium ·01 cm. in thickness, when the primary beam came direct from an X-ray tube, was found to be 72·75 per cent. When only the penetrating portion of the primary which had got through an aluminium plate ·079 cm. thick was used as the primary beam, the absorption of the secondary was found to be 71·8 per cent. Now the former primary beam used was found to be absorbed to the extent of about 35 per cent. by the same plate of aluminium, whereas the second primary beam was absorbed by only 10 or 12 per cent.

But by placing the aluminium plate ·079 cm. thick in the

primary beam before it fell on the radiator of copper, the ionization produced by the primary was reduced to 14 per cent., and that by the secondary to 19 per cent. of the original ionizations.

We thus see that though 81 per cent. of the secondary ionization was produced by the secondary rays set up by the absorbable portion of the primary beam, yet when this was cut off the enormously more penetrating portion of the primary set up secondary rays differing in absorbability by something of the order of 1 per cent.

When iron was used as the radiator, the change in character was still less, the absorption of the secondary set up by the primary beam direct from the bulb being 90·6 per cent., 90·9 per cent., 90·9 per cent., and 90·5 per cent. in successive experiments; while it was 90·2 per cent. and 90 per cent. when the thick aluminium plate was placed in the primary beam.

The absorption of the radiation from lead appeared much more variable. From several experiments the absorption of the secondary beams set up in the same way in lead as in the two experiments on copper, were found to be about 39 per cent. and 35 per cent., indicating a greater dependence on the character of the primary.

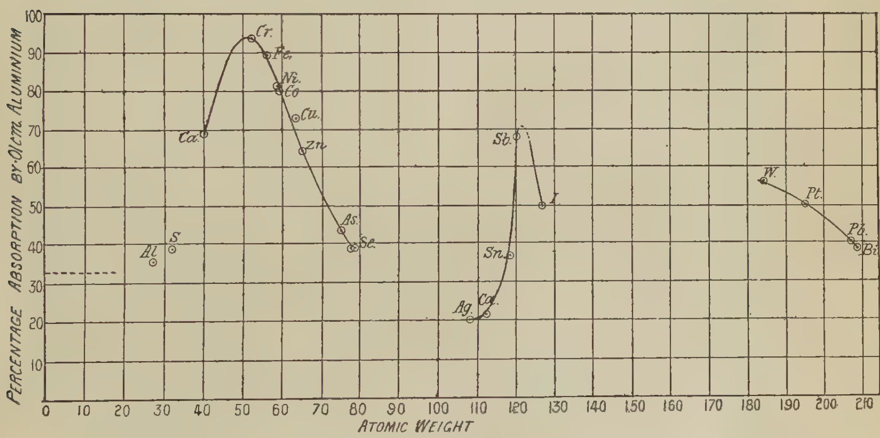
As a general rule, it was also found that experiments on those substances which emitted a very easily absorbed radiation gave much more constant results than those on substances emitting a comparatively penetrating radiation.

It is, however, important to notice the effect of heterogeneity in both primary and secondary beams, assuming the character of the secondary radiation to be independent of that of the primary. First dealing with those substances which emit a radiation of fair penetrating power, as silver, cadmium, and tin, the layer of metal from which the secondary rays proceed has a thickness comparable to that penetrated by the primary rays. Consequently, when a more penetrating primary beam is used, a greater proportion of the secondary radiation proceeds from the deeper layers, and in its passage to the surface of the metal is robbed of its more absorbable constituents. On the whole, then, the emergent secondary beam consists of a larger proportion of penetrating rays than

when set up by a more easily absorbed radiation. On the other hand, when the secondary radiation is very easily absorbed, as in the case of the radiation from iron, the layer from which secondary rays emerge is very thin, and variation in the penetrating power of the primary does not appreciably change the thickness, all the primary rays getting through with little diminution in intensity. Hence, whether penetrating or comparatively absorbable rays form the primary, the emergent secondary is from the same thickness of metal, and is therefore equally penetrating.

It appears possible then that this alone would account for the differences observed and given in detail above. But there

Fig. 1.



were found to be enormous changes in the absorbability of the secondary rays from antimony and iodine, for instance, due to changes in the primary radiation, though the absorption of some of these beams was greater than that of some secondary radiations whose character appeared very constant when that of the primary was varied.

In general the substances from silver to iodine (see fig. 1) emitted secondary rays showing remarkable variations in absorbability in different experiments. The behaviour of these substances and of those in the middle of the first long chemical period form a striking contrast.

This points to the conclusion, though the evidence is

not decisive, that at least some substances emit a radiation differing considerably in absorbability from the primary, and exhibiting considerable variation in character as the primary is varied.

[*Note*, May 14th, 1906.—More recent experiments with thin metal leaves have shown this conclusively.]

Polarization.

Further information regarding the nature and cause of the radiation was obtained by using a partially polarized beam of X-radiation for the primary, and measuring the intensity of radiation proceeding in different directions from the substances investigated. It was shown in the paper on "Polarized Röntgen Radiation," that from those substances, in which, during the passage of a Röntgen pulse, the corpuscles or electrons are accelerated in the direction of electric displacement, the secondary radiation differs in intensity in different directions, giving evidence of the polarization in the primary beam. These substances are also the origin of polarized secondary radiation. When the acceleration of electrons ceases to be in the same direction as the electric displacement in the primary pulse, evidence of polarization of the primary disappears, and the secondary radiation ceases to be polarized. Conversely, the disappearance of evidence of polarization, *i. e.* of the variation of intensity of secondary radiation in different directions at right angles to the primary, when polarization has been demonstrated, shows that the secondary radiation ceases to be polarized and that the electrons cannot now be accelerated in the direction of electric displacement during the passage of Röntgen pulses.

To test the nature of the secondary pulses from this point of view, experiments were made using different substances as radiators, and it was found that approximately equal evidence of polarization was given by all those substances of low atomic weight which emitted a radiation whose character differed little from the primary; that with those substances of lowest atomic weight which were the source of a secondary radiation differing to a greater extent from the primary, polarization was shown to a smaller extent; and all substances

experimented upon with atomic weights beyond a certain value gave no evidence of polarization at all. Thus carbon, air, cardboard, aluminium, and sulphur emitted a secondary radiation, varying in intensity in the principal directions by about the same amount. Calcium emitted radiation exhibiting about half this variation, while the radiations from iron, copper, zinc, tin, and lead exhibited no variation at all. Thus the character of the pulses changed not abruptly with an increase in atomic weight, but very rapidly between certain atomic weights.

It is significant that the polarization effect disappears with the similarity between the secondary and primary radiations. We thus see that the change in absorbability of the secondary radiation is accompanied by an acceleration of electrons in directions not those of electric displacement in the pulses producing the radiation. This is important, because the difference in character between the primary and secondary radiations might possibly be accounted for by a change in the average distance apart of the separate pulses. Here, however, we see that the change in character is accompanied by a change in what might be called the pulse structure. The rather striking independence of the absorbability and the distances of the pulses apart is shown in the case of secondary radiation from a mass of carbon for instance; for on the theory of the production of secondary radiation, each electron is the source of a secondary pulse, yet in spite of this enormous multiplication of pulses, the absorption of the radiation differs very little from that of the primary.

The compounds ammonium carbonate, lime, calcium carbonate, and copper sulphate were tested in the same manner. Ammonium carbonate emitted a radiation varying in intensity in the principal directions by about the same amount as that shown by the elements of low atomic weight. Lime and calcium carbonate emitted radiations showing a smaller variation, while from copper sulphate evidence of polarization of the primary could not be detected.

These results again are what would be given by mixtures of the radiations proceeding from the different constituents. All the elements in ammonium carbonate belong to the scattering class, and hence the full variation is produced. In

the radiations from calcium oxide and calcium cabonate, the rays from calcium produce most of the ionization on account of its greater absorbability, and hence little more than half the full variation was found, the rays from calcium itself showing only about half the variation shown by those from lighter elements.

In the radiation from copper sulphate, the rays from copper were so easily absorbed that the effect of the more penetrating rays from sulphur and oxygen was swamped, and no evidence of variation in intensity in different directions was detected.

Experiments were also made to ascertain if the more penetrating portion of the secondary radiation from some of the heavier substances gave evidence of polarization in the primary ; for if the character of the radiation depended on the relation between the pulse-thickness and the distances separating the electrons, we might expect that the effect of thin pulses passing through a heavy atom would be similar to thicker pulses passing through an atom in which the electrons are not so closely packed.

No such evidence was obtained (see paper on "Polarized Röntgen Radiation").

[These experiments on polarization were made before the absorbability of the rays from many substances had been determined. It will be interesting to learn if evidence of polarization reappears when such a substance as silver is used as the radiator (see fig. 1). The radiation from silver, however, is much more absorbable than the primary producing it, though from a *thick* plate its ionizing effect is diminished by a smaller fraction than that of the primary by passage through a thin plate of aluminium. Enormous differences are observed in the character of the radiations proceeding from thin and thick plates of the same substance.]

Connexion between Atomic Weight of Radiating Substance and Character of Secondary Radiation.

Though it is impossible to determine the true character of the radiation as it proceeds from an atom of the radiating substance except by using very thin plates, it was thought

that by using plates of sufficient thickness to absorb nearly all the incident primary radiation, or rather that portion of it which produced an appreciable effect in the electroscope, some law might be found to exist where from a few isolated results there appeared to be great irregularity.

A large number of elements were therefore in turn exposed to the primary radiation, and the penetrating power of the rays from each was observed.

The experiment was simply the following:—

The rates of leak of an electroscope receiving a narrow pencil of primary radiation and of one receiving a beam of secondary radiation from the substance experimented upon, were observed when no absorbing plates intercepted the radiations. A plate of aluminium ·01 cm. in thickness was placed in front of the electroscope receiving the secondary beam, and the two rates of deflexion of the gold-leaves were again observed. The percentage reduction of the ionization in the secondary electroscope was found, using the electroscope receiving the pencil of primary radiation to standardize the intensity of the primary. (In some cases an electroscope receiving a secondary beam was used to standardize the intensity.)

In these experiments, no particular care was taken to keep the character of the primary radiation constant; but in one or two cases several experiments were made on a pair of metals taken alternately. Bismuth and lead, silver and cadmium were treated in this way. Consequently the results can only be regarded as approximately true, and little value is attached to the absolute absorptions obtained, for under certain conditions some of the elements were found later to give results differing considerably from these. The most variable were those from silver to iodine (fig. 1). There was, however, no indication of much variation in the majority of cases. The elements were studied irregularly, and the discharge in the X-ray tube was not kept more constant than could be done by ordinary observation. The results, though incomplete, and possibly containing one or two errors in detail, are, I think, of sufficient interest to justify their publication at this stage of the investigation.

They also suggest a method of determining atomic weights

by interpolation, for a small variation in atomic weight is usually accompanied by a very considerable change in absorbability of the secondary radiation. It appears probable that a variation of atomic weight by one fourth in certain regions would be detected, but the possible error varies considerably from one region to another.

Substance.	Atomic Weight.	Percentage 'Absorption'* by ·01 cm. Aluminium.
Aluminium	27·1	35·5
Sulphur	32·06	39
Calcium	40·1	69
Chromium	52·1	93·5
Iron	55·9	89
Nickel	58·7	81·5
Cobalt	59·0	80
Copper	63·6	73
Zinc	65·4	64·5
Arsenic	75·0	43·5
Selenium	79·1	39
Silver	107·9	20
Cadmium	112·4	21·5
Tin	118·5	36·8
Antimony	120·0	68
Iodine	127·0	50
Tungsten	184·0	56
Platinum	194·8	50
Lead	206·9	40
Bismuth	208·5	38·5

Percentage 'absorption' of the primary was about 33.

* As measured by diminution of ionization in an electroscope.

It would be useless to attempt to calculate from these results absorption coefficients for the different radiations, for the primary and secondary beams consist of mixtures of rays differing enormously in penetrating power, and these constituents produce effects in the electroscope which cannot even be regarded as approximately proportional to their energies. Also the constituent of the primary beam that sets up in one metal the most effective constituent in the secondary beam, is not that which gives rise to the most effective constituent in another metal; so that the ionizations produced in the electroscope receiving the secondary beam are not strictly due to the same primary beam.

The absorption of a given radiation, however, is not a

periodic function of the atomic weight *, so that the general features of the curve (see fig. 1, p. 209) showing the relation between percentage diminution of the ionization produced by the secondary beam when a plate of aluminium ·01 cm. thick was placed in its path to the electroscope, and the atomic weight of the substance emitting that radiation, are no less significant. The diminution of the ionization by this plate has been spoken of as the "absorption," but this must not be taken as signifying the percentage diminution of energy in the beam traversing the plate.

It will be seen that as far as these experiments have gone, curves showing a rise and fall in the absorption of the secondary radiation connect the absorption and atomic weights of elements in the first and part of the second long chemical periods, and that the latter part of such a curve has been obtained with the latter part of the third long period. These are the periods shown by McClelland †, by experiments on the secondary rays from substances subject to β and γ rays from radium. His curves also show the second short period. The results obtained in these experiments from substances of atomic weights less than 32 have not been plotted, because they were made under different conditions, the radiating layer of gas absorbing only a very small fraction of the incident primary radiation. These substances were found to emit rays differing little in character from the primary.

It will be noticed that some substances emitted a radiation whose ionizing effects were diminished more by an absorbing plate than were those of the direct primary beam. The rays, however, were produced by the penetrating portion of the primary beam, so the transformation was one to greater absorbability. This was seen by placing aluminium screens in the primary beam before it fell on the radiating substance,

* What has been proved for primary radiation from an X-ray tube is here assumed for secondary radiations, viz., that there is not selective absorption, the absorption by a given mass being a steadily increasing function of the atomic weight of the absorbing substance. This will be thoroughly tested by obtaining curves similar to that shown in fig. 1, when the aluminium absorbing plate is replaced by other metals.

† Transactions of Royal Dublin Society, May 17, 1905.

for the diminution of the secondary ionization was considerably less than when the same screen was placed in the path of the secondary beam. The reason for this is obvious when we consider the different penetrating powers of the constituents of the primary beam.

[Later experiments have shown that variations in the primary beam have such an enormous effect on the character of the radiations from the substances referred to as being inconstant, that the relative positions of antimony and iodine may have to be reversed. The values first obtained with an approximately constant primary beam are, however, unaltered in this paper, as the true positions are not known with certainty.]

Theory.

The theory which has been shown to account for the phenomena of secondary radiation from certain gases and light solids may be extended to explain the results of experiments on metals.

In light atoms the corpuscles or electrons have sufficient freedom to move almost entirely independently of each other under the influence of the primary pulses, consequently to emit a radiation whose penetrating power is the same as that of the primary, and whose intensity depends on the direction of propagation with regard to the plane of polarization in the primary beam.

In heavier atoms each electron is more intimately connected with the electrons in its immediate neighbourhood, and is therefore subject to considerable disturbing forces due to the displacement of these. The period for which it is subject to considerable forces is much greater than that of passage of the primary pulse over it, hence the secondary pulse emitted is thicker and more complex in character. The greater thickness of the secondary pulse results in greater absorbability; and the interference with the simple direct acceleration due to the primary pulse prevents pure scattering, and accounts for the disappearance of polarization in the secondary beam and of evidence of polarization in the primary. An increase in thickness of the primary pulse produces an increase in thickness of the secondary pulses;

consequently an increase in absorbability of the primary results in an increase in the absorbability of the secondary.

On this hypothesis, the penetrating power of the secondary radiation is a measure of the independence of motion of corpuscles or electrons within the atom; and the relation between absorption and atomic weight exhibiting a periodicity which is obviously connected with the periodicity in chemical properties, is evidence of a connexion between chemical properties and distribution of corpuscles in the atom such as Prof. J. J. Thomson suggests.

It would be premature at this stage of the investigation to attempt a more detailed explanation of the results, as the experiments are still very far from completion.

The theory, however, appears sufficient to explain the results of experiments made up to the time of writing, without assuming appreciable disintegration of the atom to occur. Radiation due to disintegration may or may not form a portion of the secondary radiation emitted by metals and detected by means of an electroscope, but it appears probable that the radiations studied have been at the expense of the energy of primary radiation.

It should perhaps be recalled that strong evidence of the similarity in nature of the secondary radiation from copper (a substance emitting a radiation differing considerably in absorbability from the primary producing it) was given in a previous paper *.

The energy of secondary radiation from light atoms was shown to be accounted for by scattering of the primary radiation by the constituent electrons.

From calcium a radiation differing considerably from the primary was produced exhibiting, to a certain extent, the polarization in the primary beam. From tin the purely scattered radiation was entirely or almost entirely absent.

Now it cannot be supposed that more than an exceedingly small fraction of the corpuscles were displaced sufficiently to disturb the stability of the atom; hence the disappearance of the scattered radiation was not due to any instability during the passage of the primary pulses.

The disappearance of scattering and the appearance of an

* Phil. Mag. vii. pp. 543-560, May 1904.

easily absorbed radiation together point to the same cause, and as all the observed effects may be accounted for by this, it is improbable that an appreciable disintegration sets in at the same atomic weight.

Also if the energy of secondary radiation depended on there being sufficient electric intensity in the primary pulses to produce disintegration of the atom, we should have to conclude, as we find intensity of secondary radiation proportional to the ionization produced by the primary in the primary electroscope (the character remaining constant), that the disintegration produced in a given metal by a primary beam is proportional to the ionzation produced in a given gas by the same beam.

If we apply the disintegration theory to calcium, we must conclude that the scattered radiation and the radiation due to disintegration are in a constant proportion whatever be the intensity of the primary, for the absorbability of the mixture is unchanged. This appears very improbable.

Again, the change in character of the secondary with that of the primary indicates that a considerable portion at least of the secondary radiation is not due to disintegration, for the character of this we should expect to be independent of the exciting cause and to be characteristic merely of the atom.

It is significant also that the secondary rays have been invariably found to be less penetrating (yet not of an entirely different order of magnitude) than the primary producing them; a result necessary to the theory here given, and not at all likely on the disintegration theory.

If Prof. Bumstead's conclusions on the point are correct, it appears probable that investigations on the easily absorbed radiation would throw further light on the subject. I hope to make such investigations shortly.

In conclusion I wish to express my indebtedness to Mr. C. A. Sadler, B.Sc., and Mr. A. L. Hughes, for their valuable assistance in conducting some of these experiments.

George Holt Physics Laboratory, Liverpool.

XIII. *The Velocities of the Ions of Alkali Salt Vapours at High Temperatures.* By HAROLD A. WILSON, *M.A., D.Sc., M.Sc., F.R.S., Fellow of Trinity College, Cambridge, and Professor of Physics, King's College, London* *.

IN a paper on "The Electrical Conductivity of Flames containing Salt Vapours," Phil. Trans. A. vol. 192, 1899, the writer gave an account of a series of experiments which included measurements of the velocities of the ions of salt vapours in flames and in hot air. The method employed was to find the electric intensity necessary to make the ions move against a stream of gas moving with a known velocity. The following table gives the results obtained:—

(1) Positive ions of various salts of Cæsium, Rubidium, Potassium, Sodium, and Lithium in a Bunsen flame. Velocity 62 cms. per sec. for one volt per cm.

(2) Negative ions of same salts in a Bunsen flame. Velocity 1000 cms. per sec.

(3) Positive ions of salts of Cæsium, Rubidium, Potassium, Sodium, and Lithium in air at 1000° C. 7·2 cms. per sec.

(4) Positive ions of salts of Barium, Strontium, and Calcium in air at 1000° C. 3·8 cms. per sec.

(5) Negative ions of salts of Barium, Strontium, Calcium, Cæsium, Rubidium, Potassium, Sodium, and Lithium in air at 1000° C. 26 cms. per sec.

In 1900 Dr. E. Marx published an account ("Ueber den Potentialfall und die Dissociation in Flammengasen," *Annalen der Physik*, 1900, no. 8) of a valuable series of experiments on the conductivity of flames, including determinations of the ionic velocities for different salt vapours in the flame. His method depended on observations of the potential gradient in the flame, and was quite different from that employed by the writer. He obtained for the velocity of the negative ions of any alkali salt in the flame 1000 cms. per sec. For the positive ions of any alkali salt in the flame he found the velocity to be about 200 cms. per sec.

The experiments of Dr. Marx and the writer thus agree in showing that all alkali salt vapours in flames give ions having under the same conditions nearly equal velocities, and that

* Read March 9, 1906.

the velocity of the negative ions is much greater than that of the positive ions.

In 1901 the writer published a paper on the "Electrical Conductivity of Air and of Salt Vapours" (Phil. Trans. A. vol. 197, 1901), in which it was shown that, above 1350° C., and with an E.M.F. of over 1000 volts, all alkali salt vapours conduct equally well if the amount of each one taken is proportional to its molecular weight. The temperature above which this result holds good was nearly the same for every salt. Above this temperature the current was independent of the temperature and of the P.D. used, provided the latter was sufficiently great. Since the lowest temperature at which this maximum current could be obtained was the same for all alkali salts, it follows that at such high temperatures the salts tried all gave ions having the same velocity; for if some salts had given ions having a smaller velocity than others, then a higher temperature and larger P.D. would have been required to obtain the maximum current with these salts.

The writer has recently, in collaboration with Mr. E. Gold, B.A., carried out a series of experiments on the conductivity of flames containing salt vapours for rapidly alternating currents. We have found that the conductivity for rapidly alternating currents varies as the square root of the corresponding conductivity for steady currents. It is shown in the paper *, that this result can be explained by supposing that all the salts tried give ions having equal velocities in the flame.

In 1903 Prof. Moreau published an account (*Annales de Chimie et de Physique*, Sept. 1903) of some measurements of the velocites of the ions of salt vapours in flames by a method which was described by the writer in the paper referred to above in 1899. Prof. Moreau found, in good agreement with the earlier work, that all the alkali salts tried gave positive ions having a velocity of about 80 cms. per sec. For the negative ions of salts of potassium and sodium, however, Prof. Moreau obtained results which do not agree with those obtained by Dr. Marx and the writer. Prof. Moreau's results for sodium and potassium salts are as follows:—

With equal molecular concentrations the velocity of the negative ions is independent of the nature of the acid radical

* Phil. Mag. April 1906.]

of the salt. It varies with the metal in the inverse ratio of the square root of the atomic weight. For a particular salt vapour the velocity of the negative ion increases as the concentration of the vapour diminishes. Prof. Moreau gives the following numbers :—

Molecular Concentration of Solution sprayed into the Flame		1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$
Velocity of Negative Ions in cms. per sec.	K	660	785	995	1180	1320
	Na	800	1040	1280		

Thus with large concentrations the sodium ions seem to have larger velocities than the potassium ions.

The method used by Prof. Moreau was to find the P.D., between two vertical electrodes, required to make the salt ions move across from the bottom of one electrode to the top of the other. Then, assuming the electric field uniform and knowing the upward velocity of the gases between the electrodes, the velocity of the ions was calculated. Now the electric field in a flame is very far from being uniform, but it is no doubt nearly the same in different cases when the conductivity of the flame is the same. Consequently, if we compare Prof. Moreau's results for potassium salts with his results for sodium salts, taking such concentrations that the flame has the same conductivity for the potassium salts as for the sodium salts, then we ought to get the relative velocities of the potassium and sodium salt ions free from error due to variations of the potential gradient with the conductivity of the flame.

Potassium salts conduct about four times as well as sodium salts when the concentration is small, so that Prof. Moreau's numbers may be tabulated thus :—

Concentration of Potassium Salts	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$
Concentration of Sodium Salts	1	$\frac{1}{4}$	$\frac{1}{16}$
Velocity of Negative Ions for Potassium Salts	785	995	1180
Velocity of Negative Ions for Sodium Salts	800	1040	1280

Thus, comparing velocities with concentrations giving nearly equal conductivities, the velocities found by Prof. Moreau are very nearly the same for potassium salts as for

sodium salts. The apparent variation of the velocity with the concentration of the salt vapour is no doubt due to the variation of the fall of potential between the electrodes with the conductivity, and not to any real variation of the velocity with the concentration.

The mean of Prof. Moreau's results for the velocity of the negative ions is about 1000 cms. per sec., and so agrees very well with the earlier results obtained by Dr. Marx and the writer.

If Prof. Moreau were correct in supposing that the velocity of the negative ions varies inversely as the square root of the atomic weight of the metal, it would follow that salt vapours are ionized into a metal ion carrying a negative charge and a positive ion the same for all salts. Such a conclusion is altogether inconsistent with our knowledge of the ionization of salts.

If all salts give positive and negative ions having the same velocities in a flame, this can be explained by supposing that a salt molecule emits a corpuscle which is the negative ion, and that the positively charged salt molecule forms an aggregate of molecules whose size depends only on the charge, and so is the same for all salts. This view is consistent with the fact that the velocity of the positive ions is only about 60 cms. per sec. while that of the negative ions is 1000 cms. per sec. According to Prof. Moreau's results, the positive and negative ions ought to have nearly equal velocities, for the atomic weight of the metals is comparable with the molecular weight of the acid radicals.

The experiments done by the writer in 1899 showed that lithium (Li = 7) salts give ions having the same velocities as the ions from cæsium (sC = 133) salts; which proves clearly that the velocity of the negative ions does not vary inversely as the square root of the atomic weight of the metal—a conclusion which Prof. Moreau has drawn from experiments on potassium (K = 39) and sodium (Na = 23) salts.

It has been shown by Prof. Lenard that the salt vapour emitting light in flames moves in a strong electric field as though it were positively charged. I have verified this result, which clearly proves that the metal goes to form the

positive ions, for the light is undoubtedly emitted by the metal atoms and not by the acid radical of the salts.

I think therefore that the view that all alkali salts in flames give ions having nearly the same velocities is really supported by Prof. Moreau's observations as well as by those of Dr. Marx and the writer.

Discussion.

Mr. D. Owen asked Dr. Wilson how, in reference to the theory that the positive ion is the salt molecule minus an electron, he accounted for the fact, shown in his experiments, that the velocity of these ions was independent of their mass.

Dr. Wilson remarked that the positive ion probably acted as a centre of condensation and formed an aggregate of molecules whose size depended only on the positive charge.

XIV. *On the Lateral Vibration of Bars subjected to Forces in the Direction of their Axes.* By John Morrow, *M.Sc. (Vict.), D.Eng.(L'pool), Lecturer in Engineering, University College, Bristol*.*

Contents.

Section I. *Statement of the General Problem.*

§ 1. The general differential equation for a rod of uniform or varying sectional area subjected to an axial tensile force may be written, neglecting the rotatory inertia,

$$\frac{d^2}{dx^2}\left(\mathrm{EI}\frac{d^2y}{dx^2}\right)-\mathrm{P}\frac{d^2y}{dx^2}+\rho\omega\frac{d^2y}{dt^2}=0. \quad . \quad . \quad . \quad (1)$$

* Read April 27, 1906.

In which

y = lateral displacement at a distance x from the origin at time t, the axis of x being parallel to that of the bar, and vibration occurring in the xy plane ;

ρ = density of the material ;

ω = sectional area at x ;

P = total axial pull;

I = Moment of Inertia of section at x about a line through its centre of gravity at right-angles to the plane of vibration ;

$E = \frac{P}{\omega}$ + Young's Modulus (sensibly equal to Young's Modulus for the material, which is assumed homogeneous and isotropic).

§ 2. If the bar carries loads concentrated at different points in its length, equation (1) holds between every pair of singular points, that is between points of support and load or between any pair of concentrated loads.

If undashed symbols refer to values of y on the left, and dashed symbols to those on the right of any singular point, we have, in addition to the end conditions, that at all singular points

$$y = y' \quad \text{and} \quad \frac{dy}{dx} = \frac{dy'}{dx}.$$

These serve to determine the constants in the solution of (1).

When the bar is of uniform sectional area the complete integral of (1) is well known to be

$$y = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x. \quad . \quad (2)$$

In the cases dealt with in this paper the cross-section is assumed to be constant.

Section II. *Unloaded Massive Bar, Supported at each End and subjected to an Axial Pull* P.

§ 3. In the consideration of a special problem we may take, instead of equation (1), a particular differential equation expressing the conditions of that problem only. This method will be adopted as it brings out the physical significance of each term very prominently. The solution for a bar supported

at each end has been otherwise obtained. (*Cf.* Rayleigh's 'Sound,' vol. i. article 189, in which rotatory inertia is taken into account.) Taking the origin at one end, the particular form of the differential equation is

$$-\mathrm{EI}\frac{d^2y}{dx^2}=-\frac{x}{2}\int_0^l \rho\omega\ddot{y}\,dx+\int_0^x \rho\omega\ddot{y}_z(x-z)\,dz-\mathrm{P}y,$$

where y_z=deflexion at a distance z from the origin,
l=length of bar.

If y_1=displacement at centre of span, the solution is

$$y=y_1\sin\frac{\pi x}{l},$$

and

$$-\frac{\ddot{y}_1}{y_1}=\frac{\mathrm{EI}\pi^4+\mathrm{P}l^2\pi^2}{\rho\omega l^4},$$

from which the frequency is obtained.

When the force P is compressive vibration is impossible if $\frac{\ddot{y}_1}{y_1}=0$, that is when

$$\mathrm{P}=\mathrm{EI}\frac{\pi^2}{l^2}.$$

Section III. *Clamped-Clamped Unloaded Massive Bar.*
Axial Pull=P.

§ 4. When the ends are clamped, using equation (2), the conditions at the ends and centre give

$$x=0,\qquad y=0,\qquad \therefore\quad \mathrm{C}=-\mathrm{A},$$

$$x=0,\qquad \frac{dy}{dx}=0,\qquad \therefore\quad \mathrm{D}=-\frac{\alpha}{\beta}\mathrm{B},$$

$$x=l,\quad y=\frac{dy}{dx}=0;\qquad \text{and}\quad x=\frac{l}{2},\quad \frac{dy}{dx}=0;$$

from which

$$\left.\begin{aligned}
&\mathrm{A}(\cosh\alpha l-\cos\beta l)+\mathrm{B}\left(\sinh\alpha l-\frac{\alpha}{\beta}\sin\beta l\right)=0\\
&\mathrm{A}(\alpha\sinh\alpha l+\beta\sin\beta l)+\mathrm{B}(\alpha\cosh\alpha l-\alpha\cos\beta l)=0\\
&\mathrm{A}(\alpha\sinh\tfrac{1}{2}\alpha l+\beta\sin\tfrac{1}{2}\beta l)+\mathrm{B}(\alpha\cosh\tfrac{1}{2}\alpha l-\alpha\cos\tfrac{1}{2}\beta l)=0
\end{aligned}\right\}\quad(3)$$

The last of these holds only for the fundamental and even harmonics.

If we put
$$\gamma=\frac{\alpha\sinh\alpha l+\beta\sin\beta l}{\cosh\alpha l-\cos\beta l},$$

$$\eta=\cosh\tfrac{1}{2}\alpha l-\frac{\gamma}{\alpha}\sinh\tfrac{1}{2}\alpha l-\cos\tfrac{1}{2}\beta l+\frac{\gamma}{\beta}\sin\tfrac{1}{2}\beta l,$$

the constants are

$$\mathrm{A}=\frac{y_1}{\eta},\quad \mathrm{B}=-\frac{\gamma}{\alpha}\frac{y_1}{\eta},\quad \mathrm{C}=-\frac{y_1}{\eta},\quad \mathrm{D}=\frac{\gamma}{\beta}\frac{y_1}{\eta},$$

and the solution becomes

$$y=\frac{y_1}{\eta}\left(\cosh\alpha x-\frac{\gamma}{\alpha}\sinh\alpha x-\cos\beta x+\frac{\gamma}{\beta}\sin\beta x\right)\quad . \quad (4)$$

The equation of the type used in the previous case is

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\rho\omega\ddot{y}_1\left\{\int_0^x\frac{y_z}{y_1}(x-z)\,dz-\frac{x}{2}\int_0^l\frac{y}{y_1}\,dx\right\}-\mathrm{P}y+\mathrm{M};$$

and we find that for (4) to be the solution of this the following three conditions obtain

$$\left.\begin{aligned}\alpha^2&=\frac{\mathrm{P}}{2\mathrm{EI}}\left(\sqrt{1-\frac{4\mathrm{EI}\rho\omega}{\mathrm{P}^2}\frac{\ddot{y}_1}{y_1}}+1\right),\\ \beta^2&=\frac{\mathrm{P}}{2\mathrm{EI}}\left(\sqrt{1-\frac{4\mathrm{EI}\rho\omega}{\mathrm{P}^2}\frac{\ddot{y}_1}{y_1}}-1\right),\\ \alpha\tanh\tfrac{1}{2}\alpha l&=-\beta\tan\tfrac{1}{2}\beta l.\end{aligned}\right\}\quad . \quad . \quad . \quad (5)$$

From these equations elimination of α and β would give an expression for $\dfrac{\ddot{y}_1}{y_1}$.

The couples required to fix the ends are each given by

$$\mathrm{M}=-\frac{\rho\omega\ddot{y}_1}{\eta}\left\{\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\left(\frac{\gamma l}{4}-1\right)-\frac{\nu l}{4}\right\},$$

where
$$\nu=\frac{1}{\alpha}\sinh\alpha l-\frac{\gamma}{\alpha^2}\cosh\alpha l-\frac{1}{\beta}\sin\beta l-\frac{\gamma}{\beta^2}\cos\beta l.$$

§ 5. Of equations (5), the first two could have been obtained directly from the general differential equation, and the last is the result of eliminating A and B from any two of equations (3).

An approximate determination of $\dfrac{\ddot{y}_1}{y_1}$ from equations similar to (5) has been made by Donkin and by Seebeck for very long and fine wires on the assumption that the vibration is

but slightly affected by the existence of rigidity, namely

$$-\frac{\ddot{y}_1}{y_1}=\frac{i^2\pi^2}{l^2}\frac{\mathrm{P}}{\rho\omega}\left(1+\frac{2}{l}\sqrt{\frac{\mathrm{EI}}{\mathrm{P}}}\right)^2;$$

as a closer approximation Seebeck found

$$-\frac{\ddot{y}_1}{y_1}=\frac{i^2\pi^2}{l^2}\frac{\mathrm{P}}{\rho\omega}\left(1+\frac{4}{l}\sqrt{\frac{\mathrm{EI}}{\mathrm{P}}}+\frac{12+i^2\pi^2}{l^2}\frac{\mathrm{EI}}{\mathrm{P}}\right)$$

from which to estimate the period of the i^{th} tone.

An approximation of a very different character is given in the next paragraph.

§ 6. Writing

$$\phi=\tfrac{1}{2}\alpha l, \quad \theta=\tfrac{1}{2}\beta l, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we have in the case of no tension and dealing with the fundamental only

$$\phi=\theta=2\cdot365,$$

and, since for such values $\tanh\phi$ varies very slowly with ϕ, when $\frac{\mathrm{P}l^2}{2\mathrm{EI}}$ is small we may put

$$\tanh\phi=1\cdot0178.$$

The third of equations (5) becomes

$$\phi=-\frac{1}{1\cdot0178}\theta\tan\theta.$$

Under these circumstances $\frac{d\phi}{d\theta}$ does not vary rapidly with θ. Hence, when the effect of P is small,

$$\phi=\int\left[\frac{d\phi}{d\theta}\right]_{\theta=2\cdot365}d\theta+\mathrm{K}.$$

Determining the constant K by $\phi=\theta=2\cdot365$ and using (6) we find

$$\alpha l=21\cdot806-3\cdot61\beta l. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Combining this with equations (5)

$$\beta^2l^2=46\cdot08+\frac{\mathrm{P}l^2}{12\cdot03\mathrm{EI}}-23\cdot7\left(1+\cdot02536\frac{\mathrm{P}l^2}{\mathrm{EI}}\right)^{\frac{1}{2}},$$

and expanding by the Binomial as far as the second term,

$$\beta^2l^2=22\cdot38-\cdot218\frac{\mathrm{P}l^2}{\mathrm{EI}};$$

whence

$$-\frac{\ddot{y}_1}{y_1}=500\cdot6\frac{\mathrm{EI}}{\rho\omega l^4}+12\cdot6\frac{\mathrm{P}}{\rho\omega l^2}-\cdot171\frac{\mathrm{P}^2}{\rho\omega\mathrm{EI}}. \quad . \quad (8)$$

§ 7. Equation (8) is of course for the fundamental. When dealing with harmonics we may in all cases take

$$\tanh \phi = 1,$$

and, when $P=0$, we have for the i^{th} tone

$$\alpha l = \beta l = \tfrac{1}{2}(2i+1)\pi.$$

Hence

$$\left[\frac{d\phi}{d\theta}\right]_{\theta=\frac{1}{2}\alpha l} = \tfrac{1}{2}(2i+1)\pi + (-1)^i \equiv \chi \text{ (say)}$$

$$\therefore \qquad \phi = \tfrac{1}{2}\{\chi - (-1)^i\}(\chi+1) - \chi\theta.$$

Equations (5) then give

$$\theta = \xi\chi - \sqrt{\frac{Pl^2}{4EI(\chi^2-1)} + \xi^2},$$

where

$$\xi = \frac{\pi}{4}\,\frac{2i+1}{\chi-1}.$$

Expanding as before

$$\beta^2 l^2 = 4\xi^2(\chi-1)^2 - \frac{Pl^2}{EI(\chi+1)}.$$

Whence

$$-\frac{\ddot{y}_1}{y_1} = \left\{2\xi(\chi-1)\right\}^4 \frac{EI}{\rho\omega l^4} + 4\xi^2\frac{(\chi-1)^3}{\chi+1}\,\frac{P}{\rho\omega l^2} - \frac{\chi}{(\chi+1)^2}\,\frac{P^2}{\rho\omega EI}.$$

§ 8. The solution is restricted, by the use of equation (7), to small values of $\frac{Pl^2}{EI}$.

If the axial force P is compressive, vibration will become impossible when

$$P = 4EI\frac{\pi^2}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the bar will be unstable.

In practice, when using equation (9) a factor of safety of at least 10 is allowed, in which case

$$\frac{Pl^2}{EI} \not> 4.$$

It can be found by trial that in this case equation (8) is a very close approximation. Generally it may be said that for the problems met with in constructional work equation (8) may be used with fair accuracy; but when dealing with long and fine rods or wires, as in acoustical problems, the necessary conditions are not fulfilled and this solution is inapplicable.

§ 9. A solution of the present problem free from the restrictions of the previous paragraphs may be obtained by the method of continuous approximation *.

Thus in the equation

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\rho\omega\ddot{y}_1\left\{\int_0^x\frac{y_z}{y_1}(x-z)dz-\frac{x}{2}\int_0^l\frac{y}{y_1}\,dx\right\}-\mathrm{P}y+\mathrm{M}$$

assume

$$y=\frac{16y_1}{l^2}\left(x^2-\frac{2x^3}{l}+\frac{x^4}{l^2}\right).\quad.\quad.\quad.\quad.\quad(10)$$

Integrating twice,

$$-\mathrm{EI}y=\frac{\rho\omega\ddot{y}_1}{l^2}16\times10^{-3}\Big(1\cdot785715\,l^4x^2-2\cdot\dot{7}\,l^3x^3+2\cdot\dot{7}\,x^6-2\cdot\dot{3}8095\dot{2}\,\frac{x^7}{l}+\cdot595238\,\frac{x^8}{l^2}\Big)$$
$$-\frac{16\mathrm{P}y_1}{l^2}\Big(-\cdot01\dot{6}\,l^2x^2+\cdot08\dot{3}\,x^4-\cdot1\frac{x^5}{l}+\cdot0\dot{3}\,\frac{x^6}{l^2}\Big)$$

and

$$-\frac{\ddot{y}_1}{y_1}=\frac{494\cdot74(\mathrm{EI}+\cdot025\mathrm{P}l^2)}{\rho\omega l^4},$$

from which the approximate frequency may be determined. Proceeding to a second approximation we obtain a simple but lengthy expression for the vibration type, viz.:—

$$-\mathrm{EI}y=\frac{\cdot0079158}{l^6}\left(1+\frac{\mathrm{P}l^2}{40\mathrm{EI}}\right)\Big[\rho\omega\ddot{y}_1\Big(3\cdot5573\,l^8x^2-5\cdot5113\,l^7x^3$$
$$+4\cdot9603\,l^4x^6-3\cdot3069\,l^3x^7+\cdot5511\,x^{10}-\cdot3006\frac{x^{11}}{l}+\cdot0501\frac{x^{12}}{l^2}\Big)$$
$$+\mathrm{P}y_1\Big(42\cdot99\,l^6x^2-66\cdot1377\,l^5x^3+46\cdot2963\,l^2x^6-49\cdot6032\,x^8$$
$$+33\cdot0688\frac{x^9}{l}-6\cdot6138\frac{x^{10}}{l^2}\Big)\Big]+\frac{16\mathrm{P}}{\mathrm{EI}l^2}10^{-4}\Big[\rho\omega\ddot{y}_1\Big(-\cdot330685\,l^6x^2$$
$$+1\cdot488093\,l^4x^4-1\cdot3\dot{8}\,l^3x^5+\cdot496032\,x^8-\cdot330688\frac{x^9}{l}+\cdot066138\frac{x^{10}}{l^2}\Big)$$
$$-\mathrm{P}y_1\Big(3\cdot968250\,l^4x^2-13\cdot\dot{8}\,l^2x^4+27\cdot\dot{7}\,x^6-23\cdot809524\frac{x^7}{l}+5\cdot952381\frac{x^8}{l^2}\Big)\Big],$$

and

$$-\frac{\ddot{y}_1}{y_1}=\frac{500\cdot4(\mathrm{EI})^2+12\cdot1526\mathrm{EIP}l^2-\cdot01266\mathrm{P}^2l^4}{(\mathrm{EI}-\cdot0003838\mathrm{P}l^2)\rho\omega l^4}.$$

* See Phil. Mag. ser. 6, vol. x. no. 55, p. 113.

The last terms in the numerator and denominator of this expression are very small and not sufficiently accurate to be retained except as a step in the third approximation if such were required.

Hence the frequency should be given with considerable accuracy by

$$-\frac{\ddot{y}_1}{y_1}=500{\cdot}4\frac{\mathrm{EI}}{\rho\omega l^4}+12{\cdot}153\frac{\mathrm{P}}{\rho\omega l^2}. \quad . \quad . \quad . \quad (11)$$

The terminal couples are given in this approximation by

$$\mathrm{M}={\cdot}079158\left(\mathrm{EI}+\frac{\mathrm{P}l^2}{40\mathrm{EI}}\right)({\cdot}71145\rho\omega l^2\ddot{y}_1+8{\cdot}5980\mathrm{P}y_1)$$
$$-\frac{16\mathrm{P}l^2}{\mathrm{EI}}10^{-4}({\cdot}66137\rho\omega l^2\ddot{y}_1+7{\cdot}9365\mathrm{P}y_1).$$

§ 10. Another approximation may be obtained by the dynamical method, in which a type of vibration is assumed *.

Thus, taking equation (10) for the type, the potential energy at any instant is

$$\mathrm{V}=\tfrac{1}{2}\,\mathrm{EI}\int_0^l\left(\frac{d^2y}{dx^2}\right)^2dx+\tfrac{1}{2}\mathrm{P}\int_0^l\left(\frac{dy}{dx}\right)^2dx$$
$$=\left(\frac{512\,\mathrm{EI}}{5\,l^3}+\frac{256\,\mathrm{P}}{105\,l}\right)y_1^2,$$

and the kinetic energy of the system is

$$\mathrm{T}=\frac{1}{2}\int_0^l\rho\omega\left(\frac{dy}{dt}\right)^2dx$$
$$=\frac{64}{315}\rho\omega l\left(\frac{dy_1}{dt}\right)^2.$$

The equation

$$\frac{d\mathrm{T}}{dt}+\frac{d\mathrm{V}}{dt}=0$$

gives

$$-\frac{\ddot{y}_1}{y_1}=504\frac{\mathrm{EI}}{\rho\omega l^4}+12\frac{\mathrm{P}}{\rho\omega l^2}. \quad . \quad . \quad . \quad . \quad (12)$$

A solution which is based on a very reliable method, but necessarily overestimates the value of $-\frac{\ddot{y}_1}{y_1}$, since equation (10) does not exactly represent the true type of displacement.

* See Rayleigh's 'Theory of Sound,' vol. i. § 182.

Section IV. *Clamped-Supported Bar under Axial Pull.*

§ 11. When the bar is clamped at one end and merely supported at the other, the solution can be deduced from that of the previous section. Thus in the final equation of § 7 we may give to i, in succession, the values of the positive integers and at the same time write $2l$ for l.

For the fundamental

$$-\frac{\ddot{y}_1}{y_1}=\left(\frac{5\pi}{4}\right)^4\frac{\mathrm{EI}}{\rho\omega l^4}+12{\cdot}29\frac{\mathrm{P}}{\rho\omega l^2}-{\cdot}0912\frac{\mathrm{P}^2}{\rho\omega\mathrm{EI}},$$

l is the length of the bar, and the result is still confined to small values of $\frac{\mathrm{P}l^2}{\mathrm{EI}}$.

Section V. *Bar of Negligible Mass, Supported at each end, Compressive Axial Force*$=$P, *having a single load concentrated at some point in its length.*

§ 12. When dealing with bars of negligible mass it is convenient to assume the positive P to be compressive. This change of notation is therefore made in the remainder of the paper.

Let the mass m divide the bar into segments a and b.

The equations are :—

For $x<a$ $$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{P}y-m\ddot{y}_a\frac{b}{l}x,$$

for $x>a$ $$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{P}y-m\ddot{y}_a\frac{a}{l}(l-x).$$

These have respectively the solutions

$$\left.\begin{aligned} y&=\frac{c}{n^2}x+\mathrm{B}\sin nx-\mathrm{A}\cos nx,\\ y&=\frac{h}{n^2}(x-l)+\mathrm{D}\sin nx-\mathrm{C}\cos nx.\end{aligned}\right\}$$

Where $$n^2=\frac{\mathrm{P}}{\mathrm{EI}},\ c=\frac{m\ddot{y}_a}{\mathrm{EI}}\frac{b}{l},\text{ and }h=-\frac{m\ddot{y}_a}{\mathrm{EI}}\frac{a}{l}.$$

The conditions at each end are

$$\left.\begin{aligned} x=0,\quad y=0;\quad &\therefore\ \mathrm{A}=0\\ x=l,\quad y=0;\quad &\therefore\ \mathrm{D}\sin nl=\mathrm{C}\cos nl\end{aligned}\right\}\quad . \quad . \quad . \quad (13)$$

When $x=a$ the two expressions for $\frac{dy}{dx}$ are the same, whence

$$\frac{c}{n^2}+n\mathrm{B}\cos na=\frac{h}{n^2}+n\mathrm{D}\cos na+n\mathrm{C}\sin na. \quad . \quad (14)$$

Similarly the two values of y are equal at the point $x=a$.

$$\therefore \quad \frac{c}{n^2}a+\mathrm{B}\sin na=\frac{h}{n^2}(a-l)+\mathrm{D}\sin na-\mathrm{C}\cos na. \quad (15)$$

Equations (13), (14), and (15) give

$$\mathrm{B}=\frac{c-h}{n^3}(\sin na \,.\, \cot nl-\cos na)$$

$$\mathrm{C}=\frac{c-h}{n^3}\sin na,$$

$$\mathrm{D}=\frac{c-h}{n^3}\sin na \,.\, \cot nl.$$

The equations of the elastic central line are therefore for $x<a$,

$$y=\frac{m\ddot{y}_a}{\mathrm{P}}\left\{\frac{l-a}{l}x+\frac{1}{n}(\sin na \,.\, \cot nl-\cos na)\sin nx\right\}, \quad (16)$$

and the expression when $x>a$ is obtained by writing x for a and a for x in the above.

By putting $x=a$ we find

$$\frac{\ddot{y}_a}{y_a}=\frac{\mathrm{P}}{m}\left[\frac{ab}{l}+\frac{1}{n}\sin na(\sin na\cot nl-\cos na)\right]^{-1} \quad (17)$$

§ 13. As a particular case, if the mass m be at the centre of the span,

$$\frac{\ddot{y}_a}{y_a}=\frac{4\mathrm{P}}{ml}\left(1-\frac{2(1-\cos nl)}{nl\sin nl}\right)^{-1}, \quad . \quad . \quad . \quad . \quad (18)$$

and vibration is impossible when

$$\sin nl=0,$$

agreeing with the condition of instability of § 3.

Section VI. *Loaded Massless Bar Clamped at each End.*

§ 14. Here for $x<a$,

$$-\mathrm{EI}\frac{d^2y}{dx^2}=\mathrm{P}y-m\ddot{y}_a\frac{b}{l}x+\mathrm{M}_1+(\mathrm{M}_2-\mathrm{M}_1)\frac{x}{l},$$

where M_1 and M_2 are the terminal couples.

This may be written

$$\frac{d^2y}{dx^2}+n^2y=hx-k,$$

where $\qquad n^2=\frac{\mathrm{P}}{\mathrm{EI}},\quad h=\frac{m\ddot{y}_a}{\mathrm{EI}}\frac{b}{l}-\frac{\mathrm{M}_2-\mathrm{M}_1}{\mathrm{EI}l},\quad k=\frac{\mathrm{M}_1}{\mathrm{EI}}.$

The solution is

$$y=\frac{1}{n^2}(hx-k)+\mathrm{D}\sin nx-\mathrm{C}\cos nx. \quad . \quad . \quad . \quad (19)$$

The conditions at the origin give

$$\mathrm{C}=-\frac{k}{n^2},\quad \mathrm{D}=-\frac{h}{n^3}.$$

For $x>a$ we must add $m\ddot{y}_a(x-a)$ to the right-hand side of (19). Then, if

$$h'=-\frac{m\ddot{y}_a}{\mathrm{EI}}\frac{a}{l}-\frac{(\mathrm{M}_2-\mathrm{M}_1)}{\mathrm{EI}l}\ \text{ and }\ s=\frac{m\ddot{y}_a a-\mathrm{M}_1}{\mathrm{EI}},$$

the solution is

$$y=\frac{h'}{n^2}x-\left\{\frac{s+h'l}{n^2}\sin nl+\frac{h'}{n^3}\cos nl\right\}\sin nx$$

$$+\left(\frac{h'}{n^3}\sin nl-\frac{s+h'l}{n^2}\cos nl\right)\cos nx+\frac{s}{n^2}. \quad . \quad (20)$$

Equating the two values of $\left[\frac{dy}{dx}\right]_{x=a}$ and of y_a we get, after some reductions,

$$\mathrm{M}_1=m\ddot{y}_a\alpha \qquad \mathrm{M}_2-\mathrm{M}_1=m\ddot{y}_a\gamma.$$

Where

$$\alpha=\frac{a+b\cos nl-l\cos nb+\frac{1}{n}(\sin nb-\sin nl+\sin na)}{2(1-\cos nl)-nl\sin nl}$$

and

$$\gamma=\frac{(b-a)(1-\cos nl)+l(\cos nb-\cos na)}{2(1-\cos nl)-nl\sin nl}.$$

For $x=a$ we find

$$\frac{\ddot{y}_a}{y_a}=\frac{\mathrm{P}}{m\left\{\frac{a}{l}(b-\gamma)(1-\frac{1}{n}\sin na)+\alpha(\cos na-1)\right\}}. \quad . \quad (21)$$

§ 15. If the load be at the centre, $a=b=\frac{l}{2}$ in this result, and

$$\frac{\ddot{y}_a}{y_a}=\frac{4P}{ml}\left\{1-\frac{2\left(1-\cos\frac{nl}{2}\right)}{\frac{nl}{2}\sin\frac{nl}{2}}\right\}^{-1}, \quad . \quad . \quad (22)$$

vibration ceasing when

$$\sin\frac{nl}{2}=0.$$

Section VII.

Deduction of Solutions to some Static Problems.

§ 16. The calculations contained in Sections V. and VI. are the same as those involved in the corresponding statical problems of the deflexion of a bar of negligible mass carrying a concentrated load and subjected to the axial compressive force P.

If both ends are supported, instead of $m\ddot{y}_a$ we must put the magnitude of the load in the formulæ of Section V. and so obtain an expression similar to (16) for the curve of deflexion of the central line, and to (17) and (18) for the displacement at the point at which the load is concentrated.

Making the same change in notation in Section VI. we get equations (19), (20), (21), and (22) for the case of the bar clamped at both ends.

An expression equivalent (except for a numerical error) to (22) has been given by Kirchhoff*, but I am not aware that the more general solutions for any position of the load have hitherto, in either case, been recorded.

University College, Bristol,
March 1906.

* *Vorlesungen über mathematische Physik*, Bd. i. Lecture 29, § 3. See also Todhunter and Pearson's 'History of Elasticity,' vol. ii. pt. 2, § 1289.

XV. *The Dead Points of a Galvanometer Needle for Transient Currents.* By ALEXANDER RUSSELL, *M.A., M.I.E.E.**

THE effects produced by an alternating magnetic field on a magnetic needle suspended in it have been studied by several physicists †. The phenomena have been shown to be amenable to calculation in certain cases, and have been employed by Rayleigh ‡ to measure the power factor in alternating-current circuits.

As a knowledge of these effects is a great help in understanding the action of transient currents, whether direct or oscillatory, on a galvanometer-needle or coil, we shall give a brief *résumé* of the main phenomena and indicate the theory.

Effect of Alternating Currents on a Galvanometer-Needle or Coil.

Let us suppose that we have an ordinary Kelvin mirror-galvanometer, and that we connect it with the alternating-current mains through a high resistance. Let us first suppose that the mirror, and consequently the needle, is at right angles to the axis of the galvanometer coil, and that it is in stable equilibrium in this position before we close the switch. If the effective value of the alternating current be less than a certain critical value, the needle will still be in stable equilibrium after closing the switch, but the time of swing will be longer. As the current approaches the critical value the time of swing gets longer and longer, and when it equals the critical value the equilibrium is neutral. For greater values of the alternating current the spot of light moves off the scale to one side or the other. This is what

* Read May 11, 1906.

† Lord Rayleigh, Brit. Assoc. Reports, 1868; see also 'Scientific Papers,' vol. i. p. 310, and vol. ii. pp. 401 and 579; A. Schuster, "Experiments on Electrical Vibrations," Phil. Mag. [4] vol. xlviii. (1874); G. Chrystal, "On Bi- and Unilateral Galvanometer Deflexion," Phil. Mag. [5] vol. ii. p. 401 (1876).

‡ Lord Rayleigh, "On the Measurement of Alternate Currents by means of an Obliquely Situated Galvanometer-needle, with a Method of Determining the Angle of Lag," Phil. Mag. [5] vol. xliii. p 343 (1897); or 'Scientific Papers,' vol. iv. p. 299.

Professor Chrystal calls bilateral galvanometer deflexion. If the needle be in stable equilibrium, the effect of the alternating field is to make it more sensitive to magnetic impulses.

If we now twist the fibre or move the controlling magnet so that the spot of light is no longer in the centre of the scale initially, then, when we increase the alternating current, the spot of light moves steadily away from the centre of the scale to a new position of equilibrium. This is called unilateral deflexion.

Similar effects are produced by alternating currents in the coil of a d'Arsonval galvanometer. In this case, when the spot of light is in the centre of the scale the coil is in stable equilibrium. The unilateral deflexion, however, is generally towards the centre of the scale. If the moving coil be enclosed in a damping metallic cylinder, a small unilateral deflexion away from the centre of the scale can sometimes be observed.

To explain these effects let us first consider the case of a Kelvin mirror-galvanometer. Let Mk^2 be the moment of inertia of the mirror and needle about an axis through the suspending fibre, let θ_0 be the initial angular deflexion of the mirror and θ its deflexion at the time t. The equation determining the motion is

$$Mk^2\ddot{\theta} + 2b\dot{\theta} + \mu H \sin(\theta - \theta_0) = \mu G i \cos\theta + \gamma i^2 \sin\theta\cos\theta. \quad (1)$$

In this equation $2b\dot{\theta}$ represents the damping torque due to air friction, μ the magnetic moment of the needle, H the strength of the controlling field, G the coefficient* of the galvanometer, that is, the strength of the field at the magnetic poles of the needle due to unit current in the galvanometer coil, and i the instantaneous value of the alternating current flowing in the coil. Following Rayleigh, we have assumed that the magnetism of the needle is made up of a constant part and a part which is proportional to the applied magnetic force. If we suppose that the eddy currents in the needle are negligible, the torque produced by the variable component of its magnetism is easily shown to be equal to $\gamma i^2 \sin\theta\cos\theta$, where γ is a constant.

* Maxwell, 'Electricity and Magnetism,' vol. ii. § 748.

As the frequency of the alternating current is very high compared with the free period of the galvanometer-needle, and the amplitude of the forced vibration is generally very small, we see that the apparent position of equilibrium of the spot of light is given by

$$\mu H \sin(\theta-\theta_0)=\gamma A^2 \sin\theta \cos\theta,$$

where A is the effective value of the alternating current. When θ and θ_0 are small we have

$$(\theta-\theta_0)/\theta=\gamma A^2/(\mu H).$$

The author has verified this equation experimentally, and finds that for a given value of A the expression $(\theta-\theta_0)/\theta$ is practically constant. If, however, A be varied between wide limits, the agreement between this formula and experiment is not so satisfactory.

If we assume that the deflexion of moving coil galvanometers by alternating currents is due to the eddy currents in the magnets, we find that

$$(\theta_0-\theta)/\theta=kA^2,$$

where k is a constant. For a given value of A the author found experimentally that $(\theta_0-\theta)/\theta$ was practically constant, but when A was varied this ratio was only approximately constant.

The Neutral Position of the Needle for Steady Currents.

Since the moment of the applied forces acting on the needle is measured by $Mk^2\ddot{\theta}$, we get, as in (1),

$$Mk^2\ddot{\theta}=\mu Gi\cos\theta+\gamma i^2\sin\theta\cos\theta-\mu H\sin(\theta-\theta_0)$$
$$-\text{ retarding torque due to air friction}$$
$$+\text{ torque due to eddy currents in the needle.}$$

The deflexion θ is found by solving the equation

$$\mu Gi\cos\theta+\gamma i^2\sin\theta\cos\theta=\mu H\sin(\theta-\theta_0).$$

Hence, if the left-hand side of this equation be zero, $\theta=\theta_0$ will be a position of equilibrium and there will therefore be no deflexion.. In this case

$$\mu G/\gamma=-i\sin\theta_0.$$

In many types of galvanometer the neutral position for a given current can be found very easily, and hence this equation can be used to give us the ratio of μG to γ.

s 2

The Dead Points of the Needle for Transient Currents.

Lord Rayleigh, in a paper to the British Association in 1883*, points out that a galvanometer is a very imperfect instrument for indicating whether the integral sum of the transient currents through it is zero or not, and he also mentions the limitations that this imposes on Maxwell's method of comparing mutual inductances. The author finds that with many needle-galvanometers it is easy to arrange so that a, relatively speaking, gigantic charge can be passed through the coil without producing any throw at all. He also finds that all the galvanometers he has tested, whether needle or moving coil, will produce throws with certain transient currents, although their integral values are zero. A simple explanation of the effects produced by a condenser discharge can be given as follows.

Let us consider the case of a condenser K connected with the galvanometer coil in the usual manner by a charge and discharge key. Let q_0 be the initial charge in the condenser, then we have

$$\int_0^\infty i\,dt = q_0$$

and

$$\mathrm{R}\int_0^\infty i^2 dt = q_0^2/(2\mathrm{K}) - \mathrm{W};$$

where R is the *effective resistance* of the discharging circuit and W the total energy given to the needle. The latter equation is very approximately true whether the discharge be oscillatory or not. If we assume that the current has become negligibly small before the needle has moved appreciably, we find, by multiplying both sides of equation (1) by dt and integrating, that

$$\mathrm{M}k^2\Omega = \mu \mathrm{G} q_0 \cos\theta_0 + \gamma \sin\theta_0 \cos\theta_0 \{q_0^2/(2\mathrm{KR}) - \mathrm{W}/\mathrm{R}\}, \quad (2)$$

very approximately, where Ω is the initial angular velocity of the needle. Thus Ω will be zero if the right-hand side of equation (2) vanishes. In general the heat generated by eddy currents in the needle is negligibly small, and so W may

* Lord Rayleigh, Brit. Assoc. Reports, p. 444 (1883), "On the Imperfection of the Galvanometer as a test of the Evanescence of a Transient Current"; or 'Scientific Papers,' vol. ii. p. 228.

be put equal to zero when Ω is zero. Hence, in this case

$$2\mu G \cos\theta_0 + \gamma \sin\theta_0 \cos\theta_0 \{q_0/(KR)\} = 0,$$

and therefore

$$\sin\theta_0 = -2\frac{\mu G}{\gamma} \cdot \frac{R}{V}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where V, the initial P.D. between the terminals of the condenser, may be positive or negative.

At the points determined by equation (3) there will be no throw produced on charge and discharge respectively. These points, which the author calls dead-points, can be determined rapidly and accurately with low-resistance galvanometers.

If we vary the resistance x between the terminals of the condenser and find the position y of the dead-point corresponding to charge, then plot a curve of y and x, we get a straight line. This line, however, does not pass through the origin. We have to suppose, therefore, that the "effective internal resistance" of the condenser is R_1 and that R in formula (3) is given by $R = x + R_1$.

Drawing a line through the point $(-R_1, 0)$, making the same angle with the axis of X as the line-locus of the dead-points for charge, but on the opposite side from it, we get the locus of the dead-points for discharge. The distances of the dead-points also from the centre of the scale were found experimentally to be inversely proportional to the applied voltages and all the lines passed approximately through the point $(-R_1, 0)$. In these experiments, a standard 1-microfarad condenser by a well-known maker was used, and its effective internal resistance was 4 ohms. The resistance of a 0·05-microfarad condenser was found to be 32 ohms, and of a 3-microfarad condenser for alternating-current work 1·6 ohms.

In practice the energy given to the needle is negligibly small compared with that expended in heating the path of the electric current. We may consider therefore that the internal resistance R_1 of the condenser is defined by the equation

$$(R_1 + G)\int_0^\infty i^2 dt = q_0^2/(2K),$$

where G is the resistance of the galvanometer-coil and connecting leads.

The internal resistance R_1 of a condenser can be found easily as follows. Let us suppose that when only the galvanometer coil is in circuit the distance of the dead point on discharge from the symmetrical point is D_1. Let us now suppose that a resistance R is put in series with the galvanometer and that the new reading for the dead point is D_2. Then, by (3), we have

$$\frac{G+R_1+R}{G+R_1}=\frac{D_2}{D_1},$$

and thus

$$R_1=RD_1/(D_2-D_1)-G.$$

The author has found experimentally that this formula gives consistent values for R_1, both when R and the applied pressure are varied between wide limits.

Equation (3) above gives an accurate method of determining the ratio of μG to γ. It also shows that the circumstances favourable to the production of dead-points near the symmetrical position are:

(1) low resistance, (2) small galvanometer coefficient, (3) small magnetic moment of needle, (4) a high charging voltage, and (5) a large value of the Rayleigh coefficient γ.

We shall now consider the throws produced when the needle is initially in a position θ_0. Equating the change in the kinetic energy of the needle to the work done in stopping the swing, we get

$$(1/2)Mk^2(\Omega^2-\omega^2)=\mu H\{1-\cos(\theta-\theta_0)\},$$

approximately, where ω is the angular velocity of the needle when the deflexion is θ. If θ_1 be the value of θ at the extremity of its swing and if θ_1 be small, we have

$$Mk^2\Omega^2=\mu H(\theta_1-\theta_0)^2,$$

and hence

$$\theta_1-\theta_0=(Mk^2/\mu H)^{1/2}\Omega.$$

Therefore by (2)

$$\theta_1-\theta_0=(1/Mk^2\mu H)^{1/2}\{\mu Gq_0+\gamma\theta_0(q_0^2/2KR)\}, \quad (4)$$

since in most cases W is a negligibly small part of the total energy.

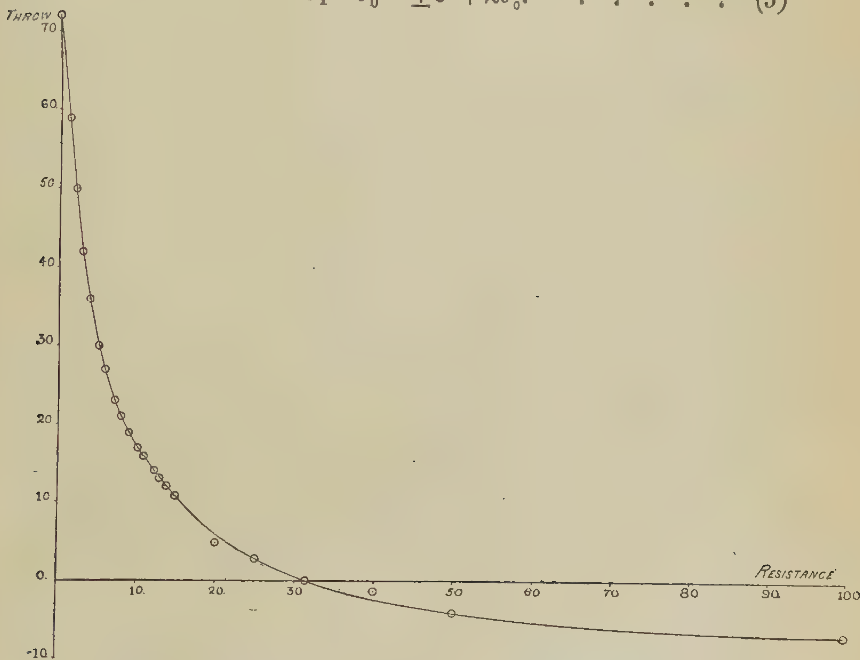
Hence $\theta_1 - \theta_0 = \theta' + \lambda\theta_0,$

where
$$\theta' = \mu G q_0/(Mk^2\mu H)^{1/2}$$
$$= \text{the throw in the symmetrical position,}$$

and
$$\lambda = \gamma q_0^2/\{2KR(Mk^2\mu H)^{1/2}\}.$$

We see that whether q_0 be positive or negative λ must be positive. Hence we may write

$$\theta_1 - \theta_0 = \pm\theta' + \lambda\theta_0. \quad . \quad . \quad . \quad . \quad . \quad (5)$$



Variation of the throw with the resistance in the discharging circuit. Capacity one microfarad. Charging voltage 105. The equilibrium position of the needle is one degree from the symmetrical position.

In the figure the throws produced when the needle was initially one degree from its symmetrical position are shown for various values of the resistance in the discharging circuit. The experimental curve in the figure is very nearly coincident with the hyperbola whose equation is

$$y = -12 + 432/(x+5).$$

The resistance of the galvanometer and leads was one ohm, the inductance of the galvanometer-coil 0·0021 henry, and the internal resistance of the condenser found experimentally by means of the dead-point line was 4 ohms. In addition to the experimental points shown the following were also found :—

Resistance in ohms ...	200	500	1000	10000	100000
Throw	−8	−10	−10	−12	−12

The throws on charge were given by

$$y=12+432/(x+5),$$

very approximately.

Equation (4) can also be written in the form

$$\theta_1-\theta_0=(1/\mathrm{M}k^2\mu\mathrm{H})^{1/2}\mathrm{KV}\{\pm\mu\mathrm{G}+(\gamma\mathrm{V}/2\mathrm{R})\theta_0\}.$$

Hence if we vary V keeping everything else constant and plot a curve showing the relation between the throw and the voltage, we get a parabola. Similarly, if we vary K (keeping R constant) we get a straight line.

The above tests could not be taken with the greatest accuracy, as the needle-galvanometers were purposely left unshielded so as not to complicate the problem, and the magnetic effects on the needle of the currents required for a neighbouring tramway were very pronounced. They, however, amply prove that the error made by the assumption that γ is constant is small.

The following simple experiment shows the effects produced by a transient current, the integral value of which is zero, on the throw of a needle galvanometer. A coil of No. 16 insulated wire, weighing about one pound, is placed on the top of a similar coil, the two coils thus forming an air-core transformer. The terminals of one coil are connected with the galvanometer terminals. The terminals of the other are connected by a Morse key with the terminals of a one-microfarad condenser in such a way that the transient currents obtained either by charging the condenser from the

100 volt mains or by discharging it pass through the coil. The integral value of the current induced in the first coil in either case will obviously be zero. When the needle is in the symmetrical position the throws produced on charge and discharge are negligibly small, but if the needle be on one side or the other of the symmetrical position the throws will be large and will always be away from the symmetrical position. Putting resistance in series with the condenser diminishes the magnitude of these throws very considerably.

With d'Arsonval galvanometers having unshielded coils similar effects are produced. The throws, however, are now towards the symmetrical position, and thus if the circuit be made and broken rapidly the spot of light moves to the centre of the scale whatever may be its initial position.

In conclusion the author wishes to emphasize the importance of the Rayleigh correction for the galvanometer equation. It explains many of the troublesome phenomena often noticed by those engaged in practical tests. When this correction is taken into account, the usefulness of the galvanometer is considerably increased, and the ease with which the dead-points can be accurately determined enables this correcting factor to be readily found. The positions of the dead-points also enable us to find the effective internal resistance of condensers. In several of the ordinary tests used in practice a knowledge of this resistance is necessary.

Discussion.

Mr. W. Duddell expressed his interest in the paper, and remarked that the phenomena described were of importance in the measurement of capacity. He suggested that since the R term in the paper included all the i^2R losses, part of it might arise from eddy-currents in the condenser. The condenser shown by the Author was constructed by winding together very long strips of tinfoil, and the resistance should therefore appear high if it was true resistance; perhaps Mr. Russell could tell him the value of the resistance of the condenser shown.

Prof. H. A. Wilson, referring to the effective internal resistance of a condenser, pointed out that if there was dielectric hysteresis there must be a loss of energy, which no doubt amounted to a resistance as determined by the Author.

Mr. Rollo Appleyard asked if it was necessary to assume that R was a resistance. If so it was difficult to see how a tinfoil condenser could have such a high resistance. R might be a physical constant of the condenser of another kind.

Mr. A. Campbell said that the effects mentioned by Mr. Russell often gave trouble in actual testing-work. Some years ago he found that in testing standard air-condensers by Maxwell's method, an ordinary needle-galvanometer gave erroneous results unless its needle was exactly in the symmetrical position. A moving-coil galvanometer, however, gave satisfactory results.

The Author, in reply to Prof. Wilson, stated that dielectric hysteresis had probably some connection with the internal resistance of the condenser, but the effect was small. The energy radiated into space was also small. Mr. Appleyard's difficulty in seeing how such a large number of sheets of tinfoil connected in parallel could have such a high resistance was a real one. It was not easy, however, to picture the distribution of the current in the sheets, and the term "resistance" was a convenient mathematical fiction. Mr. Duddell's suggestion of eddy-currents might clear up any lack of uniformity in results obtained with very rapid oscillatory discharges. The Author also showed how the resistance of a condenser could be measured in a minute or two by his method. The resistance of the actual condenser used in the experiments was found to be about 9 ohms.

XVI. *Colour Phenomena in Photometry.*

By J. S. Dow, *A.C.G.I., B.Sc.**

The discussion of Dr. Fleming's paper on Photometry, read before the Institution of Electrical Engineers in 1903, revealed great differences of opinion on the importance of colour phenomena in photometry, and it still does not seem to be generally known to what extent they are noticeable under ordinary working conditions. This may be due to the fact that most of the work done on this subject was carried out with special apparatus, such as the spectrophotometer, and not with the ordinary implements of photometry.

It therefore occurred to the author that some simple experiments on these points, carried out by him at the Central Technical College on an ordinary photometrical bench, might be of interest.

The sources of light were two similar glow-lamps, which could be screened with glass of different colours, and which were compared by means of one or other of several different photometers in the usual way. Four photometers were made use of during the experiments—the Lummer-Brodhun, the Grease-Spot, the Joly, and the Flicker.

The uncertainties which may be introduced by colour phenomena appear to be due to four separate effects:—

(1) The difficulty experienced in forming a judgment in the case of differently coloured lights, and the possibility that the judgments of different people may not be the same.

(2) The fact that the apparent relative brightness of two surfaces, illuminated by light of different colour, depends on the part of the retina on which the image of them is received.

(3) The Purkinje phenomena.

(4) The possibility, when mirrors are made use of, that the coefficient of reflexion may not be the same for different coloured lights.

(1) No doubt people differ in their capacity in this respect, but according to the author's experience it is chiefly a matter

* Read May 25, 1906.

of practice. After a considerable amount of practice, he has found that it is possible to secure fairly consistent results even when comparing such colours as ruby-red and signal-green, while anyone unused to such work would be quite unable to do so. Extraordinarily consistent results were sometimes obtained, but it was found that if the observer stopped work for an hour or so, his readings afterwards would settle down to another very consistent value, but differing by perhaps 5 or even 10 per cent. or so from those obtained before. This seems to suggest that extreme consistency in reading is partly a matter of visual memory. We recall the impression previously received by the eye and involuntarily set the photometer, the next time, so as to produce the same appearance of the field of view.

The difference in sensibility of different eyes to a particular colour certainly introduces another disturbing factor in observations made by different people. The author has not met with any serious differences in judging the colour contrasts which ordinarily occur, but it might be supposed that in such an extreme case as that quoted above considerable divergences in judgment would exist, and have, indeed, been recorded by Sir Wm. Abney, Professor Rood, and others.

It must be noted, however, that the effect mentioned in (2) must be eliminated, in order to make any satisfactory comparisons. All the figures given in this paper were obtained by the author himself, though corroborative results from other eyes were obtained in many cases.

(2) This effect has been very completely dealt with by Sir William Abney in his investigation on Colour Vision.

It has long been known that the central portion of the retina—"the yellow spot"—is much more sensitive to the red end, and less sensitive to the blue end of the spectrum, than the surrounding portion of the retina.

It has been suggested that the "yellow spot," being yellow in colour, will obstruct the blue rays, but will allow the yellow rays to pass through practically unimpeded, to the light-perceiving organs. This explanation, however, does not explain why the differences observed are distinctly more noticeable at low illuminations.

There appears to be another physiological effect, which will be referred to later.

Suppose, now, that we compare a red and a green light with a Joly photometer. An image of the illuminated blocks is formed on the retina, and we adjust the position of the photometer until the red and green appear equally bright.

But if we now observe the photometer obliquely, or if we observe it with the eye at a different distance away, the image falls on a different part of the retina where the sensibility to red and green may be different. Consequently the red and green may no longer appear equally bright.

And if a photometer with blocks of a different size were used we might again come to a different conclusion; for, even if the eye were kept at the same distance, the size of the image would be different, and a new portion of the retina would be covered by it.

The position of the photometer, for which we obtain balance, therefore depends upon :—

(*a*) The obliquity at which rays from the illuminated surface strike the eye.

(*b*) The distance away of the eye from the surface.

(*c*) The size of the surfaces.

The first point is not very important, for in focussing our eyes on the surfaces we involuntarily look straight at them. But (*b*) and (*c*) may easily affect the readings considerably. This is brought out in the curves in fig. 1 (p. 248).

These curves were obtained as follows :—Two glow-lamps, screened with red and green glass respectively, were run at a constant P.D., and compared by means of each of the three photometers referred to above.

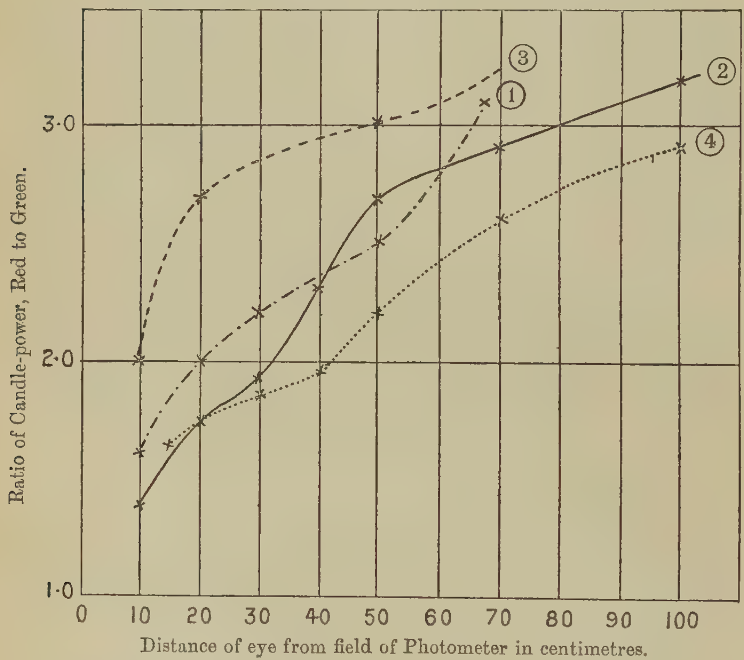
In each case a series of readings was obtained with the eye at different distances away from the illuminated surfaces. It should be mentioned that in the case of the Lummer-Brodhun photometer, the telescope was removed while the readings were taken. As will appear later, however, considerable variation is possible even with the telescope in position, as in use.

Curves (1), (2), and (3) speak for themselves. They

bring out two points. Firstly, that the ratio of the candle-powers of the red and green lights, as thus observed, is quite different for each photometer.

Secondly, that this ratio depends on the distance of the eye from the photometer, the red becoming more and more

Fig. 1.—Ruby-Red Light compared with Signal-Green.



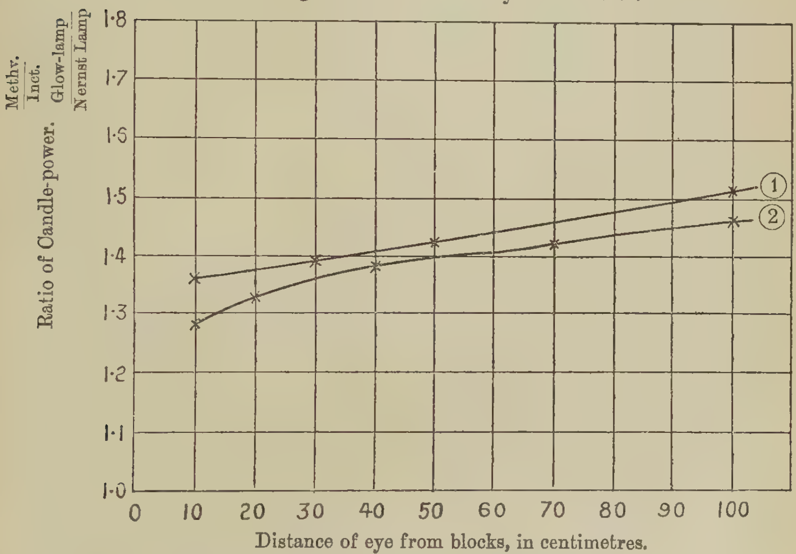
(1) Lummer-Brodhun Photometer (telescope removed).
(2) Joly Photometer.
(3) Joly Photometer (linear dimensions of blocks reduced by half).
(4) Grease-Spot Photometer.

accentuated as the eye recedes. Also the red is most accentuated and the curve is steepest for the Lummer-Brodhun photometer, in which the field is smallest.

Curve (4) was obtained by placing in front of the blocks of the Joly photometer a paper screen which reduced its linear dimensions by one half. The reduction in size of the blocks corresponds with a still further accentuation of the red.

This effect has been found to be quite distinctly observable in several commonly occurring comparisons. Fig. 2 exhibits the connexion between the apparent relative candle-power and the distance away of the eye with a Joly photometer, when a Nernst lamp was compared with a glow-lamp (running at 3·7 watts per c.p.), and when an incandescent mantle was compared with a Methven burner.

Fig. 2.—Curves for Joly Photometer.



(1) Glow-lamp compared with Nernst Lamp.
(2) Methven Gas Standard compared with incandescent mantle.

In the Joly photometer the distance away of the eye is left entirely to the inclination of the observer, and it will be seen from the above that differences of 5 per cent. or more might easily be introduced between the readings of different observers in this way.

In the Lummer-Brodhun photometer the distance of the eye is limited, to some extent, by the use of the telescope. But the position of the telescope can be varied between wide limits without putting the field out of focus, and this latitude allows of a considerable difference in the readings.

In the table below the extreme differences are given for these two limiting positions of the telescope :—

Nature of Lights compared.	Ratio of C.P.		Percentage Difference.
	Telescope in.	Telescope out.	
Ruby-Red to Signal-Green.	1·65	2·20	25·0
Glow-lamp (3·7watts per c.p.) to Nernst lamp.	1·38	1·43	3·5
Methven Gas Standard to Incandes. mantle.	0·129	0·135	4·4
Harcourt 10 c.p. Pentane Standard to Fleming Standard Glow-lamp.	0·652	0·657	0·77

Here again a distinct difference in reading is produced in several practical cases.

It is also remarkable that a small but distinct effect was produced in the last case, even though the flame of the Harcourt lamp is only very slightly redder to the eye than the light from the Fleming glow-lamp.

It is difficult, of course, to speak with certainty of such a small change as this—a change which would be produced by moving a photometer, set midway between two lights 2 metres apart, a distance of less than 2 millimetres. But the writer has usually found that the mean of a set of readings, taken with the telescope out, worked out to a value slightly different to the mean of those taken with the telescope in, and the difference was always in favour of the redder of the two lights. It need hardly be said that the difference observed might be important in such work as these two standards are used for.

There is one other point that requires mention. It is, of course, often necessary to reverse a photometer in order to correct for any differences between the two sides of the screen &c. However, for lights of similar colour, very little difference is produced by doing so, as a rule.

But when the lights differ in colour, the Lummer-Brodhun

behaves differently from the Joly and Grease-Spot photometer. In the case of the latter, the image the retina receives is unaltered by reversing. But, with the Lummer-Brodhun, the image is reversed. If, before reversing, we see a green disk with a red centre, after reversing we see a red disk with a green centre.

We should, therefore, expect a much greater difference on reversing the photometer in the case of the Lummer-Brodhun.

The following figures, obtained for similarly coloured lights, and for red and green lights, exhibit this :—

Photometers used.	Lights of similar colour.			Red to Green light.		
	Ratio of C.P.		Per cent. differ.	Ratio of C.P.		Per cent. differ.
	1st Pos.	2nd Pos.		1st Pos.	2nd Pos.	
Joly	0·93	0·91	2	2·24	2·12	5·5
Grease-Spot	0·94	0·91	3	1·81	1·70	6·0
Lummer-Brodhun	0·95	0 91	4	2·10	1·75	18·0

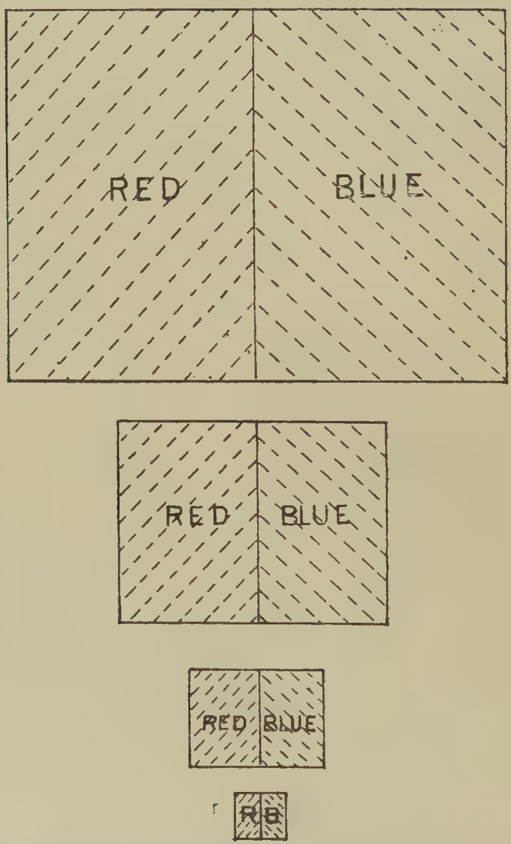
(3) The Purkinje phenomenon has often been referred to as the chief source of trouble in heterochromatic photometry, but it appears to be only troublesome at very low illuminations.

An experiment was shown illustrating the Purkinje effect. A series of coloured screens, diminishing in size from about four feet square, as shown in the diagram (fig. 3), were illuminated by a 32 C.P. glow-lamp at a distance of about ten feet away. All these screens were made from the same identical red and blue paper, but, even at normal illuminations, it could be seen that as the surfaces became smaller the red appeared brighter and brighter in comparison with the blue. The red, however, was distinctly the brighter, even in the case of the very large screen.

The illumination was now weakened by introducing resistance in series with the glow-lamp, and the blue began to appear brighter. A point was soon reached, when, for the very large screen, the blue was unquestionably brighter than the red. At this stage the difference in appearance of the

different sized screens was much more marked than at the higher illumination, and, in the case of the smaller screens, the red was still much brighter than the blue.

Fig. 3.



As the light was still further weakened the colours began to disappear until, eventually, the red appeared as black, while the blue shone out with a phosphorescent white appearance. After this point the blue also fades away until nothing can be seen.

There is, however, a distinct difference between the behaviour of the very big screen and the very small ones. In the case of the latter the Purkinje effect is much weaker.

Both colours seem to fade away together, and, by the time the Purkinje effect is really noticeable on the big screen, the very small screen can scarcely be seen at all.

In fact, to see the Purkinje effect really well, it is necessary to stand quite close up even to the big field.

A very interesting physiological explanation of these effects has been given by M. Sartori in a recent paper *.

Dotted about over the retina are two varieties of light-perceiving organs, known from their appearance as the "rods" and the "cones" respectively. The rods, it is thought, are sensitive to light but cannot perceive colour. Light of any colour appears to them white, but they are most sensitive to blue light. They are, moreover, sensitive to very weak light, but as the illumination is increased they become, as it were, saturated, and do not respond any further. The cones, on the other hand, perceive colour, but are most sensitive to yellow-green light, and while they do not respond at the low illuminations at which the rods can act, they continue to respond further to increased stimulus, once they have started, long after the rods have ceased to do so.

At normal illuminations, therefore, it is the cones which chiefly act, and we see colour. At very low illuminations the action of the rods is predominant, and we cannot see colour, while light of a bluish colour shines out with a whitish appearance. As the illumination is increased the cones suddenly begin to act and the colours appear. Then takes place what has been called "The Battle of the Rods and Cones." It is while this battle is in progress that the Purkinje effect is noticeable.

But the Purkinje effect is complicated by the fact that the rods and cones are unequally distributed over the retina. At the yellow spot the cones are predominant. Consequently, the Purkinje effect is much weaker when the field of view subtends a small angle at the eye.

This uneven distribution of the rods and cones will also explain the fact, referred to above, that at low illuminations the size of the field of view can produce much greater differences in the results.

* *Electrotechnik und Maschinenbau*, March 18, 1906.

In order to gain an idea at what illumination the effect becomes noticeable, in practice the following experiment was carried out:—

Two 100-volt 8 C.P. glow-lamps were run in series with a constant P.D. of 190 volts across them.

One was screened with red glass, and the other with green glass. The distance between the two lamps was varied from 20 to 250 inches, and the mean of a set of readings, giving their relative candle-power, taken in each case. In order to avoid the effect mentioned in (2) above, the Lummer-Brodhun photometer was used, and the telescope was kept in exactly the same position throughout the experiment.

Fig. 4.

Fig. 4.—Curve exhibiting Purkinje Phenomena for Ruby Red and Signal-Green Lights.

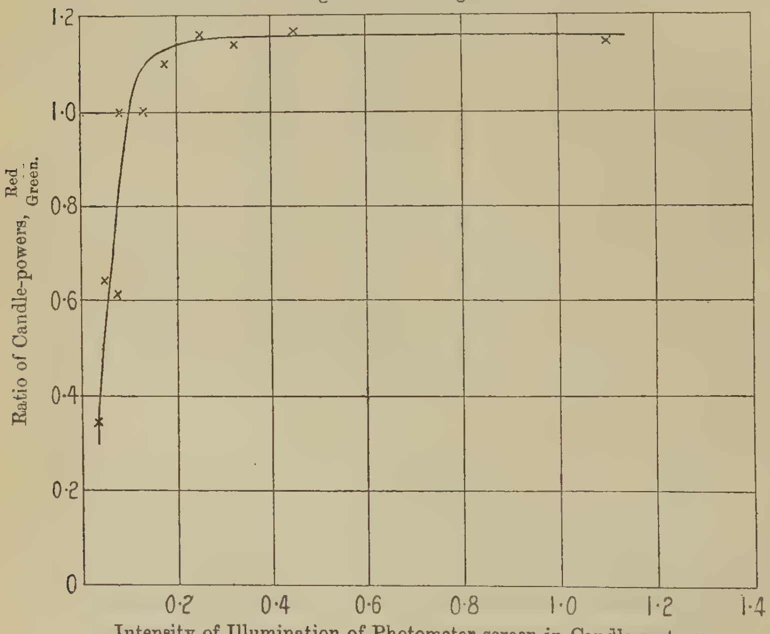


Fig. 4 shows the result of plotting the ratio of the candle-power of the two lamps against the illumination of the photometer-disk, in candle-metres.

From what has been said above, it is clear that we cannot, strictly, define the candle-power of a red light in terms of white-light standard, because this ratio depends upon the distance of the eye from illuminated surfaces.

Nevertheless, in order to give an idea of the order of illumination used, the red lamp was compared against a 2 C.P. Methven gas standard, using the Lummer-Brodhun photometer, and its candle-power worked out to something of the order of 0·25 candle-power.

The relative illuminations used in the curve are expressed in terms of this figure.

It will be seen it is only when the illumination has been reduced to about 0·2 candle-metres, that the ratio of the red to the green light begins to decrease.

The same experiment was tried on several occasions with lights of different candle-power, and it was always found that fairly consistent results were obtained until the illumination fell to something of the order specified above, when the accentuation of the green light began to appear.

At such illuminations it is very difficult to obtain readings at all, and the mean of a great many should be taken. The eye seems to be in a state of uncertainty, and one receives the impression that the lights are continually varying. In fact the eye seems to be in a state analogous with that of unsaturated iron on the steep part of the magnetization curve.

It may safely be assumed, therefore, that in all ordinary cases, where the illumination would be at least ten or twelve times as great as that employed above, where the field of view would subtend a comparatively small angle at the eye, and where we should never meet with such an extreme colour contrast as in this case, the Purkinje phenomena will not materially influence the results.

(4) In order to obtain the curve of vertical distribution of light from arc lamps, &c., the beam of light is often reflected in the desired direction by means of a 45 degrees mirror *. When this is done, the question arises, whether the coefficient

* A mirror, that is, which rotates about an axis making an angle of 45° with its plane.

of reflexion of the mirror will be the same for lights of different colour.

The following experiment was carried out with a silvered glass mirror of this type. Two 200-volt 32 C.P. lamps were run in parallel off a constant P.D. of 200 volts. The two lamps were compared against each other direct. They were then compared when the light from one of them was reflected along the bench from the mirror. In this way, the coefficient of reflexion of the mirror is easily obtained.

This was repeated when the lamps were screened with red and green glass, successively.

The mean of one set of determinations gave :—

Coefficient of reflexion,	78·8	per cent.	for unscreened lamps.
	79·0	,,	,, red light
	77·8	,,	,, green light.

Repetitions of the experiment gave slightly differing results, but it was found that the values for the three different colours did not differ among themselves by more than 2·5 per cent. Moreover, no connexion could be traced between these differences and the colours, for the difference was sometimes in favour of the red light, and sometimes in favour of the green.

We may safely conclude therefore, that in all ordinary cases, where the difference in colour is less pronounced than that adopted here, the effect is inappreciable.

In any case, the adoption of adjustable photometers of the Simmance-Abady type will probably render mirrors unnecessary in obtaining curves of distribution of light, in the future.

The only really important effect, under ordinary working conditions, therefore, is that described under the second heading. It has been shown that this may, apparently, give rise to a perceptible discrepancy even when lights so similar in colour as the Harcourt 10-candle Petane standard and the Fleming glow-lamp standard are compared.

Flicker Photometers.

The interesting question now arises whether photometers of the Flicker type are also influenced by these colour-phenomena.

If, as is claimed, the disappearance indicates that the two surfaces appear equally illuminated to the eye, one would suppose that all the conditions which determine this impression must also determine the point of disappearance of the flicker. On the other hand Mr. T. C. Porter, and others, have come to the conclusion that the flicker, at ordinary illuminations, is independent of colour.

Through the kindness of Messrs. Everett & Edgcombe the author has been able to make some experiments on the points referred to, with a photometer of this kind.

It is clear, in the first place, that when comparing lights of different colour, the readings of an ordinary photometer and a Flicker photometer may not agree even if the latter be independent of all colour effects. For instance, when two lamps giving light of exactly the same colour were compared, the results with the Joly photometer and the Flicker photometer were exactly the same. The two lamps were now screened with red and green glass respectively, and the ratio of the red to the green, obtained by the Flicker photometer, was about 1·34 ; but with the Joly photometer it was found possible to get readings from 0·7 to 1·4 for distances of the eye up to half a metre. Agreement between the photometers occurred when the eye was about 40 centimetres away from the blocks of the Joly.

One difficulty met with in this investigation was as follows. As the effects observed were comparatively small, it was advisable to use lights widely differing in colour. On the other hand the position of minimum flicker, though sufficiently sharply defined in most practical cases, is far from being so when such colours as red and green are observed. In such cases, the method followed was to note the position of the photometer in which a flicker was just visible on either side of balance, and take the mean. There is, however, another method of judging the position of balance for these two colours.

When the photometer is too near the red light, the field of view in the photometer appears reddish in tint. Similarly, a greenish tinge shows that the green illumination is the stronger. When the illumination of the two surfaces, illuminated by light of these two complementary colours, is the

same, an intermediate greyish tinge is produced. To the writer's eye the transition from red to grey to green was sharper than the disappearance of the flicker, and a series of tests showed that the result was the same in each case. The method, however, is only applicable to complementary colours.

Some experiments were first made to discover whether a difference of reading could be produced by altering the telescope, as in the case of the Lummer-Brodhun photometer, but no distinct difference was observable. Next the telescope was removed, and readings were taken with the eye about 20 centimetres from the aperture. A brass tube was then inserted which allowed the aperture to be inspected from a distance of 60 centimetres from the eye. It was then observed that, as the eye was withdrawn, the field became distinctly redder, and readings taken by the "disappearance of flicker" method also showed a change of relative candle-power in favour of the red. The following table exhibits some of the results obtained :—

Nature of Lights compared.	Eye 20 cms. from aperture.	Eye 60 cms. from aperture.	Percentage change.
Ruby-Red to Signal-Green.	$\frac{\text{Red}}{\text{Green}}=1{\cdot}52$	$\frac{\text{Red}}{\text{Green}}=1{\cdot}70$	12
Ruby-Red to Light from unscreened glow-lamp (white).	$\frac{\text{Red}}{\text{White}}=0{\cdot}70$	$\frac{\text{Red}}{\text{White}}=0{\cdot}75$	6
Methven Gas Standard to Incandescent Mantle.	$\frac{\text{Methv.}}{\text{Inct.}}=0{\cdot}28$	$\frac{\text{Methv.}}{\text{Inct.}}=0{\cdot}29$	3

The effect is very noticeable when observed in the following way :—Supposing we are comparing red and green, and have placed the photometer so as to secure balance. Move the photometer until a distinct flicker can just be seen owing to the green being too bright. It will be found that as the eye is removed the flicker gradually disappears. But if the photometer is put out of balance on the red side, the flicker does not disappear but becomes if anything more distinct as the eye is removed.

It appears therefore that this Flicker photometer is affected by the distance of the eye, but, apparently, not to a sufficient extent to be noticeable when the telescope is used, as in ordinary work.

It is difficult to see why these effects should be so much less noticeable than with ordinary photometers. It almost seems as though, when two differently coloured objects are placed side by side, any change in their relative illumination becomes exaggerated, thus creating a different impression to that received when they are viewed alternately (as in a Flicker photometer).

An attempt was also made to discover whether the Flicker photometer was subject to the Purkinje phenomenon. Now it is well known that the speed required to just make the flicker disappear depends upon the illumination, and the writer has found it impossible to judge the point of disappearance of the flicker with any certainty at the extremely weak illuminations necessary to produce the Purkinje effect.

The plan of "colour-reading" referred to above was therefore adopted. It may be objected that in doing so the real question at issue is avoided altogether. But the method, while admittedly not so satisfactory as the "disappearance of flicker" method from this point of view, has been shown to give the same results at ordinary illuminations and, to the writer, it seems impossible that the green illumination could be brighter and yet the field of view appear red.

Two 100-volt 8 candle-power lamps screened with red and green glass in the usual manner were used for the experiments. They were first run at 100 volts, and compared against each other at different distances. The two lamps were then run off 50 volts only, so as to produce a very low illumination, and the experiment repeated. The following table shows very clearly the influence of the Purkinje effect.

At the higher illumination the readings differed among themselves considerably (as was only to be expected with such a great colour contrast), but they do not seem to be connected in any way with the illumination.

The readings at the low illumination can only be regarded as very approximate, but they bring out very clearly the accentuation of the green as the illumination gets weaker.

Lamps run at 100 volts.		Lamps run at 50 volts.	
Distance between Lamps.	Ratio $\frac{\text{Red}}{\text{Green.}}$	Distance between Lamps.	Ratio $\frac{\text{Red}}{\text{Green.}}$
inches. 50	2·32	inches. 40	2·0
60	2·26	50	1·5
70	2·25	60	1·1
120	2·15	70	0·7
150	2·25	90	0·6

The conclusion the writer draws from these experiments is that Flicker photometers are affected by the same colour phenomena which affect ordinary photometers.

The interesting assertion was made by Messrs. Simmance and Abady, in a paper before the Physical Society *, that a colour-blind person obtained practically the same results with their Flicker photometer as people with normal sight. On the other hand, Sir Wm. Abney, speaking of colour-blindness, remarks † :—

"We cannot hope for instance that the red-blind, who sees no red in the extreme end of the spectrum, would show any luminosity in that region. . . . One of the most striking experiments in colour-vision is to place a bright-red patch on the screen and to ask a red-blind to make a match in luminosity with the white. The latter will have to be reduced to almost darkness—a darkness, indeed, that makes the match almost incredible."

It seems incredible that such a person, when comparing red and green with a Flicker photometer, would obtain the same results as if he had normal sight. However, it appears that, according to Dr. Edridge Green ‡, colour-blindness is of two kinds. A person may be unable to distinguish, say, red-light by colour, but nevertheless a red

* Phil. Mag. vii. p. 341 (1904). † 'Colour Vision,' p. 83.

‡ "The Physical Aspects of a Theory of Colour Vision," by F. W. Edridge Green, M.D., British Association, 1902.

object may appear as *luminous* to him as to anyone else. On the other hand, the colour-blindness may be due to the fact that the eye is incapable of perceiving red light at all. A colour-blind person of the first variety would presumably make normal photometrical readings. A person of the second class must, surely, obtain abnormal readings with all photometers, Flicker or otherwise.

This is borne out by some of Professor O. N. Rood's experiments on flicker*. He found that those of his students who were colour-blind obtained abnormal results with his Flicker photometer. Indeed, he actually used the Flicker photometer to investigate not only cases of colour-blindness but also the difference in sensibility to light of different colours, of the eyes of persons with normal sight.

In conclusion the writer wishes to express his great indebtedness to Professor Ayrton, and also to Mr. J. M. McEwan, for their assistance and for many valuable suggestions.

Discussion.

Mr. A. P. Trotter expressed his interest in the paper, and said he thought the difficulty of comparing the intensities of two lights of different colours was an imaginary one. It was possible for anyone accustomed to photometric measurements to get reliable comparisons of lights of different colours. The Author, for the sake of precision, had dealt with the comparison of red and green, but he pointed out that in practice the variation in colour between the lights under test was not so great as this; ranging from the light of the Hefner lamp to that of the arc-light or daylight. He agreed with the Author that when taking photometrical measurements it was best to get as far away as possible from the comparison disc.

Mr. A. Russell considered Mr. Dow's explanation of the phenomena of heterochrome photometry by means of the irregular distribution of rods and cones on the retina quite

* American Journal of Science, 1899, p. 258.

satisfactory. Whatever theory we adopt as to the difference between the functions of the rods and cones, it is obvious that their relative numbers on the portion of the retina covered by the image of the photometer-disc is a factor of primary importance in determining the balance. It is a physiological fact that there are no rods on the yellow spot, and this had led Professor Blondel to suggest that if we arranged so that the image always fell on the yellow spot, consistent results might be obtained. Reference was also made to M. Lauriol's paper in the Bulletin of the *Société Internationale des Électriciens* on the effect of the speed of the rotating part of the Flicker photometer on the balance obtained. The experiments proved that many of the results obtained depended merely on the speed of rotation. When comparing green with red, the numbers obtained varied between the wide limits of 1 to 5. Even when comparing a Nernst with an ordinary glow-lamp, a variation of 20 per cent. with speed was noticed. It was pointed out that it was easier to obtain a balance when comparing red with green by using a grease-spot photometer, than by using a Lummer-Brodhun contrast-photometer. The colours of the two sides of the grease-spot in the position of balance, owing to the transmitted light, were not pure red and green, but were mixtures which it was much easier to compare. The experience of the speaker was that the Lummer-Brodhun photometer was the one most favoured by those engaged in accurate photometric work. For comparing lights of different colours, however, a modified form of grease-spot photometer would have many advantages.

XVII. *Exhibition of a Bifilar Galvanometer free from zero creep, by* Mr. Campbell.*

For measuring direct currents and voltages of ordinary range (from 0·1 ampere or volt and upwards) moving-coil galvanometers with shunts or added resistances are convenient. The usual instruments, however, are affected by gradual displacement of zero (due to elastic set) when a deflexion is maintained for some time. This difficulty is got over by replacing the ordinary torsional suspension by a bifilar system in which the two wires are so far apart that the gravity control quite swamps that due to the torsion of the wires. In the instrument shown, the wires are more than 1 cm. apart, and the sensitivity with 40 ohms resistance is 400 mm. at 1 metre for 0·001 ampere; this is sufficient for many purposes. The full deflexion may be maintained for hours without causing a zero creep of 1 part in 2000. To attain good damping a very powerful magnet is used. The arrangement of this and the general design are due to Mr. R. W. Paul, who has adapted to the purpose Mr. Evershed's system of coil and magnet.

Discussion.

Prof. H. A. Wilson asked if the zero creep might not be due to magnetic material in the coil. If the field was unsymmetrical the presence of magnetic material would produce a creep.

Mr. Campbell replied that in very sensitive moving-coil galvanometers the presence of magnetic dirt was a source of trouble, but in the instrument which he had exhibited the control was so strong that the creep due to this cause was practically nothing.

* May 25, 1906.

XVIII. *The Theory of "Moving Coil" and other kinds of Ballistic Galvanometers. By* Prof. HAROLD A. WILSON, *M.A., D.Sc., M.Sc., F.R.S.**

A FORMULA which is usually given for ballistic galvanometers is

$$Q=\frac{HT}{G\pi}\sin\frac{\theta}{2},$$

where Q = quantity of electricity passed through galvanometer.
H = magnetic field controlling the galvanometer-needle.
G = magnetic field at needle, supposed perpendicular to H, due to unit current in the coil.
T = time of a complete oscillation of the needle.
θ = angle of swing of the needle from rest corrected for damping.

If ϕ is the steady angular deflexion due to a current i, then $i=\frac{H}{G}\tan\phi$; so that

$$Q=\frac{Ti\sin\theta/2}{\pi\tan\phi}.$$

The above formulæ are of course, strictly speaking, only applicable to ballistic galvanometers consisting of a coil of wire having a single small magnet freely suspended at the centre of the coil, and arranged so that at its equilibrium position the axis of the magnet is in the plane of the coil.

In the following paper the proper formulæ for ballistic galvanometers of several types in general use are obtained, and it is found that in several cases they differ appreciably from the above.

In Prof. Fleming's 'Handbook for the Electrical Laboratory and Testing Room,' vol. ii., the formula for a moving-coil galvanometer is worked out; but owing to an approximation used in the calculation it is not obtained exactly, and it is afterwards given as applying to both moving-coil and moving-needle galvanometers.

The first type of galvanometer that will be considered is

* Read May 25, 1906.

the moving-coil type with a rectangular coil, cylindrical iron core and cylindrical pole-pieces symmetrically arranged. In galvanometers of this type the magnetic field is approximately radial, so that the couple on the coil due to a steady current is proportional to the current and independent of the deflexion. The controlling couple is due to the torsion of the wire by which the coil is suspended, so that the work required to turn the coil through an angle θ from its zero position is $\frac{\alpha\theta^2}{2}$, where α is the couple exerted by the wire when the coil is turned through the unit angle. Let Ci denote the couple due to a current i, then we have $Ci=\alpha\phi$. When a current i is passed through the galvanometer for a short time dt, we have

$$Ci\,dt=C\,dQ=K\,d\omega,$$

where K is the moment of inertia of the coil and ω its angular velocity. Hence

$$Q=\int i\,dt=\int\frac{K\,d\omega}{C}=\frac{K\omega}{C}.$$

We have also

$$T=2\pi\sqrt{\frac{K}{\alpha}}\text{ and }\frac{\alpha\theta^2}{2}=\tfrac{1}{2}K\omega^2;$$

hence

$$T=2\pi\sqrt{\frac{\theta^2}{\omega^2}}=\frac{2\pi\theta}{\omega}.$$

Thus

$$Q=\frac{K\omega}{C}=\frac{\alpha\theta^2}{C\omega}=\frac{T\alpha\theta}{2\pi C}.$$

But $\frac{\alpha}{C}=\frac{i}{\phi}$; so that finally

$$Q=\frac{Ti\theta}{2\pi\phi}\text{ exactly.}$$

Since the couple on the coil due to the current is independent of θ, it is not necessary with this type of instrument that the time during which the transient current passes should be very small compared with T.

Another type of moving-coil instrument in common use has a narrow coil suspended between the poles of a magnet and no iron core. In this case, the couple on the coil due to

a current i will be nearly $Ci\cos\phi$, so that $Ci\cos\phi=\alpha\phi$ or $\frac{\alpha}{C}=\frac{i\cos\phi}{\phi}$; hence the exact formulæ for this type of instrument are

$$Q=\frac{T\alpha\theta}{2\pi C}=\frac{Ti\theta\cos\phi}{2\pi\phi}.$$

In these formulæ it is assumed that the plane of the coil at its zero position is parallel to the magnetic field.

It appears, therefore, that the first type of moving-coil ballistic galvanometer considered is superior to the second in respect of the simplicity of the exact formulæ to be used with it, and the absence of error when the transient current lasts an appreciable time.

Moving-needle instruments will now be considered. The formulæ

$$Q=\frac{HT\sin\theta/2}{G\pi} \text{ and } Q=\frac{Ti\sin\theta/2}{\pi\phi}$$

apply to what may be called a tangent ballistic galvanometer. The time of the transient current must of course be small compared with T, and its greatest value must not be sufficient to appreciably change the magnetism of the needle.

In many ballistic galvanometers an astatic system of needles is used. First, suppose the system is exactly astatic and that the torsion of a wire or quartz fibre supplies the controlling couple. In this case, $Ci\cos\phi=\alpha\phi$ and $K\omega^2=\alpha\theta^2$, so that the formulæ are

$$Q=\frac{T\alpha\theta}{2\pi C}=\frac{Ti\,\theta\cos\phi}{2\pi\phi},$$

as with the second type of moving-coil instrument. Also $C=M(G_1+G_2)$, where M is the moment of each needle and G_1 and G_2 the fields due to unit current at the two needles respectively.

One other case will be considered, viz. when the needle system is more or less astatic and is controlled by magnetic fields at each needle. Let the horizontal component of the controlling field at one needle be H_1 making an angle α_1 with the plane of this needle, and let M_1 be the moment of this needle. Let M_2, H_2, and α_2 be the corresponding quantities at the

other needle. Then in the equilibrium position,

$$O=M_1H_1\sin\alpha_1+M_2H_2\sin\alpha_2.$$

The controlling couple when the needle is deflected through an angle ϕ is

$$M_1H_1\sin(\phi+\alpha_1)+M_2H_2\sin(\phi+\alpha_2)$$
$$=\{M_1H_1\cos\alpha_1+M_2H_2\cos\alpha_2\}\times\sin\phi;$$

and if ϕ is the deflexion due to a current i, then

$$iM_1G_1\cos\phi+iM_2G_2\cos\phi=\sin\phi\{M_1H_1\cos\alpha_1+M_2H_2\cos\alpha_2\}.$$

Hence

$$i=\frac{M_1H_1\cos\alpha_1+M_2H_2\cos\alpha_2}{M_1G_1+M_2G_2}\tan\phi.$$

It can now be easily seen that the formulæ for a ballistic galvanometer of this type are

$$Q=\frac{T(M_1H_1\cos\alpha_1+M_2H_2\cos\alpha_2)\sin\theta/2}{\pi(M_1G_1+M_2G_2)}\text{ and }Q=\frac{Ti\sin\theta/2}{\pi\tan\phi}.$$

So far the correction of the observed deflexion θ for damping has been neglected. The method usually described depends on the "logarithmic decrement," and is cumbrous, and moreover inexact unless the damping is very small. The following method has the great advantages of simplicity and exactness; it is not new: the writer learned it about ten years ago from Prof. W. Stroud.

Let θ_1, θ_2, θ_3, &c. be the successive swings of the galvanometer-needle. Then

$$\frac{\theta_1}{\theta_2}=\frac{\theta_2}{\theta_3}=\frac{\theta_3}{\theta_4}=\&\text{c.}=f\text{ say.}$$

To find f observe, say, θ_1 and θ_3 or θ_1 and θ_5.

Then
$$f^2=\frac{\theta_1}{\theta_3}\text{ or }f^4=\frac{\theta_1}{\theta_5}.$$

Also the corrected value of the first swing is

$$\theta_1\sqrt{f}=\theta_1\left(\frac{\theta_1}{\theta_3}\right)^{\frac{1}{4}}=\theta_1\left(\frac{\theta_1}{\theta_5}\right)^{\frac{1}{8}}.$$

In most cases the angular deflexions may be taken as proportional to the deflexions on the scale. Let these be δ_1, δ_2, δ_3, &c. Then the corrected value of the first swing is

$$\delta_1\left(\frac{\delta_1}{\delta_3}\right)^{\frac{1}{4}} \text{or } \delta_1\left(\frac{\delta_1}{\delta_5}\right)^{\frac{1}{8}}.$$

If the damping is small the corrected value is very nearly $\delta_1+\frac{1}{4}(\delta_1-\delta_3)$, and this formula is sufficiently exact for most purposes in practice.

The following table contains a summary of the results obtained :—

Type of Galvanometer.	*Formulæ.*
(1) Tangent Galvanometer with small needle at centre of coil.	$Q=\frac{HT}{G\pi}\sin\frac{\theta}{2}$, $Q=\frac{Ti\sin\theta/2}{\pi\tan\phi}$.
(2) Astatic Galvanometer with purely torsional control.	$Q=\frac{T\alpha\theta}{2\pi M(G_1+G_2)}$, $Q=\frac{Ti\,\theta\cos\phi}{2\pi\phi}$.
(3) Astatic Galvanometer with purely magnetic control.	$Q=\frac{T(M_1H_1\cos\alpha_1+M_2H_2\cos\alpha_2)}{\pi(M_1G_1+M_2G_2)}\sin\frac{\theta}{2}$, $Q=\frac{Ti\sin\theta/2}{\pi\tan\phi}$.
(4) Moving-coil Galvanometer with iron core and radial magnetic field.	$Q=\frac{T\alpha\theta}{2\pi C}$, $Q=\frac{Ti\,\theta}{2\pi\phi}$,
(5) Moving-coil Galvanometer with narrow coil and no core.	$Q=\frac{T\alpha\theta}{2\pi C}$, $Q=\frac{Ti\,\theta\cos\phi}{2\pi\phi}$.

It will be seen that type (4) is superior to all the others as regards the simplicity of the exact formula to be used with it. It is also superior in most other respects. Galvanometers are often met with in practice which strictly speaking do not belong to any of the simple types considered in this paper. In such cases the exact formulæ are more complicated, and the best plan to adopt is to determine experimentally the relation between θ and Q.

XIX. *On the Solution of Problems in Diffraction by the Aid of Contour Integration.* By HENRY DAVIES, *B.Sc., Technical Institute, Portsmouth* *.

THE general problem of diffraction consists of finding solutions of the equation

$$\frac{\delta^2 V}{\delta t^2}=a^2\nabla^2 V. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solutions must remain finite throughout the space considered and must satisfy certain specified boundary conditions.

This equation is modified when assumptions are made concerning the light vector.

Consider the case of a wedge of angle α, and assume that the electric force—taken as the light vector—is parallel to the edge of the wedge. Assume also that $V \propto \epsilon^{ikt}$. Then the general equation reduces to

$$\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+m^2 V=0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

provided the origin of co-ordinates is taken in the edge of the wedge, the latter being assumed to occupy the space $\alpha<\theta<2\pi$.

The boundary conditions are that V shall vanish at $\theta=0$ and at $\theta=\alpha$, and shall become infinite at a point $(r' . \theta')$.

2. The proper solution for unbounded space is

$$V=K_0(mR), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$R=\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

and $K_n(x)$ is Bessel's function of the second kind and of order n.

At this point it is necessary to introduce the relations which hold between the various functions which will be used.

The Bessel's functions of the second kind are related to those of the first kind by the equation

$$K_n(x)=\frac{\pi}{2\sin n\pi}\cdot\left[J_{-n}(x)-\epsilon^{-in\pi}J_n(x)\right]. \quad . \quad (4)$$

* Read June 8, 1906.

When n is large the value of $J_n(x)$ is given very approximately by

$$J_n(x)=\left(\frac{x}{2}\right)^n \frac{1}{\Pi(n)}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $\Pi(n)$ represents an infinite product.

Taking the asymptotic value of $\Pi(n)$ we have

$$J_n(x)=\left(\frac{x}{2}\right)^n \frac{1}{\sqrt{2\pi n}\,\epsilon^{n\log n-n}}. \quad . \quad . \quad . \quad (6)$$

From (6) it is evident that $J_n(x)$ vanishes at infinity when the real part of n is positive.

To obtain a value for $J_{-n}(x)$ proceed as follows:—

$$\Pi(n)\,\Pi(-n)=\frac{n\pi}{\sin n\pi}$$

$$\therefore \quad \frac{1}{\Pi(-n)}=\frac{\sin n\pi}{n\pi}\Pi(n).$$

Therefore

$$\begin{aligned} J_{-n}(x) &=\left(\frac{x}{2}\right)^{-n}\frac{1}{\Pi(-n)} \\ &=\left(\frac{x}{2}\right)^{-n}\frac{\sin n\pi}{n\pi}.\Pi(n) \\ &=\left(\frac{x}{2}\right)^{-n}\frac{\sin n\pi}{n\pi}.\sqrt{2\pi n}\,\epsilon^{n\log n-n}. \quad . \quad . \quad . \quad (7)\end{aligned}$$

This does not vanish at infinity when the real part of n is positive. This difficulty can be overcome as follows.

From (4) by multiplication by $J_n(x')$

$$K_n(x)\,J_n(x')=J_n(x')\,.\,\frac{\pi}{2\sin n\pi}\left[J_{-n}(x)-\epsilon^{-in\pi}J_n(x)\right]$$

$$=\left(\frac{x'}{2}\right)^n\frac{1}{\Pi(n)}.\frac{\pi}{2\sin n\pi}\left\{\left(\frac{x}{2}\right)^{-n}\frac{\sin n\pi}{n\pi}.\sqrt{2\pi n}\,.\,\epsilon^{n\log n-n}\right.$$

$$\left.-\epsilon^{-in\pi}\left(\frac{x}{2}\right)^n.\frac{1}{\sqrt{2\pi n}\,\epsilon^{n\log n-n}}\right\}. \quad . \quad . \quad (8)$$

With substitution of the value of $\frac{1}{\Pi(n)}$ this reduces to

$$K_n(x)\,J_n(x')=\frac{1}{2n}.\left(\frac{x'}{2}\right)^n\left(\frac{x}{2}\right)^{-n}$$

$$-\epsilon^{-in\pi}\left(\frac{x}{2}\right)^n\left(\frac{x'}{2}\right)^n\frac{\pi}{2\sin n\pi}\left\{\frac{1}{\sqrt{2\pi n}\,\epsilon^{n\log n-n}}\right\}$$

Hence when the real part of n is positive this becomes

$$\mathrm{K}_n(x)\,\mathrm{J}_n(x') = \frac{1}{2n}\left(\frac{x'}{x}\right)^n, \quad . \quad . \quad . \qquad (9)$$

since the second part vanishes at infinity.

The expression

$$\frac{\cos n\{\pi-(\theta-\theta')\}}{\sin n\pi}$$

vanishes at infinity if $\theta-\theta'$ lies between 0 and 2π.

3. Consider the integral

$$\int_{\mathrm{A}} \frac{\cos n(\pi-\overline{\theta-\theta'})}{\sin n\pi}\,\mathrm{J}_n(mr')\,\mathrm{K}_n(mr)dn, \quad . \quad (10)$$

where $\quad r>r' \quad$ and $\quad \theta>\theta'$;

the integral being taken over the path A in the n-plane.

The integral is equal to $2\pi i\,\Sigma\mathrm{B}$, where $\Sigma\mathrm{B}$ is the sum of the residues of this function.

Fig. 1.

The path A (fig. 1) has a small semicircle at the origin.

If, however, we remove the small semicircle then we easily find that

$$\frac{1}{i\pi}\int_{\mathrm{A}'} \frac{\cos n(\pi-\overline{\theta-\theta'})}{\sin n\pi}\,\mathrm{J}_n(mr')\,\mathrm{K}_n(mr)dn$$

$$= \frac{1}{\pi}\left\{\mathrm{J}_0(mr')\,\mathrm{K}_0(mr) + 2\sum_{n=1}\mathrm{J}_n(mr')\,\mathrm{K}_n(mr) \times \cos n(\theta-\theta')\right\}. \quad . \quad (11)$$

The path A′ is that with the circle removed. There is a well-known addition theorem for $\mathrm{K}_0(mr)$ which gives

$$\mathrm{K}_0(m\mathrm{R}) = \mathrm{J}_0(mr')\,\mathrm{K}_0(mr) + 2\sum_{n=1}\mathrm{J}_n(mr')\,\mathrm{K}_n(mr) \times \cos n(\theta-\theta'). \quad (12)$$

From (11) and (12) there results

$$K_0(mR)=\frac{1}{i}\int_{A'}\frac{\cos n(\pi-\overline{\theta-\theta'})}{\sin n\pi}\,J_n(mr')\,K_n(mr)\,dn. \quad (13)$$

This equation gives a solution for the unbounded space. It is necessary now to add terms which shall satisfy the boundary equations while introducing no new singularities. After many trials I have found that the following equation satisfies the conditions completely:—

$$V=\frac{1}{i}\int_{A'}\left[\cos n(\pi-\overline{\theta-\theta'})-\cos n(\pi-\overline{\alpha-\theta'})\frac{\sin n\theta}{\sin n\alpha}\right.$$
$$\left.-\cos n(\pi-\theta')\frac{\sin n(\alpha-\theta)}{\sin n\alpha}\right]$$
$$\times\frac{J_n(mr')\,K_n(mr)}{\sin n\pi}\,dn. \quad . \quad . \quad (14)$$

If this is tested term by term it will be found to satisfy the differential equation and the boundary conditions. With some laborious work the trigonometry can be simplified, and the final result appears as

$$V=2i\int_{A'}\frac{\sin n(\alpha-\theta)\sin n\theta'}{\sin n\alpha}J_n(mr')\,K_n(mr)dn \quad . \quad (15)$$

$$r>r' \quad \text{and} \quad \alpha>\theta>\theta'.$$

Since in (15) there is no pole at the origin, then A and A are identical.

By Cauchy's residue theorem the whole solution is now obtainable as an infinite series for all values of θ in the space $0>\theta>\alpha$.

The series is

$$V=\frac{4\pi}{\alpha}\,.\,\Sigma\sin\frac{s\pi}{\alpha}\theta\sin\frac{s\pi}{\alpha}\theta'\,J_{\frac{s\pi}{\alpha}}(mr')\,K_{\frac{s\pi}{\alpha}}(mr), \quad . \quad (16)$$

when $r>r'$, and for the case when $r<r'$ it is only necessary to interchange these quantities.

A solution of the same problem is given by Macdonald in his book on Electric Waves, which depends on a theorem in an earlier portion of his book. That solution is in agreement with the above.

The method can be applied to three-dimensional problems, and some interesting results are being obtained which I hope soon to send in.

XX. *The Effect of Radium in Facilitating the Visible Electric Discharge in Vacuo.* *By* A. A. CAMPBELL SWINTON *.

As has been shown by Edison, Fleming, and others, the passage of the electric discharge in vacuo is much facilitated by heating the cathode.

More recently, as has been shown by Owen (Phil. Mag. viii. 1904) and by Wehnelt (Phil. Mag. x. 1905), the passage of the discharge is still further facilitated by coating the heated cathode with oxides of the alkaline metals; the effect in this case being so great that large and highly luminous discharges amounting to several amperes can, under favourable conditions, be passed through the vacuum tube, using pressures of only 30 to 300 volts.

It is generally held that the efficacy of the hot oxides in this direction is due to their giving off negatively charged ions or corpuscles.

It therefore occurred to the writer to ascertain whether similar effects could not be obtained by painting the cathode with radium, and as radium gives off corpuscles when cold, it was anticipated that it might not be necessary to heat the cathode.

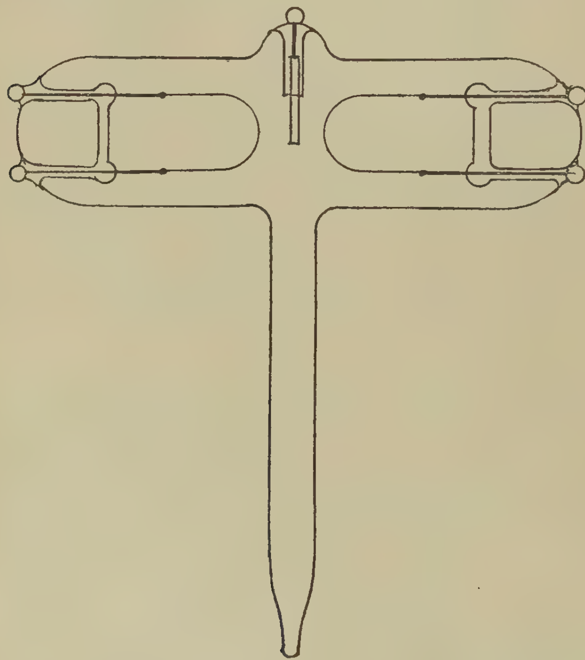
In the first experiment, however, this was found not to be the case, as with a cold cathode, and using continuous current up to 400 volts pressure, the radium did not seem to have any appreciable effect in producing a visible discharge. When, however, the radium-coated cathode was heated to redness, the radium was found to have a very marked action in facilitating the production of a luminous discharge.

In the experiments a strip of platinum foil was used for the cathode, which, before mounting in the tube, was dipped into a solution of radium bromide and dried. The amount of radium on the foil must have been extremely small, but with a suitable vacuum, and with the cathode heated to a bright red colour, the discharge passed, and the gas in the tube lighted up brightly with electrical pressures as low as about 80 volts.

In order to determine whether it was necessary that the

* Read June 22, 1906.

radium should be on the cathode itself or whether the mere presence of radium in the tube would be sufficient to produce the effect, a tube was constructed, as shown in the illustration, with two similar platinum-foil electrodes, either of which could be used as a cathode, and both being arranged so that they could be heated to redness by passing a current through them, the anode being mounted midway between the two cathodes. One of the cathodes was dipped before mounting in a solution of radium bromide, while the other was not so treated.



Under these circumstances, it was found that with pressures up to 400 volts a visible discharge passed only when the platinum strip that had been treated with radium was used as cathode and was heated to redness, and that when both platinum strips were heated to redness there was no visible discharge when the untreated strip was used as a cathode. Furthermore, it was found that the tube would only allow visible discharges to pass in the direction that made the

treated platinum strip cathode, the tube acting as a unidirectional valve in the same way as do tubes with cathodes coated with oxides.

It was not found possible to obtain as large currents through these radium tubes as through oxide tubes under similar conditions as regards voltage, degree of exhaustion, and temperature of cathode; but, having regard to the very small quantity of radium employed in comparison with the quantities of oxide, this is not surprising.

Experiments were next made without any heating of the cathode, but using alternating currents of higher voltages than were available in continuous current.

Employing the tube with one central electrode and two symmetrically placed loop electrodes, one of the latter being treated with radium and the other being plain, the following results were obtained, the voltages being measured by means of a Duddell thermo-galvanometer.

It was found that, using the untreated electrode, it required from 800 to 900 volts to get a visible discharge to pass; whereas, using the radium treated electrode, a visible discharge could be got to pass with from 700 to 800 volts. The exact voltages required in each case to cause a visible discharge were somewhat variable; but it was always found that a visible discharge could be got to pass, using the radium treated electrode, with about 100 volts less than when the untreated electrode was employed. It was further very noticeable that, using the treated electrode and gradually reducing the voltage, a much fainter luminosity of discharge could be obtained without actual extinction of the discharge than was the case when the untreated electrode was employed, in which case the transition from a visible discharge to no luminosity was much more abrupt. This fact goes to show that the differences in the minimum voltage required to produce a visible discharge in the two cases was actually due to the presence of radium, and not to any slight difference in the distances between the electrodes or other want of symmetry in the tube.

A d'Arsonval mirror galvanometer was next inserted in the circuit. With a plain alternating current this galvanometer would, of course, give no deflexion. It was, however,

found that whether the radium treated or the untreated electrode was used, there was always a slight deflexion of the galvanometer, due no doubt to some unidirectional valve action on the part of the vacuum-tube. It was found, however, that using the treated electrode, the galvanometer deflexion was between two to three times as great as when the untreated electrode was used, this being evidence that the presence of radium increased the valve action to a considerable extent.

The thermo-galvanometer was next substituted for the d'Arsonval mirror galvanometer, and it was then found that, using the radium treated electrode, the amount of current that passed through the tube was from one and a half times to twice as large as when the untreated electrode was employed, this showing that the presence of radium on the cathode materially increases the amount of current that passes through the tube with any given voltage.

The writer is indebted to Mr. J. C. M. Stanton and Mr. R. C. Pierce for their assistance in making the above investigations.

XXI. *The Effect of Electrical Oscillations on Iron in a Magnetic Field.* By W. H. Eccles, *D.Sc., A.R.C.S.**

THE following investigation of the action of high-frequency oscillations upon iron held magnetized in a magnetic field was carried through in July and August of last year, and was undertaken to supplement in a quantitative manner the rather vague information available. Since Marconi † announced in 1902 that feeble electrical oscillations were capable of altering the flux of induction in a piece of magnetized soft iron, a great deal of work has been done in the endeavour to determine what it is that really happens in this process. A glance through the chronicles of these labours reveals a surprising lack of precise data, and shows, in fact, that most of our knowledge of this subject is merely qualitative. This is due, no doubt, to the complicated possibilities

* Read June 22, 1906.

† Roy. Soc. Proc. lxx. p. 341, July 1902.

accompanying any experimental method that can be devised. One, for instance, among these difficulties, is the shielding of the greater mass of an iron wire, or other sample, by the eddy currents that are generated in the metal by the oscillations. This skin-effect was, in a way, overcome by Maurain by using oscillations sufficiently vigorous to penetrate to the very core of the iron or steel wire experimented upon. By this means he has shown that if the I H curve of a sample be drawn in the ordinary manner while these very vigorous oscillations are in operation, the curve obtained is not looped but is the single path obtained by the coalescence, as it were, of the branches due to increasing and decreasing fields. The single curve thus obtained he called, after Duhem, the "normal curve of magnetization." Duhem had previously shown from thermodynamical reasoning that the effect of a long train of slowly damped oscillations superposed on iron at any stage of the cyclic process would be to bring the representative point on the I H diagram towards a unique locus—a locus, he concluded, exactly the same as that obtained by mechanically agitating the iron throughout the cycle. This theory of Duhem's takes no account, of course, of the shielding of the inner parts of the iron from the action of the oscillations; and thus cannot be demonstrated experimentally except by using, as Maurain* has done, very powerful oscillations.

In the experiments to be described, an endeavour has been made to turn the difficulty arising from the skin-effect. Oscillations so feeble have been used that they affected only the outermost layers of the iron wires employed. The cores of the iron wires have therefore not been used.

Other and great difficulties arise in the matter of producing oscillations of determinate and invariable character. Maurain, in the greater part of his work on this subject, appears to have used the oscillations that passed through a helix in series with a Leyden jar kept sparking strongly and continuously by means of an induction-coil. Russell †, in some recent experiments, applied to his iron the oscillations passing through a coil connected directly in series with a small

* Maurain gets slightly different curves by mechanical agitation and by oscillations.

† Proc. Roy. Soc. of Edin. Nov. 20, 1905.

induction-coil. This last method appears to the writer to subject the iron to very violent treatment of a nature not easily described accurately; for how far the mere surgings of secondary current overwhelm in importance the genuine oscillations, it will be difficult to say. Piola *, again, worked with highly damped oscillations, because, he found, such oscillations produce the highest effects. He has, indeed, following Rutherford, used this fact in determining the damping factor of a circuit. But in the present investigation these sources of indeterminateness have, as far as might be, been avoided by using a single train of waves instead of a continued torrent of such trains, and by using oscillations as little damped as possible.

It would not be to the point to extend this brief survey of previous work till it included that of Walter, of Ascoli, of Arnò, and of Foley; for all these have paid especial attention to oscillations whose magnetic field was in some degree transverse to the main magnetic field. The present paper deals solely with oscillations whose magnetic field is along the direction of the principal field.

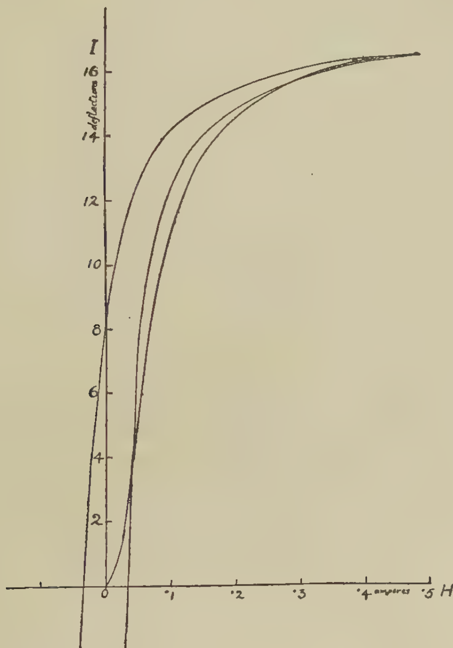
In these experiments fairly soft Swedish charcoal iron, aged, and not freshly annealed, was taken a large number of times round any chosen magnetic cycle (as in ordinary magnetic testing) till what may be called a cyclic state was attained. The field was then given any desired value, and the iron submitted to a single train of oscillations. This was managed by generating the oscillations on a helix surrounding the iron wire. The consequent alteration in pole strength was observed by the deflexion of a magnetometer mirror. These processes were all repeated a number of times for each selected point of the cyclic curve. The figures given in the tables below are thus each the mean of a number of observations.

In detail, the apparatus adopted consisted of two straight solenoids each of 3270 turns of No. 20 copper wire wound in 6 layers on a length 59 cm. of split brass tube. They were placed horizontally on one and the same magnetic east-west line, but on either side of a mirror magnetometer. The magnetometer-needle was in the common axis of the two solenoids.

* *Elettricista*, iv. p. 145, May 15, 1905.

The solenoids were connected in series, and so adjusted in position that when a large current was passed through them the magnetometer-needle was undisturbed. A liquid resistance, a Weston milliammeter, suitable switches and commutators, and a battery of six secondary cells, completed the solenoid circuit. The coil destined to be the seat of the oscillations consisted of 1252 turns of No. 26 copper wire wound in a single layer on 70 cm. of a glass tube 0·5 cm. in external

Fig. 1.



diameter, and had a resistance of about 3 ohms. The whole coil was wrapped in paraffined paper and pushed into the brass tube of the east solenoid.

The iron wire examined was unannealled Swedish charcoal iron, diameter 0·749 mm., and was used always in lengths of 56 cm. Its characteristic curve is given in fig. 1. The magnetometer was a silk-suspended mirror carrying four very small magnets ; readings were taken on a scale distant 88 cm.

The object in employing twin solenoids is clear. By placing equal amounts of iron wire in each solenoid and adjusting their positions carefully, the magnetometer deflexion could be kept very slight whatever magnetic variations the iron was taken through, the iron in the one selenoid compensating that in the other.

It was then permissible to exalt greatly the sensibility of the magnetometer. This was done by reducing the controlling field at the needle by means of an auxiliary permanent magnet. As a matter of fact, slight inequalities existing in the construction of the solenoids made perfect balance unattainable; there was always a remanent deflexion, which was different in sign and magnitude for different values of the current through the solenoids, and which in the main proved rather useful by furnishing a check on the proper carrying out of the commutating operations involved in a magnetic cycle.

In the experiments, three iron wires insulated from one another and tied in a bundle were used in each solenoid, the group in the east coil lying, of course, in the glass tube on which the oscillation-coil was wound. The inner ends of the wires were 9·3 cm. distant from the magnetometer-needle. The field at the needle was reduced to 0·034 C.G.S. units. With these arrangements, such a sensibility was attained that on certain days of August last the strokes of distant lightning, not visible in London *, were easily perceptible in the laboratory by their effects on the iron in the oscillation-coil.

The oscillations used were produced by leading the free east end of the oscillation-coil, that is the end distant from the magnetometer, to one side of a micrometer spark-gap. The other end of the oscillation-coil was left insulated. The poles of the spark-gap were connected to the terminals of a diminutive influence machine, that pole not connected to the oscillation-coil being, besides, earthed. Thus, when the handle of the influence machine was turned through a certain angle (depending on the spark-gap) at a speed easily learned, a spark occurred which set up oscillations in the coil. The spark-length finally settled upon was about half a millimetre. It was found possible in this way to get over and over again practically the same magnetometer deflexion for every spark, provided the effect of previous oscillations was wiped out by

* The newspapers gave accounts of thunder-storms in Hertfordshire.

taking the iron through a cycle. As these small sparks were usually inaudible, their occurrence was recognized by the sudden deflexion of the magnetometer-needle.

The calculated period of the stationary waves on the oscillation-coil is $5{\cdot}7 \times 10^{-7}$ second.

A typical set of operations was as follows. The iron was demagnetized by reversals. An amplitude of cycle being then decided upon, the iron was taken a number of times through that cycle. This last operation was stopped at the point settled upon for the observations, and the magnetometer reading taken. A spark was passed, and the magnetometer again read after 30 seconds. A complete cycle was now performed ending at the same point as before, and another spark passed, and so on. Usually, about twenty points on a cycle were examined by 4 or 6 observations at each. In this paper, however, the effects of the spark are recorded as if the observations had been taken only on the ascending half of the hysteresis curve; the figures given being in fact the means of the measured effects at points symmetrical with regard to the origin on the ascending and descending branches.

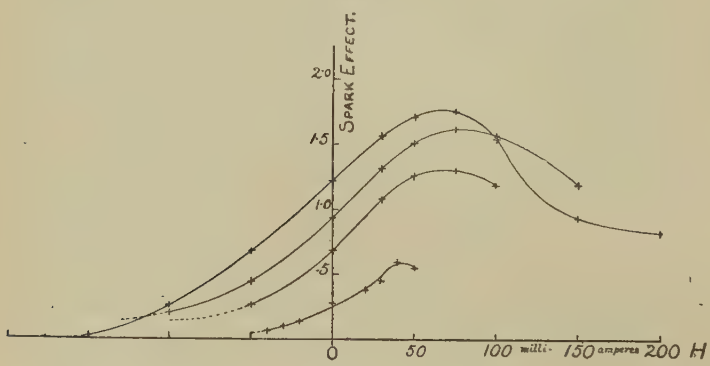
TABLE I.

Current in milliamperes.	−150	−100	−75	−50	−40	−30	−20	0	20	30	40	50	75	100	150	200
50 cycle	...	...	...	...	0·06	0·10	0·13	0·28	0·38	0·45	0·60	0·55				
100 cycle ...	...	...	...	0·27	...	...	...	0·68	1·08	...	...	1·26	1·30	1·18		
150 cycle ...	...	0·20	...	0·44	...	...	...	0·92	...	1·32	...	1·51	1·62	1·57	1·19	
200 cycle ...	0·20	0·36	...	0·68	...	...	...	1·22	...	1·57	...	1·71	1·75	1·55	0·93	0·83

In this table the figures given can be reduced to absolute measure as follows:—The numbers at the top of each column and down the left column, representing the current through the solenoids in milliamperes, yield when multiplied by 0·0696 the magnetic field in C.G.S. units applied to the iron. The deflexions in the body of the table give the change in pole-strength, due to the spark, by multiplying by 0·017; or yield the volume-average change of intensity of magnetization by the factor 1·27; or give change of total magnetic moment of the affected specimen by the reducing factor 0·940.

Fig. 2 is plotted from the above table. The curves show clearly how for increasing cyclic amplitudes, within the range here attempted, the effect of the same spark is increased. The curves show very distinct maxima. It is evident, moreover, that the magnitude of the effect at any point is closely connected with the slope of the hysteresis curve. On the whole the curves tend to corroborate the fact—first noted by

Fig. 2.



E. Wilson in his 1902 patent specification—that the sensibility of the iron to oscillations is greatest when in the magnetic condition represented by the point of inflexion on the hysteresis curve. To examine this matter more closely these curves representing the effect of the spark are repeated in figs. 3, 4, 5, 6, each alongside the gradient curve, that is, first derived curve, of the corresponding cyclic curve. To assist in picturing the magnetic state of the iron there is plotted on each diagram the proper hysteresis loop. These loops were obtained by independent experiments with the earth's field controlling the magnetometer-needle. (Reducing factor for I is 67·5).

In these figures the abscissæ for all the curves are milliamperes (multiply by 0·0696 to get C.G.S. field). The figures have been so constructed that, except in the cyclic curves, equal ordinates on the different figures represent absolutely equal effects. The reducing factors for the spark-effect curves are as given in connexion with Table I. In all the slope curves the ordinates give at once very approximately

Fig. 3.—The dotted curve shows spark-effect.

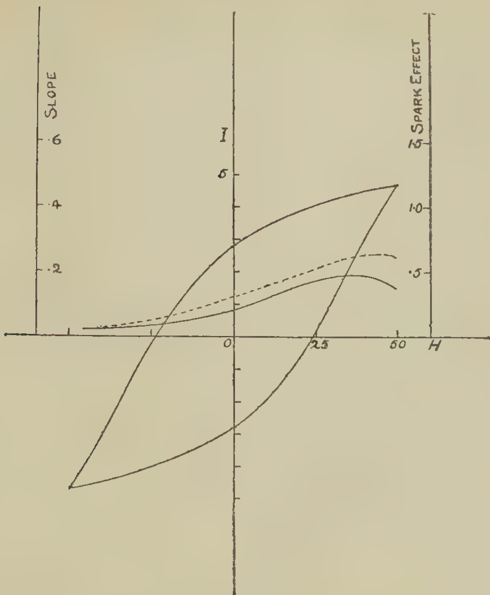


Fig. 4.—The dotted curve shows spark-effect.

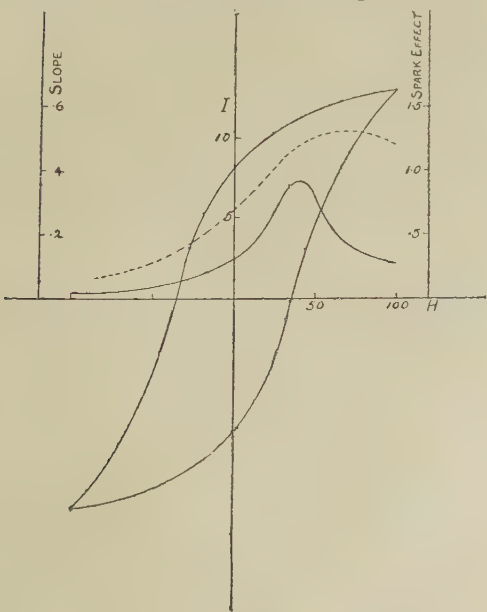


Fig. 5.—The dotted curve shows spark-effect.

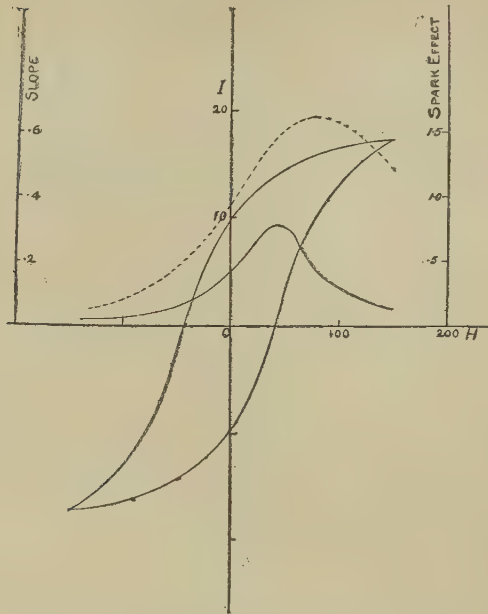
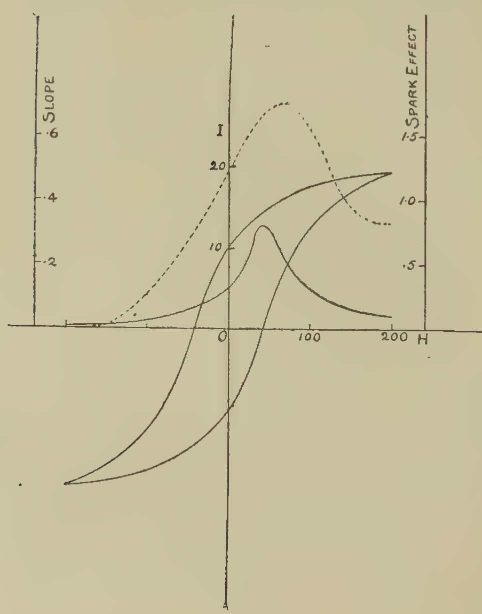


Fig. 6.—The dotted curve shows spark-effect.



the change of intensity of magnetization per ·001 C.G.S. increment of field. But the ordinates of the cyclic curve are reduced 21 times in fig. 3, 42 times in fig. 4, 63 times in fig. 5, 84 times in fig. 6; and are thus not directly comparable with one another or with the companion curves.

Returning to the spark-effect and the gradient curves, however, it is seen that the spark-effect is by no means a simple function of the gradient of the cyclic curve. The curves show, moreover, that the maximum effect of the spark is greater in the cycles of larger amplitude; and not only because the hysteresis curves are steeper in large cycles than in small, but also because in small cycles the effect of the oscillations is, for some unknown reason, much smaller than the diminished gradient would lead one to expect.

It must be mentioned here that during the above measurements the effect of changing the sign of the initial charge given to the oscillation-coil was repeatedly tried. On no occasion was there any perceptible difference in the magnetometer deflexion. The damping of the oscillations must therefore have been very trifling.

A matter quite different, but one of considerable interest, is examined in the following short table of the effect on the magnetometer deflexion of varying the length of the spark-gap from which the oscillation-coil was excited. The observations here recorded were all made at that point of the 0·200 ampere cycle where the magnetic field is zero; observations made at other points of this or other cycles gave results the same in character.

Table II.

Length of Spark-gap in centimetres.	Magnetometer Deflexions.
0·024	0·8
039	0·8
053	1·1
070	2·6
087	4·4

This table is plotted in fig. 7. On the same diagram appears the curve of Paschen's* observations of potential difference and spark-length.

* Table in J. J. Thomson's 'Recent Researches,' p. 78.

The most important question to which the results set forth in this paper can be applied, is that bearing on the real nature of the phenomena observed. A number of observers have summed up their views by the phrase "hysteresis is annulled by oscillations." The present experiments at first sight bear out that conclusion anew. The effect of the oscillations at every point of the cyclic curve is such as to carry the representative point towards the curve that would be yielded by a substance devoid of hysteresis. This might be interpreted in the same manner as is, usually, the effect of mechanical vibration on iron undergoing varying magnetic forces—the intermolecular bonds that hold the magnetic molecules in their instantaneous configuration are loosened for a moment so that the resultant field can have full play in rearranging the molecular groupings. Looking further into the matter, however, on this view that hysteresis is temporarily annulled, it is evident that on the ascending loop of the cyclic curve the external applied field is permitted to do work when the magnetic shaking occurs. On the other hand, similar alterations of intensity of magnetization occur when the applied field is zero; and, in addition, on the descending loop of the cyclic curve there are alterations in intensity opposite in sense to the applied field. It seems clear, then, that there must exist in the magnetized material intrinsic forces that tend to drag the substance into that presumably more stable magnetic state defined by the "normal curve of magnetization" before mentioned. Thus this view that hysteresis is annulled, taken with the experimental facts of this paper, implies that the store of intrinsic energy of magnetization may possibly be allowed to run down through and during the action of the oscillations. In other words, the action of the oscillations may be analogous to the pulling of a trigger. These considerations threaten to lead us into the often discussed question of the intrinsic energy of a magnetized material having hysteresis. In the present case there is, besides, the difficulty arising from the fact that different layers of the wire used are doubtless differently affected. Notwithstanding, it is perhaps worth while attempting to answer the question: Is the alteration of stored magnetic energy greater than the initial energy of the oscillations producing the alteration?

The latest word on the subject of the intrinsic energy of magnetized iron is contained in a paper by Larmor*. There the expression $2\pi\int I^2 d\tau$, where I is the intensity of magnetization and $d\tau$ an element of volume, is deduced for the part of the energy we are here concerned with. Taking the observations recorded in this paper for that point of the ·200 ampere cycle where the field is zero, the wastage of the intrinsic energy instigated by a spark is, if it be legitimate to apply here this formula, 3600 ergs. Now the initial energy of the oscillations, which exists as the electrostatic energy of the charge on the oscillation-coil just before the spark-gap breaks down, is about 5700 ergs. These figures are too close to enable a decisive conclusion to be reached; that is, to say finally whether or not any considerable amount of magnetic energy is available and drawn upon. Allowing mentally for the oscillation energy lost in Joule effects and in radiation, it seems probable that magnetic work is done by the oscillations—whose efforts, therefore, would appear to be attended with hysteresis loss, just as are those of slower variations of magnetic field.

Fig. 7.

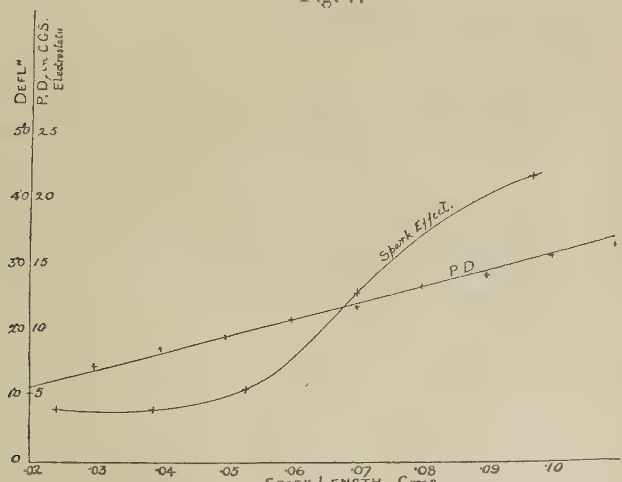


Table II. and fig. 7 give, in some degree, support to this conclusion. The bent (experimental) curve may be taken as representing the magnetic work done by sparks of length

* Roy. Soc. Proc. lxxi. p. 229, Feb. 1903.

shown by the abscissæ, other circumstances being the same. The almost straight line represents the potential to which the oscillation-coil was charged at the corresponding spark-length. The relation between the two curves shows that for some portion, at any rate, of the diagram the magnetic work done varies as the square of the charging potential, that is, the electrostatic energy of the initial charge. The bending away of the first curve towards the right is explicable by the consideration, that as the applied oscillations get more violent the outermost layers of the iron wire attain, so to speak, a saturated condition; that is, become incapable of being more greatly impressed—with the result that the magnetic effect of the oscillations begins to fall off in amount. It was for this reason that in the experiments of Table I. the spark-gap was kept at the value 0·53 millimetre, well within the part of the curve free from this saturation effect.

I have to thank my colleague Mr. J. Lister for deducing the gradient curves and drawing all the figures in this paper.

Discussion.

Mr. A. Campbell expressed his interest in the paper, and asked for the magnitude of the pole-strength produced by the passage of a spark.

Mr. Walter congratulated the Author upon the method of the research, and said that the precautions which had been taken rendered the results free from suspicion.

Dr. Eccles, in reply to Mr. Campbell, said that the deflexions produced by the change in pole-strength consequent on the passage of a spark were about $\frac{1}{84}$ of the maximum deflexions obtained in the hysteresis curves.

XXII. *The Disruptive Voltage of Thin Liquid Films between Iridio-Platinum Electrodes.* Part I. *Voltages* 25–400. *By* P. E. Shaw, *B.A., D.Sc.**

Electric-touch measurements can readily be made under oil. If a telephone be included in the circuit, the sounds made in contact are sharp and clear for all the liquids mentioned below. They even excel air in this respect. This property of oils to render contact distinct has been used in potentiometers and other instruments. The whole instrument, or at least the slide-wire, is immersed in paraffin oil (see a paper by R. A. Lehfeldt and discussion following, in Phys. Soc. Proc. p. 479, 1901–1903). This previous use of oils suggested the present research.

The relation of spark-length to small P.D. of the electrodes has been experimentally tested for gases by Earhart (Phil. Mag. [6] i. p. 147, 1901), Shaw (Proc. Roy. Soc. vol. lxxiii. p. 337, 1903), Kinsley (Phil. Mag. May 1905, p. 692), and Hobbs (Phil. Mag. [6] x. Dec. 1905).

Earhart discovered the knee occurring in the curve at 350 volts. Shaw pushed the investigation further by using small P.D. down to ½ volt. Kinsley threw doubt on the constancy of potential for a given gas and given electrodes. Hobbs, working more exhaustively than his predecessors, has added much definite and useful information.

The discharge in gases must bear a close relationship to the discharge in liquids, but little precise work has been done on the latter, though Hughes (Proc. Inst. Elec. Eng. 1892), Swinton (Discussion on the above paper), Steinmetz (Trans. Amer. Inst. Elec. Eng. 1893), before 1903, and recently, C. E. Skinner (Nat. Elec. Light Assoc., Boston, Mass., 1904), Przibram (*Akad. Wiss. Wien*, *Sitz.-Ber.* Nov. 1904), and Voege (*Elektrotechn. Zeitschr.* Dec. 1904) have tested the relationship for spark-lengths greater than $\frac{1}{20}$ mm.

In the following paper the P.D. varies from 25 volts to 400 volts, and the corresponding spark-lengths vary from $\frac{1}{10}$ micron (micron = $\frac{1}{1000}$ mm.) to about 10 microns.

Recently a book on the subject has been published giving

* Read June 22, 1906.

useful information ('The Insulation of Electric Machines,' by H. W. Turner and H. M. Hobart, Whitaker & Co., Paternoster Sq., E.C.). It is made clear in this book that water, acids, and alkalies are very destructive of the insulating powers of oils; thus Skinner has shown that $\frac{1}{50}$ per cent. of water reduces the dielectric strength of transformer oil to half its value when quite dry.

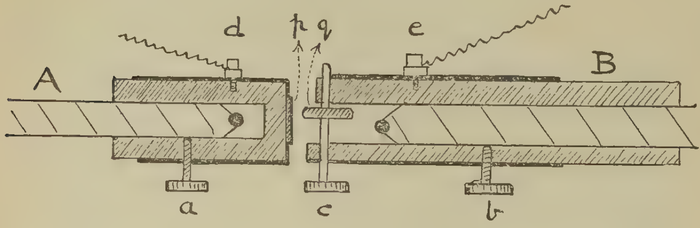
In the present work all the commercial liquids used were shaken up and allowed to remain over potassium carbonate, filtered, and then heated to 110° for several hours, then kept in a glass-stoppered bottle till used.

Method.

In a research by the present writer on the sparking-distance in air (*loc. cit.*), the electric micrometer with levers was used to measure spark-length; in the present case the electric micrometer, without levers, is used. It has been already described (Proc. Roy. Soc. April 1906) for measuring gauges; it has proved very convenient for the present experiments.

The only parts of the apparatus which need be here mentioned can be understood from fig. 1. Two spindles, A and B, are in one line, and can be worked to or from one another

Fig. 1.



by micrometer-screws not shown, and their distance apart can be read and controlled to $\frac{1}{100}\,\mu$ (*i. e.*, $\frac{1}{100,000}$ mm.). The divisions on the micrometer-head show movements of $1\,\mu$, and these, when viewed by a microscope, can be each subdivided into 100 parts.

On A and B are fitted two caps of ebonite clamped on by screws a, b. Mounted on each ebonite is a sheath of steel.

On the left cap is a plate of iridio-platinum p, and on the right is a disk of iridio-platinum q, set on a spindle turning on an axis in the plane of the paper and perpendicular to B. This disk projects, as shown, so that when B advances to A, the surfaces q and p are brought into contact. Since q is a diametral section of a sphere, its edge touches the plate p at a point. Discharge is made to occur between p and q. By causing p to rotate on a horizontal axis along A and q to rotate on a horizontal axis perpendicular to B, fresh surfaces of p and q can be presented to one another continually, since q touches p not at the centre.

The P.D. on surfaces p, q is maintained by connecting wires to terminals d e. The full circuit is shown in fig. 2. Here a 400-volt lighting circuit is joined through key K_4 to a water resistance. Between the fixed zinc electrodes Z_1, Z_3 is the full P.D., while between the moveable Z_2 and the fixed Z_1 is the P.D., say V, used in discharging the apparatus. The leads to the electrodes Z_2 and Z_3 are insulated from the liquids by being enclosed in glass tubing.

If switches K_2 and K_3 are put over to the right, the above voltage V will act through the voltmeter V_0 or on the discharge surfaces p, q according as the keys K_5 and K_6 are put over to the left or the right.

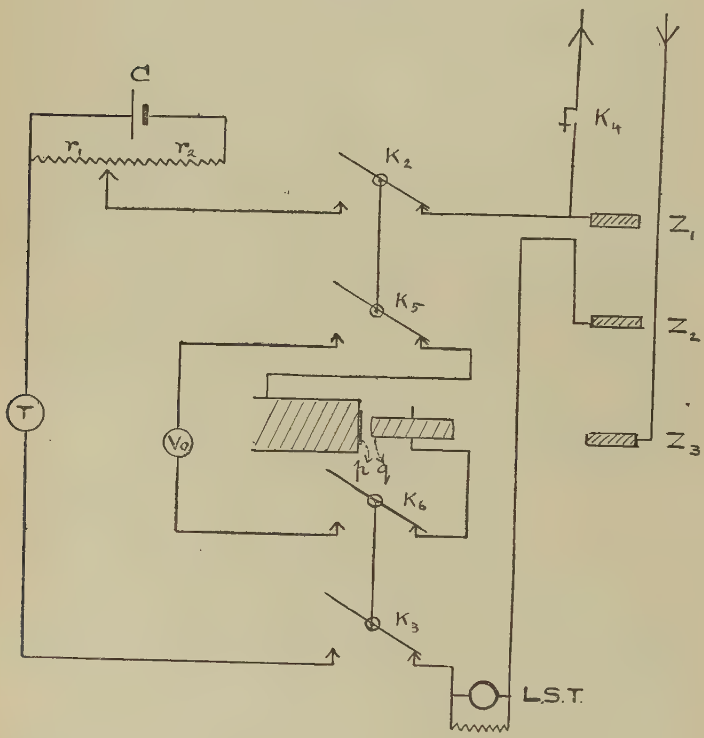
It is easy to vary the P.D. used by moving Z_2.

Another circuit is on the left and is brought into action by putting switches K_2 and K_3 over to the left, whilst K_5 and K_6 are kept to the right. The single cell C has resistance r_1, say 100 ohms, and r_2, say 4000 ohms, in circuit. The telephone T is in the circuit and sounds when p and q touch. This circuit is used to determine the position of contact of p and q, the P.D. between them being very small, in this case $\frac{1}{40}$ volt. The actual discharge for high potential experiments, say 100 volts, from the right circuit, takes place when p and q are separated by a small gap; but the discharge-distance for the small voltage $\frac{1}{40}$ volt of the left circuit is so minute that it can be ignored. The above apparatus can be used for discharge through gases, liquids, or solids.

The *modus operandi* for liquids is as follows:—The caps on spindles A B are removed from the spindles, the iridio-platinum surfaces ground with finest emery to remove the

old surface, polished with rouge or leather, and cleaned. The caps are placed on A and B and p, q made to approach. One drop of the liquid to be tested is placed on q, and as p approaches q the liquid settles by capillarity between p and q. Then the spindle B is moved slightly away, the liquid being thus drawn out into a thread connecting the surfaces, and there is a guarantee that liquid without air-bubble connects

Fig. 2.



the surfaces. The switches K_2 and K_3 are put to the left, and K_5 and K_6 put over to the right. If now p is advanced and touches q, the telephone T sounds ; the reading of the divided head of the right micrometer-wheel is taken. This is the first reading. The surface q is drawn away from p about twice as far as the spark-length about to occur. Switches K_2 and K_3 are put to the right, and K_5 and K_6 put

to the left. The position of Z_2 is arranged so that the voltmeter shows the desired P.D. Now K_5 and K_6 are put over to the right, and q is made *very slowly* to approach p, the micrometer-scale being watched so that at the moment when discharge occurs and is indicated by the sounding of telephone L.S.T., the scale is read a second time. The discharge-distance is the difference between the first and second readings. These distances are entered in the tables following in units which are 0·01 μ (μ = micron = $\frac{1}{1000}$ mm.).

The surfaces of p, q presented to one another are renewed, as shown above, by rotations and another discharge is made.

Most of the liquids used are of commercial quality, and are purified as shown above; others were obtained specially pure. The following liquids were tested :—

Vegetable Oils.—Olive oil, castor oil, linseed oil, rape oil. Also turpentine, fusel oil, and oil of resin.

Animal Oils.—Cod-liver oil, neats-foot oil.

Mineral Oils.—Paraffin, transformer oil.

A Homologous Series.—Pentane (C_5H_{12}); hexane (C_6H_{14}); heptane (C_7H_{16}); octane (C_8H_{18}). These were obtained from C. A. F. Kahlbaum, Berlin. Hexane and octane were specially pure, being derived from propyl iodide and octyl iodide respectively; while heptane and pentane were obtained from petroleum.

Varnishes.—Of these armacell and ohmaline are obtained from Griffiths Bros., Bermondsey, London; while Sterling varnish is made by The Sterling Varnish Co., Manchester. Though these are made to insulate when dry, they all have great dielectric strength when liquid, as shown in the curves (fig. 5, p. 297).

As all experimenters on gaseous discharge have found, the discharge-distance is an uncertain measurement. The same order of error occurs for liquids. In the case of gases, dust or want of polish increases the distance of discharge for a given P.D. In liquids, solid or flocculent substances or minute air-bubbles may be floating, or the presence of water, acid, alkali, or salts will set up electrolysis and change the discharge-distance.

TABLE I.

l = spark-length stated in units of $0{\cdot}01\mu$.

P.D. in volts.	l.						
	Olive Oil.	Linseed Oil.	Castor Oil.	Rape Oil	Turpentine.	Fusel Oil.	Air.
375	560	550	570	...	...	550	1300
350	510	475	520	610	520	...	1100
300	445	350	400	430	420	...	160.250 480.320
275	...	...	...	...	...	...	120
250	400	300	350	400	340	300	118
200	...	220	...	300	220	255	100
186	340	...	318	...	165		
150	290	190	275	220	120	205	80
125	235	150	210	...	100	...	48
100	200	95	170	115	90	150	46
75	145	...	150	115	80	110	36
70	140						
68	115						
66	93						
62	80	...	110				
50	60	56	58	45	55	80	14
25	27	27	35	...	27	35	8
10	11						

TABLE II.

P.D. in volts.	l.						
	Paraffin Oil.	Transformer Oil.	Cod-liver Oil.	Pentane.	Hexane.	Heptane.	Octane.
390	...	...	450				
345	breaks	breaks	395	400	260	320	350
300	down	down.	350	250	...	260	310
250	readily.	...	290	220	220	275	
200	...	350	...	170	220	235	240
175	350						
150	300	210	175	145	170	140	180
125	220	170	150				
100	180	110	120	90	150	100	105
75	155	80					
62	125						
56	95						
50	75	70	50	50	80	50	
25	38						

In the curves below (figs. 3, 4, 5) some liquids may be represented roughly by straight lines; other liquids certainly require

irregular curves for proper representation. They may be classed as follows :—

Straight Curves.	*Irregular Curves.*
Linseed Oil.	Paraffin Oil.
Rape "	Transformer "
Fusel "	Castor "
Cod-liver "	Olive "
Pentane.	Turpentine.
Octane.	Hexane.
Armacell Varnish.	Heptane.
Ohmaline "	
Sterling "	

Though this classification is convenient, there is no sharp

Fig. 3.

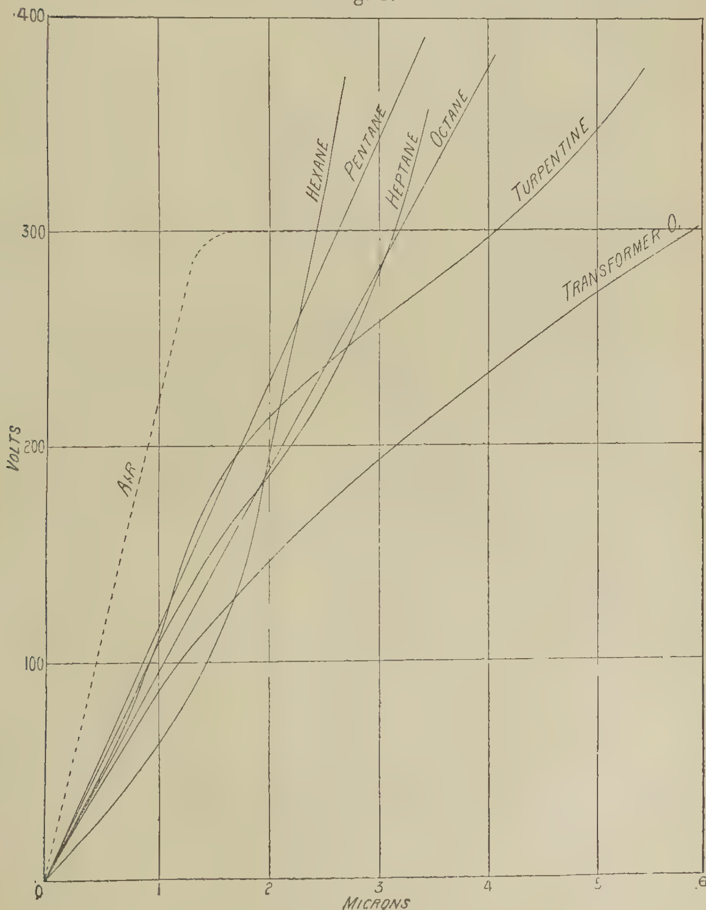


Fig. 4.

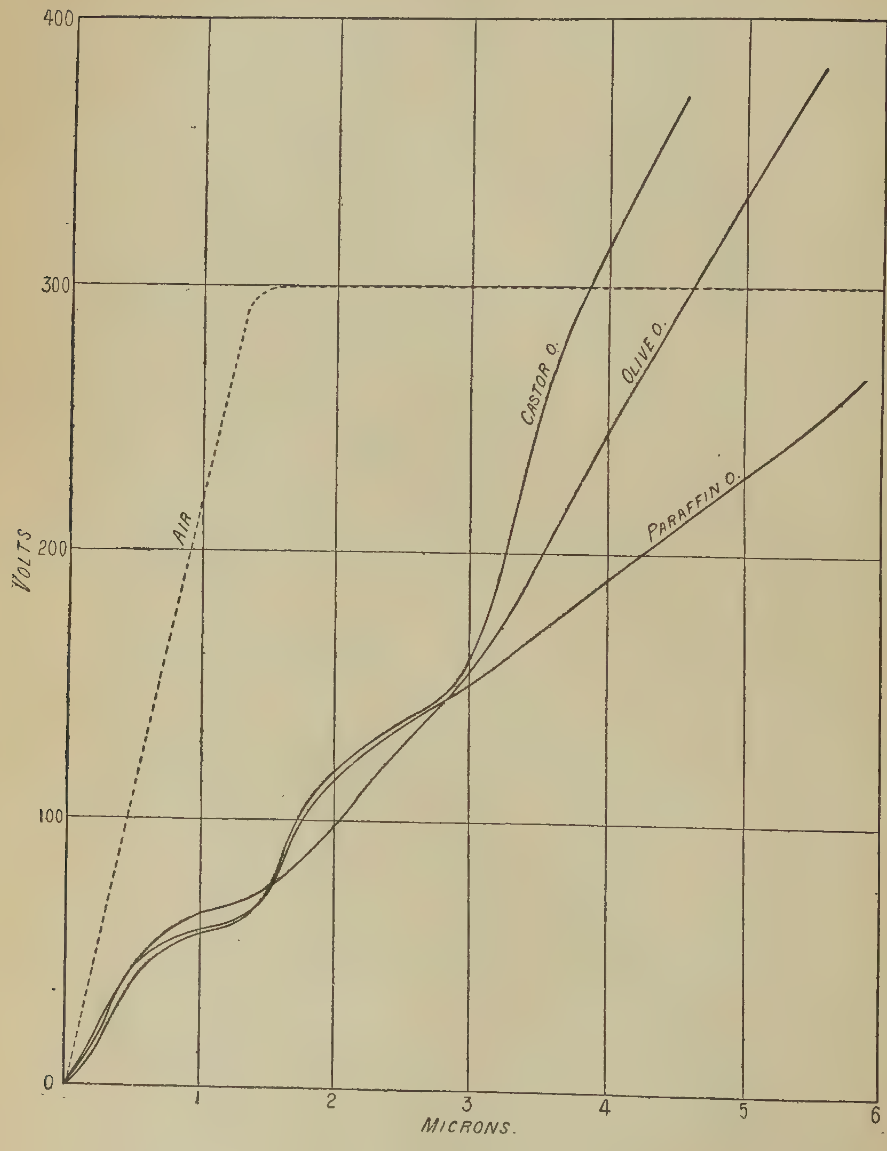
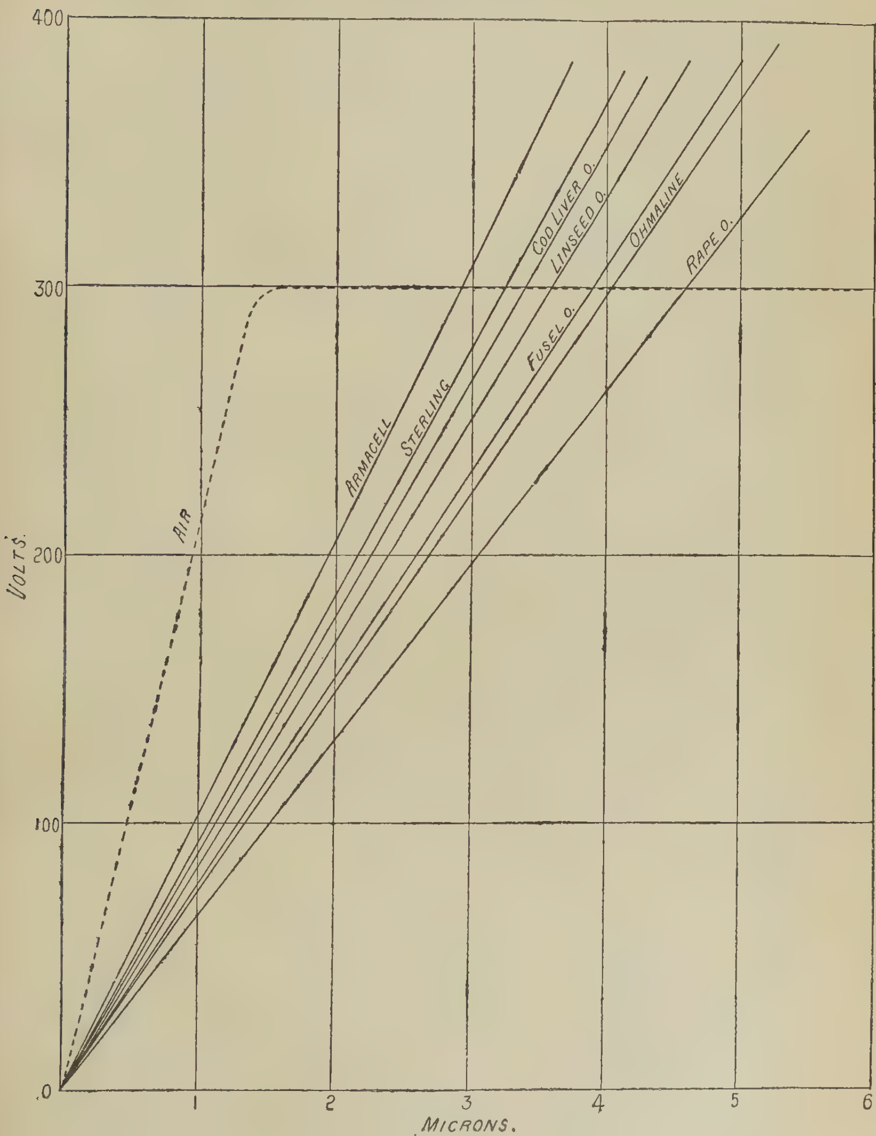

400
300
200
100
0
VOLTS
AIR
CASTOR O.
OLIVE O.
PARAFFIN O.
1
2
3
4
5
6
MICRONS.

Fig. 5.


400
300
200
100
0
1
2
3
4
5
6
VOLTS.
MICRONS.
AIR
ARMACELL
STERLING
COD LIVER O.
LINSEED O.
FUSEL O.
OHMALINE
RAPE O.

demarcation between the two classes above. Only those liquids are placed in the second class for which it would be absurd to suppose the departure from a straight line could be due to experimental errors.

In the cases of castor oil, olive oil, and paraffin there appears to be a distinct and similar variation in potential gradient for discharge-distances between $1\,\mu$ and $2\,\mu$ (see fig. 4).

The air curve is given for reference.

Hobbs (*loc. cit.*) has shown that potential gradient near the origin depends on the metal of which the electrodes are made, every metal having a different gradient from every other.

The electrodes used throughout these researches are iridio-platinum. The gradient for this alloy is steeper (220 volts per micron) than any found by Hobbs, though he found platinum to have a high gradient (160 volts per micron).

The "knee" occurs at 300 volts, the horizontal part extending to 8μ, *i. e.* well to the right of the plotting shown. After 8μ the gradient is very small. Air has greater strength than any liquid for small distances; but the liquid curve in all cases crosses the air curve for discharge lengths varying between $2\frac{1}{2}\mu$ and 6μ. That is, for any greater distances the liquids have much greater strength than air.

The order of strength for the higher voltages used is roughly:—

Hexane, heptane, armacell, pentane, octane, Sterling varnish, cod-liver oil, linseed oil, fusel oil, ohmaline, castor oil, turpentine, olive oil, rape oil. Paraffin and transformer oil are both weak. Resin oil gave very irregular results, not showing, as might be expected, great strength.

Notes and Precautions.

1. The voltmeter was carefully standardized.

2. The loud-speaking telephone in the discharge circuit was shunted with non-inductive resistance of a few ohms. Some experimenters in previous researches have kept the discharge surfaces shunted with a voltmeter, so that the discharge is shown by the fall of the voltmeter-pointer. This seems faulty; for when discharge commences, it will be at once violently retarded by an inductive rush from the voltmeter. In the present work the shunted telephone of low

resistance indicates discharge, the voltmeter being then cut out and no inductive resistance left in circuit.

3. The members of the homologous series are very volatile; the method employed is to cause drops of them to fall slowly on the discharge surfaces, a fresh drop arriving before the last has evaporated.

4. The discharge is of the nature of a volcanic eruption, the cathode loses material and a distinct pit or crater forms there, while fine dust is scattered on both electrodes. The surfaces approach, but do not always meet on discharge. To avoid dust the surfaces are wiped after each discharge.

5. The discharge surfaces should be treated with care whilst the zero is being obtained. "Make" and "break" should be made as little as possible, and the zero should not be much overrun. The highly polished surfaces, even of the very hard iridio-platinum, are easily damaged for discharge work.

6. If a contact is "wheezy," it is rejected at once; this indicates bad polish, and always produces electrolysis or premature discharge.

7. The contact sound is invariably clearer and sharper for liquids than for air, and discharge is more definite. Air discharge often occurs partially, *i. e.* there is a rush of material from the cathode and the contacts approach one another but do not touch. Discharge does not continue; in fact, this partial discharge seems to retain the tension and for a time stop the complete and final breakdown.

Theory.

Very short spark-lengths in gases have now been studied by Earhart, Shaw, Kinsley, and Hobbs. The accumulated results of these and the present experiments on short spark-lengths in liquids should lead inductively to a theory of the conditions of matter at and between the discharge surfaces. Recently T. Schwedoff (*Ann. d. Physik*, pp. 918–934, April 1906) has propounded a ballistic theory of the spark-discharge. The fundamental principle he uses is that the discharge between the surfaces is due to the shattering of the molecules of the gas into free ions; this occurs when the electric strength between the electrodes jointly with the kinetic energy of the ions already existing free, reaches sufficient magnitude. He uses the ballistic law of resistance, in which

the frictional force opposing the passage of an ion depends on its speed and on the density of the gas. An expression is finally derived

$$v = \frac{300h_0 \,.\, 10^{-4} \,.\, l}{1 + \frac{c}{\lambda}\left(1 - e^{\frac{l\delta}{c}}\right)},$$

where v = potential difference on the surfaces;

l = spark-length;

δ = density of the gas;

h_0, c, λ are constants.

The values of these constants are derived from consideration of the experimental results of Earhart and Shaw. Putting in the constants and the density of the gas, the P.D. required for any given spark-length is derived.

Schwedoff thus obtains a curve which fairly closely follows that of Earhart, having a knee in the curve at voltage about 350.

At first sight the similarity between Earhart's experimental curve and Schwedoff's theoretical curve appears to be a striking confirmation of the theory; but careful comparison shows that the two curves are far from coincident near the "knee," one set at this important point being only 50 per cent. of the other set. Again, since Earhart's values are used to give the constants in the formula, there is naturally a certain degree of coincidence. But this is not all: the most recent results by Hobbs (*loc. cit.*) are opposed to Schwedoff's theory in each of the following particulars:—

(1) The critical potential gradients for different gases are identical below the knee.

(2) The critical potential gradients vary very greatly according to the metal used in the electrodes.

(3) Although Earhart did not take note of the fact, his results indicate a short horizontal portion to the curve, the potential gradient there being nil. Hobbs has thoroughly established the horizontal portion for every gas and for every kind of electrode.

As Hobbs points out, his results in conjunction with the previous work of Peace (Proc. Roy. Soc. p. 99, 1892), Strutt (Phil. Trans. p. 377, 1900), Carr (Proc. Roy. Soc. p. 374, 1903), which showed in every case minimum spark-potential

in a gas (unless the voltage is very small), seem to indicate that at atmospheric pressure the carriers of the discharge are the ions of the gas for distances greater than 6μ; while they are the ions of the metals of the electrodes for distances less than $2\frac{1}{2}\mu$; for distances between $2\frac{1}{2}\mu$ and 6μ the P.D. for discharge remains constant, and discharge occurs not at the extreme points of the opposing surfaces, but between points at the distance corresponding to the minimum sparking-potential, *i. e.* about 350 volts in air.

J. J. Thomson had previously suggested ('Conduction of Electricity through Gases,' p. 386, Camb. Univ. Press) that the metal corpuscles might play a part in these effects. A corpuscle about to leave the electrode feels an electrostatic attraction towards the electrode; if the electric intensity between the electrodes exceeds this, the corpuscle escapes. This critical field works out, from the principle of electric images, to be 8×10^3, which is the order of field in Earhart's experiments. Thus Thomson accounts for the fact that the first part of the curve is straight. This theory, however, may need some modification or amplification in view of the fact, mentioned above, discovered by Hobbs that each metal has its own gradient. Hobbs' curves indicate that great gradient goes with great atomic weight, *i. e.* the heavy metals lose their corpuscles with greatest reluctance.

Results.

(1) The insulating liquids do not differ very much from one another in power of resisting discharge, the strength lying between the limits 110 volts per micron and 70 volts per micron, though some liquids seem to have slightly less strength. The range of voltage here spoken of is between 25 and 400.

(2) Some liquids have a peculiar variation in potential gradient for distances between 1μ and 2μ. Castor oil, olive oil, and paraffin are cases in point.

(3) All insulating liquids have greater strength than air for P.D. greater than 300 volts; for less P.D. liquids have much less strength than air.

(4) The results for a homologous series of hydrocarbons do not indicate any simple connexion between composition and power of resisting discharge; the differences found are as

likely to be due to accidental impurities as to specific qualities of the liquids.

But these simple substances have specially great dielectric strength. It may be that simplicity of composition makes for strength.

(5) Whereas in the case of gases there is always a horizontal portion in the curve, in liquids there is no such distinct break in the curve between 25 and 400 volts, unless the region at about 60 volts in castor oil, olive oil, and paraffin is of this nature.

(6) No simple connexion can be observed between power of resisting discharge and specific inductive capacity, in liquids.

(7) The temperature throughout was from 15° C. to 18° C.

Discussion.

Mr. A. Russell considered the method of research a valuable one. He doubted, however, whether the results obtained would be much help in determining the relative values of the "dielectric strengths" of thin films of insulating materials. The numbers given in the paper only refer to thin films between iridio-platinum electrodes. Hobbs has shown that for small distances the sparking potentials are practically independent of the nature of the gas between the electrodes. They depend mainly upon the metal of which the electrodes are made. At these microscopic distances, owing to the corpuscles, the electrostatic field between them was far from uniform. The disruptive voltage for a film of air one micron thick, if it could be subjected to a uniform electrostatic stress, would only be about 3·8 volts. When liquids were introduced between the electrodes, it was extremely difficult to determine the maximum electric intensity at the instant of discharge, as they had a different dielectric coefficient from air, and so there would be refraction of the Faraday tubes.

Dr. Shaw, in reply to Mr. Russell, said that the air-curve mentioned was that for iridio-platinum, and was obtained by a distinct set of experiments on air. The gradient was steeper than any found by Hobbs in his recent work. As regards the use of the term "dielectric strength," Dr. Shaw used it in its common sense as a general term implying strength to resist discharge, and no numerical value was given to it in the paper.

XXIII. *Experiments on the Propagation of Longitudinal Waves of Magnetic Flux along Iron Wires and Rods.* By THOMAS R. LYLE, *M.A., Sc.D., Professor of Natural Philosophy in the University of Melbourne,* and J. M. BALDWIN, *M.A., B.Sc.**

1. THE following experiments were undertaken in order to investigate the changes in the Fourier characteristics of waves of magnetic flux as they travel along iron wires and rods. In all cases these flux-waves were started at the middle points of the rods or wires by sending alternating currents through short coaxial solenoids placed there.

The subject has already been attacked in different ways by several experimenters, and from the difference in phase of the flux oscillations at two points a so-called "velocity of magnetization" has been obtained from the formula,

$$v = \frac{2\pi l}{\phi \mathrm{T}},$$

where v = the "velocity of magnetization,"
T = the period of the oscillations,
ϕ = the difference in phase at two points l centimetres apart.

Oberbeck † worked with rods and bundles of wires, in some cases 40 cm., in others 1 metre long, and by means of a dynamometer measured the phase-difference of the currents induced in two short solenoids placed at various distances apart on the rod. This was equal to the phase-difference of the two resultant fluxes at the points, provided the time-constants of the circuits of the two solenoids were negligible.

The "velocity" deduced varied from 232,300 cm./sec. for 64 steel wires each 1·8 mm. diam., to 4410 cm./sec. for a steel rod 12 mm. diam., a result which indicates the great influence of eddy currents. He found that the "velocity" was not directly influenced by the character of the iron of the rod.

The amplitude F_x of the resultant flux at different points

* Read January 26, 1906.

† A. Oberbeck, Wied. *Ann.* vol. xxii. p. 73 (1884).

he found to be given by the formula

$$F_x = F_0 e^{-\lambda x},$$

where λ is to a first approximation a constant depending on the material of the bar, but independent of the area of its cross-section.

The values he found for λ when x, the distance from the centre of the magnetizing solenoid, is in centimetres, were

·1027, ·1017, ·1007 for soft iron,
·148 for hard iron,
·1451, ·1616, ·1637 for steel.

One frequency only was used.

Trouton * attempted to obtain stationary flux-waves by the interference of two trains of waves travelling in opposite directions round a ring of iron wires, the two trains being produced by an alternating current of known frequency circulating in a short solenoid looped on the ring. He concluded, however, that the effects which he observed were not due to interference but to some peculiarity of the ring.

Zenneck †, assuming the permeability μ and the leakage coefficient $\lambda = -dF/Fdx$ constants, where F is the flux at x, obtained from theory expressions for the flux-waves at different points along iron cylinders in terms of the initial flux where $x=0$, and to verify his theoretical conclusions used a Braun tube to determine the relative amplitudes and phases of the flux oscillations at any two points.

For thick wires (some millimetres in diameter) he found that the "velocity" and the leakage coefficient λ increased with frequency, while for wires less than 1 mm. in diam. they were, to a first approximation, independent of the frequency.

More recently Perkins ‡, working at a frequency of 60, has measured by means of a quadrant-electrometer the amplitude and phase, at different points along an iron bar, of the flux oscillation which had been started in the usual

* F. T. Trouton, 'Nature,' vol. xlv. p. 42 (1891).

† J. Zenneck, *Ann. der Phys.* vol. ix. p. 497 (1902), and vol. x. p. 845 (1903).

‡ H. A. Perkins, Amer. Jour. Sci. vol. xviii. p. 165 (1904).

way at the centre. The bar used was 1 metre long, 2·83 cm.^{2} in cross-section, and observations were made over a range of 25 cm. from its centre.

The phase was found to change less rapidly as the distance from the magnetizing-coil increased, and the "velocity of magnetization" obtained varied from

7500 cm./sec. at the centre
to 24000 cm./sec. at 24 cm. from the centre.

The curve he obtained showing the relation between the lag in phase of the flux at any point and the distance of the point from the magnetizing-coil agrees in general character with the corresponding small portions of the curves obtained by us.

The same problem has been theoretically discussed by Thomson *. He assumes that the permeability is constant for all values of the induction, and takes no account of the lag in phase of the induction behind the magnetizing force. As the flux-density or induction diminishes with great rapidity as we pass down the bar, and as both the permeability and the lag in phase of the induction vary within wide limits for different values of the latter, the theoretical conclusions arrived at on the assumption that one of these two quantities is constant and the other negligible, can scarcely be expected to be realized in practice.

2. In all the experimental investigations hitherto carried out on this subject, the range along the rod over which it was possible to determine even roughly the amplitude and phase of the flux oscillations at different points was limited, and the apparent behaviour over a short range afforded a very doubtful and misleading description of the whole phenomenon. No attempt, moreover, was made to determine the change of form of the flux-waves as they passed along the bar.

For this investigation the wave-tracer † designed by one of us is peculiarly well suited, as it enables us to determine completely, not only current and E.M.F. waves, but also flux-waves over a wide range of amplitudes.

* J. J. Thomson, 'Recent Researches,' p. 302.

† T. R. Lyle, "Wave-Tracer and Analyser," Phil. Mag. vol. vi. p. 549 (1903).

For instance, by its means can be accurately determined flux-waves whose amplitudes range from very large values down to a fraction of a single magnetic line; and by subjecting its readings to harmonic analysis we have been able to obtain the Fourier characteristics of the resultant flux oscillations at all points along rods of iron or steel, $\frac{1}{4}$ in. in diam. and 10 feet long, which have been magnetized by an alternating current circulating in a short solenoid placed at their middle points, when the amplitude of the flux-density at the centre was as low as 10,000.

3. The experiments, of which the results will be given below, were made on:—

A. A long rod of Lowmoor iron $\frac{1}{4}$ in. in diameter.

B. A long rod of silver steel $\frac{1}{4}$ in. in diameter.

C. A long iron wire $\frac{1}{6}$ in. in diameter.

D. A bundle of 185 charcoal iron wires, each ·079 cm. in diameter, 370 cm. long, tightly taped together so as to form a circular cylinder.

d_1. A bundle of 46 of the wires from D.

d_2. A bundle of 12 of the wires from D.

In each case at the middle point of the rod or bundle a short coaxial solenoid was placed, and through it an alternating current obtained from a rotary converter supplied with direct current from storage-cells was sent.

The magnetizing-current wave and the resultant flux-waves, crossing the sections of the bar or bundle at different points along its length, were quantitatively determined by the wave-tracer using the galvanometer method * described in the paper already quoted. The wave forms so obtained were subjected to harmonic analysis, the results of which are given in the Tables below. Some of the more interesting of the results of the investigation are also exhibited by means of curves.

The arrangement of the apparatus, the method of experiment, and the reduction of the observations were the same in all essential features as that fully described in a former paper † by one of us. Instead of the fixed secondary coil of

* T. R. Lyle, "Wave-Tracer and Analyser," Phil. Mag. vol. vi. p. 549 (1903).

† T. R. Lyle, "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. vol. ix. p. 102 (1905).

5 or 10 turns wound on the iron rings used in the experiments just quoted, three small search-coils were used. One of these, wound close to the rod or bundle, was fixed at the centre of the magnetizing solenoid; while the other two, of 166 and 805 turns respectively, could be moved from point to point along the rod, the one with the larger number of turns being used for exploring the rod at points near its ends.

During each series of experiments the magnetizing current and the rate of alternation were kept as constant as possible, and a continuous record was made by a chronograph of the times taken for every 200 periods.

The method of procedure in any one series of experiments was as follows:—The speed of the rotary converter having been adjusted to the desired value by means of resistances in the field circuit, or, if necessary, by changing the number of storage-cells used on its direct-current side, the magnetizing current drawn from its alternating side was adjusted by means of resistances to the desired value, and allowed to run until all conductors had attained a steady temperature.

By means of the wave-tracer equi-spaced ordinates of the magnetizing current wave sufficient in number to determine it were now taken. Then the corresponding ordinates of the flux-waves at the centre and at different points along the rod were obtained, and finally the ordinates of the current wave were redetermined to serve as a check on the constancy of the magnetizing force during the run.

Each set of ordinates was now subjected to harmonic analysis and the results affected by their proper factors * to reduce them either to current or flux, as the case might be, in absolute measure.

Three frequencies were used with specimens A and C, and two with specimen D, the central or initial flux being approximately the same for the different frequencies in any one specimen. With A and D additional series were obtained at one of the frequencies already used, but with different initial fluxes. For each of the other specimens B, d_1, and d_2, the results of one series of experiments only are given below.

For all the specimens but d_1 and d_2, by means of a ballistic galvanometer, using the method of reversals, the fluxes at

* T. R. Lyle, "Wave-Tracer and Analyser," Phil. Mag. vol. vi. p. 549 (1903).

different points along their lengths, due to a continuous magnetizing current in the central solenoid, were determined when the continuous current was such as produced at the centre a flux approximately equal to the amplitude of the central alternating flux previously used.

In addition the statical permeability of the material of each of the specimens for different values of the flux density was obtained by the method of reversals, a long solenoid being used to carry the magnetizing current.

The specific resistance of each specimen was also measured.

In the case of specimen A, after all the observations that were deemed necessary for its full length had been taken, it was shortened by successive steps by cutting off each time equal lengths from both ends ; each reduced length was then investigated for one frequency and one central flux.

4. We now proceed to give the results of the different experiments.

For every series the Fourier characteristics of the magnetizing current wave and of the resultant flux wave at different distances from the centrally placed magnetizing solenoid will be given in tabular form, and curves will be given showing how the phase, and the natural logarithm of the amplitude, of the first harmonic of the resultant flux varies as we pass along the specimen from the centre to the end. In the tables will also be given for different points the values of the leakage coefficient λ_1 of the amplitude f_1 of the first harmonic of the flux, where

$$\lambda_1 = -\frac{1}{f_1}\frac{df_1}{dx} = -\frac{d}{dx}(\log_e f_1)$$

$$= \frac{(\log_e f_1)_x - (\log_e f_1)_{x'}}{x' - x} \text{ (q. p.)}$$

at the point $\frac{1}{2}(x + x')$, where x and x' are two adjacent points at which f_1 has been determined. By means of the last expression the values of λ_1 have been calculated, and they are placed in the tables between the two adjacent rows of characteristics from which they have been obtained.

In the tables T will represent the period of the alternating current and flux in any one experiment, x the distance from

the centre of the specimen (and of the magnetizing solenoid) of the section across which the resultant flux F (in absolute measure) passes, and C the magnetizing current used during one series (also in absolute measure).

No harmonics higher than the 5th will be given, though they were determined in all cases, unless their amplitudes were below one per cent. of the amplitude of the corresponding first harmonic.

The galvanometer circuit of the wave-tracer when search-coil No. 3 was used had a very small time constant which was allowed for.

[*Note.*—The figures given in the tables are the results from the individual experiments. No smoothing of values has been resorted to, though a small error in the setting of the search-coil would make a large error in the result.]

5. *Details of Specimen* A.

A straight cylindrical rod of Lowmoor iron.
Length = 267 cms.
Diameter = ·644 cm. Section = ·3260 sq. cm.
Length of magnetizing solenoid .... 2·54 cm.
Radial depth of " " .... 2·40 cm.
Mean radius of " " .... 1·70 cm.
No. of turns on " " .... 240.

Specific resistance of the iron = $1{\cdot}140 \times 10^4$.

Statical Permeability for different Inductions.

B ...	25	50	100	250	500	750	1000	2000	3000	5000	7000	9000	11000	13000	15000
μ ..	290	325	370	451	550	650	750	1160	1490	1960	2215	2300	2165	1720	900

The analysed results of the alternating current experiments with this specimen are given in the following tables (I. to IV.), and the values of the flux and leakage coefficients at different points due to a continuous magnetizing force at the centre of the rod are given in Table V.

Table I.—Specimen A. T=·053 q.p.

$C = \cdot 1371 \sin \omega t + \cdot 00642 \sin 3(\omega t - 55\cdot 28) + \cdot 00057 \sin 5(\omega t - 68\cdot 23).$
$F = f_1 \sin (\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$

x.	T.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0...	·0537	3798	51·7	23·3	30·64	93·35	31·85	0	
5...	·0536	2618	70·1	14·0	41·42	92·04	32·19	10·78	·0744
10...	0534	1710	54·8	7·97	53·59	92·35	29·17	22·95	·0852
15...	·0531	1131	37·6	5·54	64·41	94·68	31·68	33·77	·0826
20...	·0532	738	23·4	1·85	74·46	98·09	31·03	43·82	·0855
30...	·0518	269·1	6·83	...	93·21	106·06	...	62·57	·1009
40...	·0528	83·8	1·50	0·49	105·42	109·52	38 67	74·78	·1167
50...	·0530	27·0	0·28	...	107·27	...	...	76·63	·1134
60...	·0526	9·51	...	...	101·97	...	...	71·33	·1042
70...	·0515	4·383	...	...	92·31	...	...	61·67	·0775
80...	·0526	2·583	...	...	84·80	...	...	54·16	·0529
100...	·0527	1·175	...	...	78·00	...	...	47·36	·0394
120...	·0528	0·604	...	...	74·61	...	...	43·97	·0333
130...	·0528	0·323	·0058	·0035	69·64	133·3	...	39·00	·0626

Table II.—Specimen A. T=·053 q.p.

$C = \cdot 2154 \sin \omega t + \cdot 01077 \sin 3(\omega t - 55\cdot 82) + \cdot 00072 \sin 5(\omega t - 72).$
$F = f_1 \sin (\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$

x.	T.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0...	·0530	5009	125·6	43 05	30·72	96·80	37·6	0	
10...	·0530	2232	109·3	23·7	54·80	88·69	25·2	24·08	·0809
20...	·0530	985·1	40·5	4 38	76·18	89·84	28·0	45·46	·0818
30...	·0532	393·4	9·46	...	94·87	95·80	...	64·15	·0918
40...	·0532	132·1	1·64	...	109·27	106·73	...	78·55	·1091
50...	·0532	42·0	...	...	113·64	...	...	82·92	·1146
60...	·0532	14·32	0·15	...	109·13	158·4	...	78·41	·1076
70...	·0531	6 16	0·10	...	98·23	177·2	...	67·51	·0843
80...	·0527	3·48	0·06	...	89 37	186·6	...	58·65	·0571
100...	·0525	1·572	0·027	...	80·84	195·9	...	50·12	·0397
120...	·0526	0·797	0·017	...	75·94	196·8	...	45·22	·0340
130...	·0526	0·428	0·009	...	71·04	203·0	...	40·32	·0622

TABLE III.—Specimen A. T = ·032 q.p.

$$C = {\cdot}1649 \sin \omega t + {\cdot}00655 \sin 3(\omega t - 58{\cdot}61) + {\cdot}00096 \sin 5(\omega t - 36{\cdot}1).$$
$$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$$

x.	T.	f_1.	f_3.	f_5.	θ.	β_3	β_5.	$\theta x - \theta_0$.	λ_1.
0...	·0320	3730	78·14	26·8	33·04	89·39	32·02	0	·0891
10...	·0320	1530	66·2	14·2	59·70	86·88	24·30	26·66	·0942
20...	·0319	596·6	17·1	...	84·89	91·19	...	51·85	·1134
30...	·0320	192·0	3·6	...	107·21	104·68	...	74·17	·1289
40...	·0320	52·9	0·83	...	122·74	110·30	...	89·70	·1251
50...	·0321	15·14	0·12	...	121·88	136·54	...	88·84	·1027
60...	·0321	5·42	0·056	...	111·21	161·53	...	78·17	·0667
70...	·0321	2·783	0·029	...	99·47	178·23	...	66·43	·0470
80...	·0322	1·740	0·019	...	93·06	182·94	...	60·02	·0351
100...	·0322	0·862	...	...	88·20	...	...	55·16	·0325
120...	·0323	0·450	...	...	84·40	...	...	51·36	·0641
130...	·0322	0·237	...	...	77·3	...	...	44·3	

TABLE IV.—Specimen A. T = ·0216 q.p.

$$C = {\cdot}1776 \sin \omega t + {\cdot}00728 \sin 3(\omega t - 58{\cdot}3) + {\cdot}00087 \sin 5(\omega t - 30{\cdot}55).$$
$$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$$

x.	T.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta x - \theta_0$.	λ_1.
0...	·0216	3537	90·6	37·7	33·75	88·57	30·68	0	·0932
5...	·0216	2220	96·3	25·6	48·04	85·30	25·96	14·29	·1014
15...	·0216	805·5	29·3	3·4	77·30	85·64	22·53	43·55	·1162
25...	·0216	251·9	4·35	...	104·37	99·48	...	70·62	·1401
35...	·0217	62·05	0·67	...	125·69	111·39	...	91·94	·1426
45...	·0217	14·90	0·20	...	132·64	124·01	...	98·89	·1088
55...	·0217	5·02	0·06	...	117·00	158·35	...	83·25	·0723
65...	·0217	2·437	0·038	...	103·53	169·67	...	69·78	·0453
80...	·0217	1·236	0·015	...	94·73	178·70	...	60·98	·0364
100...	·0217	0·597	0·009	...	90·59	180·57	...	56·84	·0287
120...	·0216	0·336	0·006	...	87·13	183·43	...	53·38	0547
130...	·0217	0·194	0·005	...	79·47	179·73	...	45·72	

Table V.—Specimen A. T = ∞.
(Static Leak.)

x.	C.	F.	λ.	x.	C.	F.	λ.
0 ...	·0751	3863		40...	·0749	261·3	
5 ...	·0750	2992	·0511	50...	·0752	100·8	·0953
10 ...	·0752	2254	·0566	60...	·0748	38·7	·0955
20 ...	·0751	1237	·0600	80...	·0750	7·44	·0825
30 ...	·0751	609·4	·0708	100...	·0750	2·60	·0525
40 ...	·0749	261·3	·0847	110...	·0750	1·49	·0557

6. In order to exhibit graphically some of the more striking results contained in the preceding tables, the curves in figs. 1 and 2 have been plotted. In fig. 1 the abscissæ are distances x from the origin or magnetizing solenoid and the ordinates of one set of curves are the natural logarithms of the amplitudes of the first harmonic of the flux at different distances, while the ordinates of the other set of curves are the retardations in phase of the first harmonic of the flux at different distances behind that of the initial flux where $x=0$.

It will be seen that the $\log f_1$, x, curves are far from being straight lines as Zenneck assumed for the basis of his theoretical work. This is still more clearly shown in fig. 2, where the leakage coefficients $\lambda_1 = -d(\log f_1)/dx$ are plotted against the distances from the origin at which they occur, as well as against $\log f_1$, the logarithm of the corresponding flux. Thus in all cases, including the statical, as we move away from the magnetizing coil, λ_1 first increases, attains a maximum, and then diminishes until near the end of the rod when it again increases. For the same initial flux both the points at which max. λ_1 occurs and the corresponding fluxes are different for different frequencies.

Figs. 1 and 2 show the effects of the end of the rod on the characteristics of the fluxes in its neighbourhood. Thus in fig. 1 the downward turn of the flux and phase curves due to the end effect can be recognized 30 cm. from the end of the rod, and the corresponding rise in the λ curves

Fig. 1.

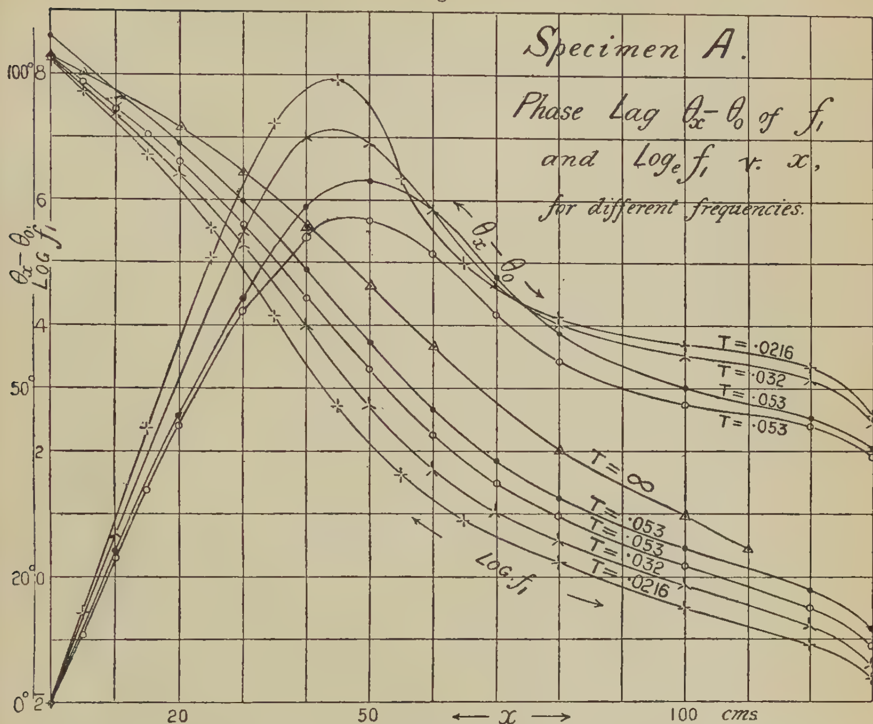
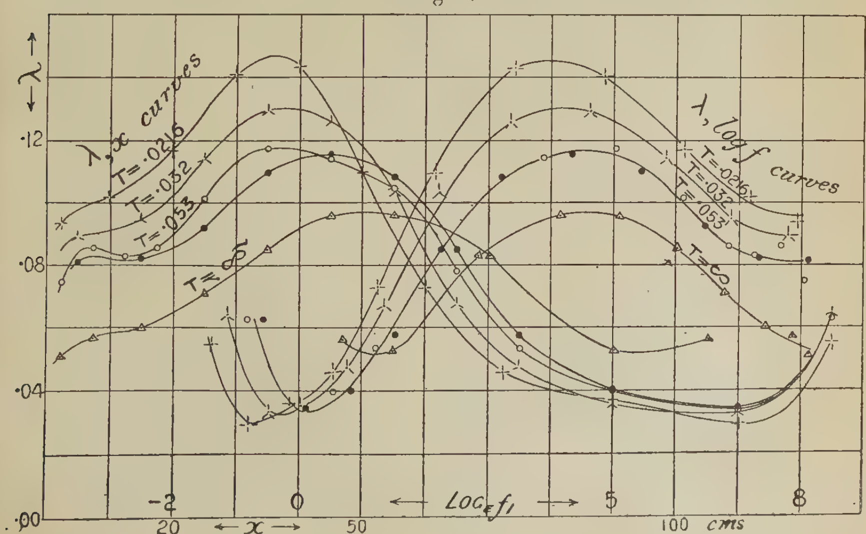

Specimen A.
Phase Lag $\theta_x - \theta_0$ of f_1 and $\log_e f_1$ v. x, for different frequencies.
$\theta_x - \theta_0$
$\log f_1$
T = ·0216
T = ·032
T = ·053
T = ·053
T = ∞
T = ·053
T = ·053
T = ·032
T = ·0216
$\theta_x - \theta_0$
$\log_e f_1$
100°8
6
4
50°
2
20°0
0°2
20
50
← x →
100 cms

Fig. 2.


← λ →
λ, x curves
T = ·0216
T = ·032
T = ·053
T = ∞
λ, log f curves
T = ·0216
T = ·032
T = ·053
T = ∞
·12
·08
·04
·00
-2
0
$\log_e f_1$
5
8
20
← x →
50
100 cms

in fig. 2 is very marked. The end effect always manifests itself in this way, and in very long rods this change in the direction of curvature does not appear.

From the λ_1, $\log f_1$ curves in fig. 2 we see that for the same value of f_1, λ_1 increases with the frequency. In the same set it will be noticed that one curve can be drawn (q.p.) through the two series of points for λ_1 when $T = \cdot 053$ arising from two different initial fluxes. This would suggest that λ_1 is independent of x and is a function of f_1 only. In the sequel it will be seen that λ_1 does depend on x as well as on f_1, and the coincidence of the above two curves is due to the fact that with the initial values used (Tables I. and II.) there is never much difference between the abscissæ at which equal fluxes occur in the two series.

Returning to fig. 1 we see how, for different frequencies, the phase of the first harmonic of the flux at any point x lags behind that of the initial central flux at $x=0$.

Thus as the flux moves away from the magnetizing coil its phase is at first retarded at a fairly regular but diminishing rate for any one frequency, while for the higher frequencies the rate of retardation $d\theta/dx$ is higher. To this part of the phenomenon the observations of Oberbeck, Zenneck, and Perkins have been limited, and following them we would deduce (see § 1) from our results the following mean values for the "velocity of magnetization" over the first 30 cms. of the rod in question:—

3380 cm./sec. when $T = \cdot 053$ sec.
4790 cm./sec. „ $T = \cdot 032$ „
6230 cm./sec. „ $T = \cdot 0216$ „

Continuing along the rod, however, we see that the space-rate of retardation ($d\theta/dx$) of the phase becomes zero, and the retardation itself a maximum at distances between 40 and 50 cms. from the starting-point, depending upon frequency and initial flux. After this the retardation diminishes, that is, the phase of the flux advances for the remainder of the length of the rod, and near the end the rate of advance ($-d\theta/dx$) is greatly increased, this latter being due to the end effect.

From this it would seem that the deduction by Oberbeck and others of a "velocity of magnetization" from the rate of

retardation $(d\theta/dx)$ of the phase of the flux close to the source by means of the formula

$$v = \frac{2\pi}{T\frac{d\theta}{dx}}$$

can hardly be legitimate.

For if so, we should have to allow that the "velocity" is infinite where the phase is stationary and $d\theta/dx=0$, and negative beyond this point, where the phase advances and $d\theta/dx$ is negative.

It is interesting to note that the points at which maximum phase retardation and maximum leakage coefficient occur are always situated very near each other.

7. It was thought that perhaps for the same frequency the phenomena at points in the rod beyond x might depend in great part only on the flux at x; and if that were so, by making the initial flux as small as that at which maximum retardation occurred in any one of the preceding series, we should get an advance of phase right from the start at the magnetizing coil. To test this a rod of $\frac{1}{4}$ in. Lowmoor iron 12 ft. long, taken from the same stock as specimen A, was used *.

The phase of the flux-wave at any point was taken as given directly by the divided-circle readings of the wave-tracer when the ordinate was zero, and the amplitude was obtained by applying the proper factor to the maximum ordinate of the same flux-wave. This was sufficiently accurate for the purpose in hand. The results obtained are exhibited graphically in figs. 3, 4, and 5. The initial values of the fluxes used were 69, 163, 288, 687, 1732, 2994, and 4100, while the period was ·052 sec. (q.p.) in all cases. In fig. 3 the logarithms (nat.) of the fluxes are plotted as ordinates against the corresponding distances from the origin as abscissæ, the points pertaining to any one initial flux being joined (as also in figs. 4 and 5) by a continuous curve. In fig. 4 against the same abscissæ are plotted the retardation in phase of the flux at any point behind that of the initial central flux; while in fig. 5 the leakage coefficients at different

* When we thought of this, test specimen A had been cut up for the purpose of investigating change of length.

Fig. 3.

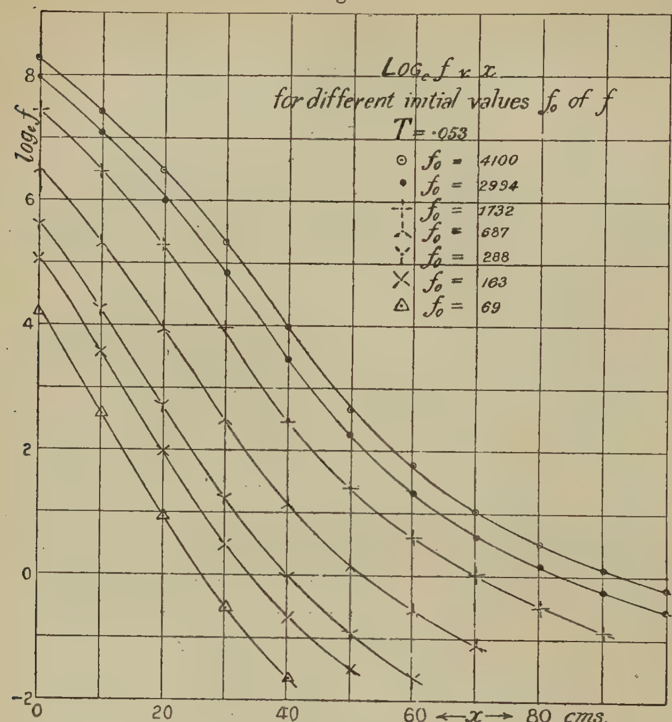
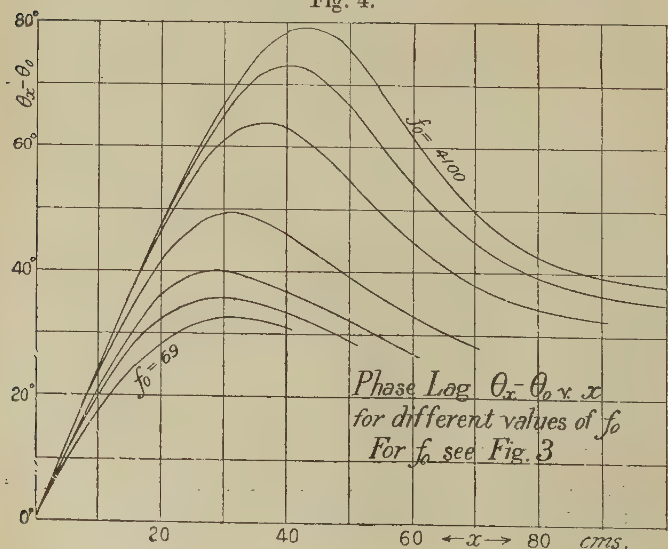

$Log_e f$ v. x
for different initial values f_0 of f
$T = \cdot 053$
$f_0 = 4100$
$f_0 = 2994$
$f_0 = 1732$
$f_0 = 687$
$f_0 = 288$
$f_0 = 163$
$f_0 = 69$
$log_e f$
8
6
4
2
0
-2
0
20
40
60 ←x→ 80 cms.

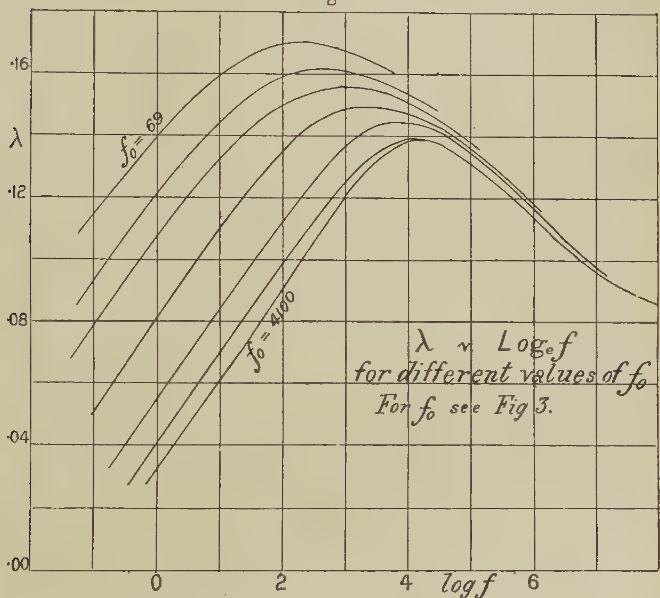
Fig. 4.


Phase Lag $\theta_x - \theta_0$ v. x
for different values of f_0
For f_0 see Fig. 3
$\theta_x - \theta_0$
$f_0 = 4100$
$f_0 = 69$
80°
60°
40°
20°
0°
20
40
60 ←x→ 80 cms.

points in the several series are plotted as ordinates against the logarithms of their corresponding fluxes.

It will be seen that in all cases over the wide range of initial fluxes used, the general nature of the phenomena is the same as that exhibited by figs. 1 and 2 and detailed in Tables I.–IV. Always as the flux moves away from the origin its phase is at first retarded, the retardation attains a

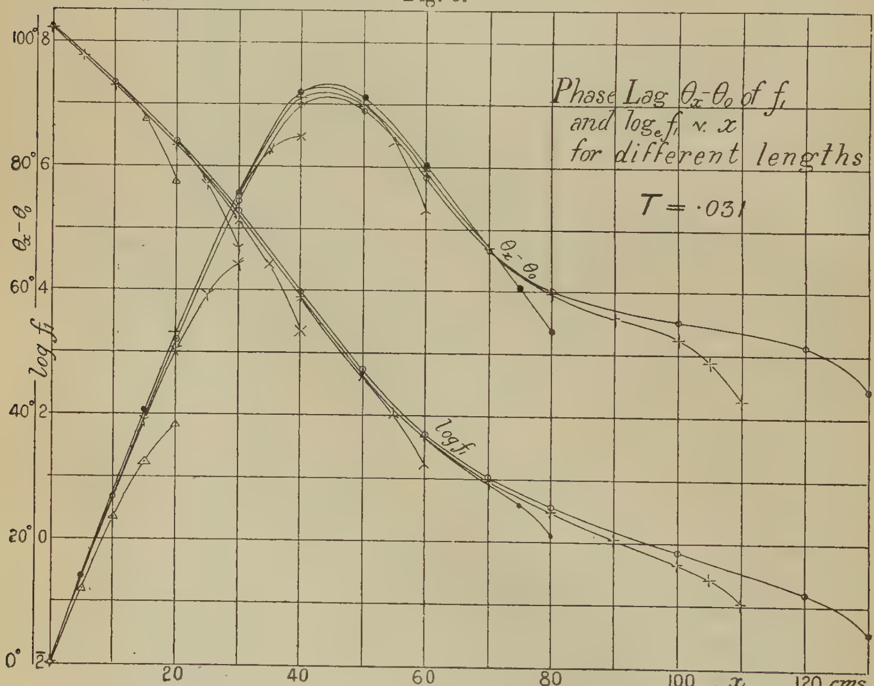
Fig. 5.



maximum, and then the phase advances and keeps advancing (as the rod is long) until the flux is completely dissipated. The leakage coefficient also always begins by increasing, attains a maximum and then diminishes ; and we found in this case as in all others that the points on the rod at which maximum retardation and maximum leakage coefficient occurred were practically coincident. A glance at figs. 4 and 5 will show the effect of change of initial flux in the same specimen at the same frequency on the retardations of phase and the leakage coefficients. Thus increase of initial flux increases the maximum retardation, diminishes the maximum leakage coefficient, and increases the distance from the origin at which these two maxima concur.

As a result of this series of experiments it is plain that what may happen at any one point of the rod, both as regards rate of retardation (or of advance as the case may be) of phase and leakage depends both on the position of the point and on the value of the resultant flux at the point, and if we followed previous investigators in computing a "velocity of magnetization" we should obtain different velocities for the same flux and frequency, provided the points for which the velocity is computed were at different distances from the magnetizing coil.

Fig. 6.



8. Specimen A was now reduced in length in successive steps by cutting each time equal portions off its two ends. For each length one series of observations was taken in which the initial flux and the frequency were approximately the same as in the series already given for the full length in Table III. The results obtained are given in detail in Tables VI. to XI., and the general effect of reduction in length on the lag in phase and on the logarithm of the amplitude of the first harmonic of the flux is exhibited in fig. 6.

TABLE VI.—Specimen A reduced to $l=225$ cms.
$T=\cdot 0316$.

$C=1595 \sin \omega t+\cdot 0054 \sin 3(\omega t-62\cdot 4)+\cdot 0007 \sin 5(\omega t-32\cdot 4)$.
$F=f_1 \sin(\omega t-\theta)+f_3 \sin 3(\omega t-\theta-\beta_3)+f_5 \sin 5(\omega t-\theta-\beta_5)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x-\theta_0$.	λ_1.
0.........	3605	99·7	26 0	32·40	89 83	37·18	0	·0913
10.........	1446	60 0	9·0	59·98	86·18	28·68	27·58	·0961
20.........	553·3	15·8	1·5	85·28	91·70	35·00	52·88	·1078
30.........	188·3	3·0	...	106·80	100·78	...	74·40	·1355
40.........	48·56	0·42	...	123·28	118·29	...	90·88	·1206
50.........	14·53	0·13	...	121·93	137·48	...	89·53	·1027
60.........	5·201	·062	...	111·64	171·46	...	79·24	·0701
70.........	2·580	·034	...	99·18	173·02	...	66·78	·0474
80.........	1·607	·017	...	92·01	180·20	...	59·61	·0430
90.........	1·045	...	...	88·19	...	...	55·79	·0388
100.........	·709	...	...	84·81	...	...	52·41	·0451
105.........	·566	...	...	81·20	...	...	48 80	·0795
110.........	·380	...	...	74·94	...	...	42·54	

TABLE VII.—Specimen A reduced to $l=165$ cms.
$T=\cdot 0309$.

$C=\cdot 1683 \sin \omega t+\cdot 0063 \sin 3(\omega t-60)+\cdot 0011 \sin 5(\omega t-33)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x-\theta_0$.	λ_1.
0.........	3744	85·5	18·1	31·23	87·9	31·37	0	·0908
5.........	2378	95·9	17·7	44·70	88·08	28·94	13·47	·0922
15.........	945·7	37·6	4·84	71·78	89·15	33·77	40·55	·1087
30.........	185·0	1·9	...	106·68	102·12	...	75·45	·1332
40.........	48·84	0·57	...	123·06	115·74	...	91·83	·1224
50.........	14·37	0·15	...	122·20	141·90	...	90·97	·1025
60.........	5·153	·076	...	111·41	165·4	...	80·18	·0719
70.........	2·510	·028	...	97·95	173·9	...	66·72	·0676
75.........	1·790	·021	...	91·85	181·0	...	60·62	·1145
80.........	1·010	·011	...	84·91	183·0	...	53·68	

TABLE VIII.—Specimen A reduced to $l=125$ cms.
$T=\cdot0305$.

$C=\cdot1722 \sin \omega t+\cdot0066 \sin 3(\omega t-57\cdot5)+\cdot0008 \sin 5(\omega t-27)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x-\theta_0$.	λ_1.
0.........	3787	89·7	20·7	32·86	90·88	29·23	0	·0917
10.........	1514	61·4	11·6	60·17	87·42	23·57	27·31	·0969
20.........	574·2	17·3	...	85·23	91·36	...	52·37	·1156
30.........	180·8	2 9	...	108·09	103·31	...	75·23	·1360
40.........	46·42	0·61	...	124·62	110·18	...	91·76	·1238
50.........	13·46	0·13	...	122·71	140·8	...	89·85	·1212
55.........	7·34	0·07	...	116·67	152·6	...	83·81	·1596
60.........	3·31	0·04	...	105·61	163·5	...	72·75	

TABLE IX.—Specimen A reduced to $l=85$ cms.
$T=\cdot0294$.

$C=\cdot1711 \sin \omega t+\cdot0065 \sin 3(\omega t-59\cdot8)+\cdot0007 \sin 5(\omega t-31\cdot7)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta-\theta_0$.	λ_1.
0.........	3684	91·5	26·8	32 60	91·4	30·14	0	·0918
10.........	1471	66·6	13·6	59·91	87·31	25·79	27·31	·0974
20.........	555·5	15·9	...	85·32	92·21	...	52·72	·1212
30.........	165·3	2·76	...	107·53	99 57	...	74·93	·1436
35.........	80·62	1·00	...	115·06	100·70	...	82·46	·2114
40.........	28·01	0·26	...	117·12	107·58	...	84·52	

TABLE X.—Specimen A reduced to $l=65$ cms.
$T=\cdot0311$.

$C=\cdot1718 \sin \omega t+\cdot0067 \sin 3(\omega t-58\cdot1)+\cdot0008 \sin 5(\omega t-26)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x-\theta_0$.	λ_1.
0.........	3794	87·6	26·0	32·66	89·0	31·67	0	·0903
10.........	1539	65 6	12·1	59·72	85·97	27·87	27·06	·0922
15.........	970·4	35·9	5·3	71·41	88·43	25·40	38 75	·1046
20.........	575·1	19·42	1·85	82·82	92·76	...	50·16	·1309
25.........	298·9	7·71	...	91·79	96·36	...	59·13	·2087
30.........	105·3	2·38	...	96·67	96·33	...	64·01	

TABLE XI.—Specimen A reduced to $l=45$ cms.
T = ·0311.

$C = \cdot 1734 \sin \omega t + \cdot 0077 \sin 3(\omega t - 58\cdot 3) + \cdot 0003 \sin 5(\omega t - 27)$.

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0.........	3820	94·9	29·1	32·01	88·37	27·95	0	·0862
5.........	2480	106·5	21·9	43·93	86·38	28·88	11·92	·1032
10.........	1502	75·3	15·4	55·44	86·16	28·13	23·43	·1139
15.........	850	41·1	7·2	64·26	89·41	31·94	32·25	·2048
20.........	305·3	13·9	2·5	70·11	91·65	30·36	38·10	

From the tables and from fig. 6 it will be seen that shortening the rod makes little perceptible difference either to the leakage of the flux or to its phase-lag until a point is reached at which the end effect begins to assert itself. Thus for the different lengths, flux-waves that are equal at the magnetizing coil are practically equal in all their characteristics after travelling equal distances along the rod, provided they have not reached to within 30 cms. or so from the ends.

9. The next specimen (B) tested was a rod of silver-steel of low permeability of the same section as specimen A. One frequency with one initial flux only was used, these being (q. p.) the same as in one of the series (Table III.) of experiments with specimen A, so that a direct comparison of the behaviour of the two materials could be readily made.

Details of Specimen B.

A straight cylindrical rod of silver-steel.
Length = 286·6 cms. Diameter = ·653 cm.
Section = ·3351 cm².
Magnetizing solenoid and search-coils the same as were used with specimen A.
Specific resistance = $1\cdot 859 \times 10^4$.

Statical Permeability for different inductions.

B	50	500	1000	2000	4000	6000	8000	10000	12500	15000
μ	73	96	122	170	248	298	316	290	222	140

Table XII.—Specimen B. T=·0305.

$C = \cdot 224 \sin \omega t$; $F = f_1 \sin(\omega t - \theta)$.

x ……	0	5	10	15	20	30	40	60	80	100	110	120	130	140	145
f_1 ……	3735	1855	700	236	80·3	16·6	5·35	1·34	0·53	·265	·190	·142	·106	·071	·053
θ ……	23·4	35·7	47·5	53·7	54·7	47·7	42·8	39·0	38·6	38·5	38·5	38·2	39·5	37·4	37·6
$\theta_x - \theta_0$…	0	12·3	24·1	30·3	31·3	24·3	19·4	15·6	15·2	15·1	15·1	14·8	16·1	14·0	14·2

λ_1 ……	·140	·195	·217	·216	·157	·114	·069	·047	·035	·033	·030	·029	·040	·058

Table XIII.—Static Leak. Specimen B. T=∞.

Magnetizing Current C=·2027 (abs.).

x ……	0	5	10	15	20	25	30	35	40	45	50	55	60
F ……	3592	1777	695	235	81·5	37·5	17·6	9·35	5·63	3·72	2·57	1·91	1·34

λ ……	·141	·188	·217	·211	·155	·151	·127	·101	·083	·074	·060	·067

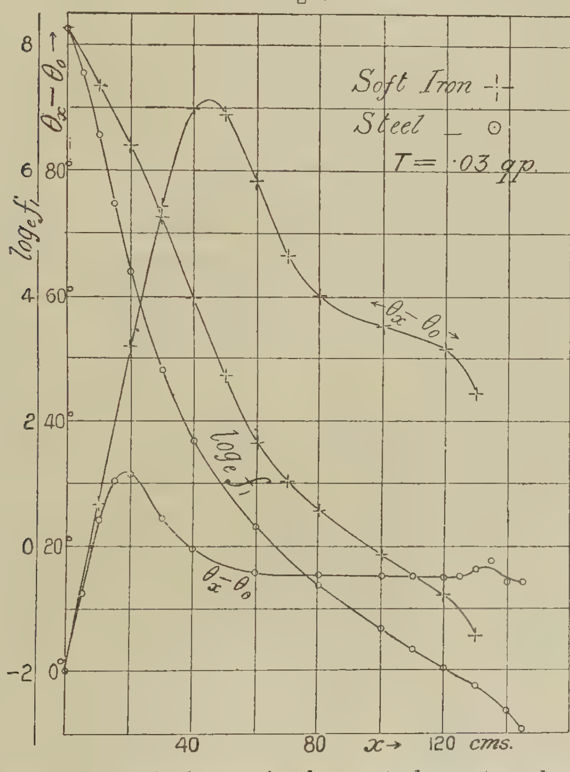
Tables XII. and XIII. give the results of the experiments with this specimen, the characteristics of the first harmonic of the flux only being given.

By comparing Tables III. and XII. one can see the difference in the behaviour of similar rods of iron and steel for the same initial flux and the same frequency. Fig. 7 illustrates the difference graphically. Thus, though near the origin the rate of retardation of phase ($d\theta/dx$) is nearly the same for both specimens (hence giving nearly equal fictitious velocities of magnetization), yet in the steel rod the retardation attains its maximum at less than half the distance from the origin at which it attains its maximum in the iron rod, and the maximum value in the latter rod is about three times what it is in the former. Again, for the steel rod the leakage coefficients are at first very much larger than for the iron one (see fig. 8), either at equal distances from the origin, or for equal fluxes; but when very low values of the flux have been arrived at, the leakage coefficients are quite as small in the steel as in the iron.

It will be noticed (see Table XII.) that in this specimen also the positions of max. retardation and max. leakage coefficient very nearly if not quite coincide.

10. The cause of the difference in the leakage-curves for the two rods is naturally to be looked for in their very different permeability-curves. Of the flux F which crosses any section

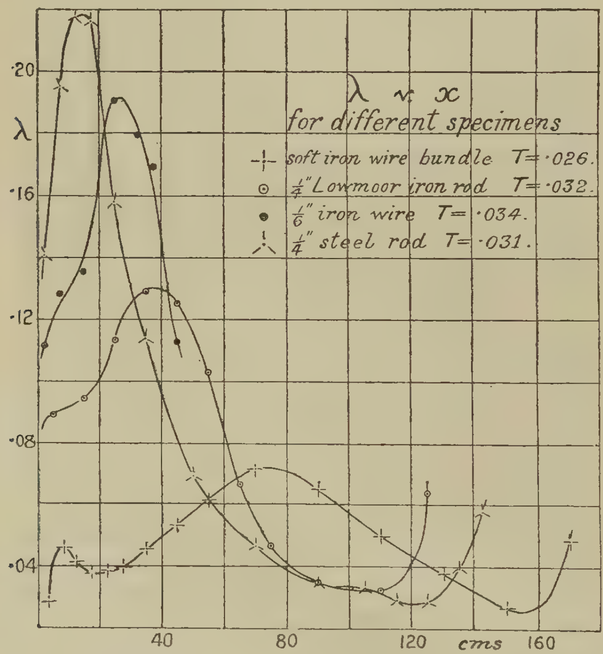
Fig. 7.



of the rod, $-dF$ leaks out in the next elementary length dx, and $F+dF$ continues in the rod, and the ratio of $-dF$ to $F+dF$ (*i.e.* $-dF/F$ in the limit) will, when the magnetizing force is constant, be equal to the inverse ratio of the reluctances of the two corresponding paths. When the magnetizing force is periodic, the phenomenon is further complicated by the eddy currents set up in the rod, giving rise to back magnetomotive forces. If, for two different rods, the reluctances of two elementary leakage-paths (air) were equal, then

$-dF/F$ for the corresponding elements (and therefore λ, as $\lambda dx = -dF/F$) would be proportional to the reluctances of the second paths (partly in the materials), and thus would be greater for the less permeable substance. However, as λ depends on the ratio of two reluctances, it is quite conceivable that this ratio might be smaller for points on a rod of low permeability than for the corresponding point on a rod of higher permeability. (See fig. 8.)

Fig. 8.



Again, λ at a point near the end of a finite rod for a given initial flux and frequency must be greater than λ for a point equidistant from the magnetizing solenoid in an infinite rod of the same material and cross-section; for the reluctances of the elementary leakage paths through the air will be equal, while the second path for the infinite rod must have a lower reluctance than that for the finite rod, seeing that iron in the former takes the place of some of the circuit, which is air for the latter. Thus the "end effect" on the leakage coefficient is to make its value larger.

11. C, the next specimen tested, was a long thick iron wire. Three series of experiments were made with it for three different frequencies and approximately the same initial flux.

Details of Specimen C.

A straight iron wire.
Length = 324 cms. Diameter = ·398 cm.
Section = 0·1244 cm².
Specific resistance = 1·006 × 10⁴.
Magnetizing solenoid and search-coils the same as were used with specimen A.

Statical Permeability for different inductions.													
B......	100	500	1000	2000	4000	5000	6000	8000	10000	12500	15000	17500	20000
μ	230	435	625	990	1385	1470	1455	1320	1100	800	540	330	170

In Tables XIV., XV., and XVI. are given the analysed results of the alternating flux experiments with this specimen, and in Table XVII. the results with the continuous flux.

TABLE XIV.—Specimen C. T = ·0545.

$C = \cdot 1742 \sin \omega t - \cdot 0091 \sin 3(\omega t + 1\cdot 15) - \cdot 0011 \sin 5(\omega t - 26\cdot 1).$
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	2360	189·0	41·8	22·05	3·28	45·08	0	·0898
5	1505	100·9	19·0	34·50	− 6·10	45·07	12·45	·1150
10	847·5	59·8	10·2	47·18	−11·81	40·74	25·13	·1245
20	244	16·5	1·6	69·82	−14·88	48·26	47·77	·1760
30	42·0	1·9	0·3	83·97	− 8·82	63·34	61·92	·1774
35	17·3	0·44	0·12	79·09	− 1·85	68·62	57·04	·1800
40	7·02	0·09	0·05	71·05	− 7·73	79·06	49·00	·1207
50	2·10	0·09	...	53·54	−11·45	...	31·49	·0759
60	0·98	0·06	...	46·23	− 5·57	...	24·17	·0510
70	0·59	0·05	...	39·72	+ 1·90	...	17·67	

TABLE XV.—Specimen C. T=·034.

$C = \cdot 1737 \sin \omega t - \cdot 0096 \sin 3(\omega t - 0 \cdot 24).$
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	2220	87	34	27·00	− 1·33	42·56	0	·1034
5	1326	54	15	40·55	−16·36	36·21	13·55	·1272
10	702	36	7	56·07	−20·51	36·24	29·07	·1351
20	181·7	8·9	...	81·94	−18·71	...	54·94	·1904
30	27·1	0·6	...	96·95	− 6·62	...	69·95	·1792
35	11·05	0·07	...	88·46	+16·98	...	61·46	·1687
40	4·75	0·04	...	77·14	66·04	...	50·14	·1123
50	1·55	0·04	...	56·41	130·2	...	29·41	

TABLE XVI.—Specimen C. T=·0234.

$C = \cdot 1803 \sin \omega t - \cdot 0072 \sin 3(\omega t + 3 \cdot 32) + \cdot 0016 \sin 5(\omega t - 6 \cdot 6).$
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$

x.	f_1.	f_3.	f_5.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	2115	75	28	30·40	−15·46	42·70	0	·1148
5	1192	59	16	45·26	−26·67	35·00	14·86	·1388
10	595·4	29·4	6·4	62·07	−27·00	30·72	31·67	·1515
20	130·9	4·9	...	91·13	−22·21	...	60·73	·2053
30	16·80	0·18	...	104·61	+ 5·08	...	74·21	·1763
35	6·96	0·10	...	90·53	49·92	...	60·13	·1494
40	3·30	0·07	...	76·41	66·49	...	46·01	·1057
50	114	0·06	...	58·76	94·31	...	28·36	

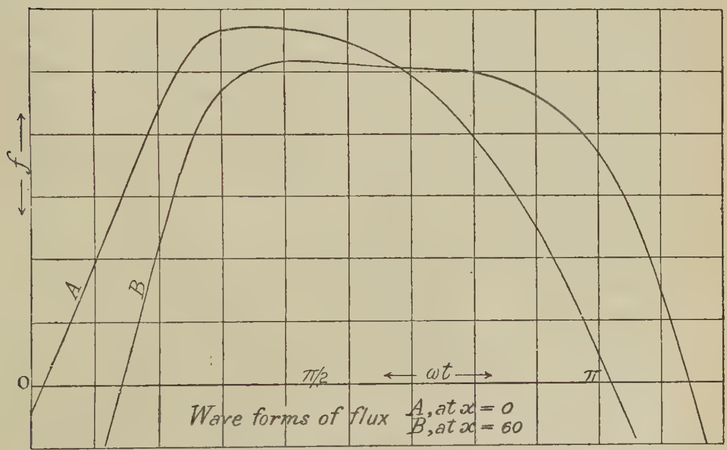
TABLE XVII.—Static Leak. Specimen C. T=∞.

Magnetizing Current, C=·1844 (abs.).

x.........	0	5	10	15	20	25	30	35	40	45	50	55
F	2220	1476	919	559	318	162	77·5	31·6	12·6	5·83	3·01	1·98
λ	·082	·095	·099	·123	·135	·147	·179	·183	·154	·132	·084	

In the series of experiments with this specimen the flux density at the centre was much higher (18,000 instead of 10,000) than in the corresponding series with specimen A. The series with rod A, given in Table II., in which the induction at the centre was about 15,000, will, however, afford a comparison under more nearly similar conditions, and we can see that the leakage coefficients for the thinner rod are considerably higher than for the other (see fig. 8). On account of the smaller sectional area and greater leakage, observations were of necessity limited to a shorter range than for the other specimens, but within this range the phenomena

Fig. 9.



observed and recorded in the preceding tables are similar in all respects to those observed with the previous specimens. In particular the leakage coefficient again attains a maximum value at or very near to the position for maximum phase-lag.

It is also worthy of notice that the value of $d\theta/dx$ near the origin is larger than for specimen A, the mean values of the fictitious "velocity of magnetization" for the first 20 cms. (a range within which in this specimen $d\theta/dx$ is fairly uniform) being

2750 cms./sec. when T = ·0545,
3850 cms./sec. „ T = ·034,
5060 cms./sec. „ T = ·0234,

values considerably smaller than those for specimen A. This agrees with the results of some experiments that will be described further on, which show that reducing the diameter of the specimen, other things being equal, increases the space rate of retardation of the flux near the origin.

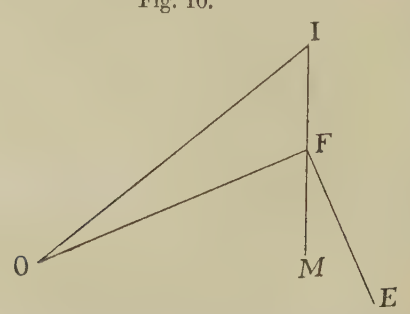
There might seem to be a discrepancy when the values of f_1 for $x=0$ in Tables XIV., XV., XVI. relative to the first harmonics of their magnetizing currents are compared with the value of F_0 in Table XVII. relative to the continuous current producing F_0. The iron seems to be quite as permeable under alternating magnetization as under continuous. This apparent discrepancy will disappear (and this remark will apply to the other specimens) when it is remembered (*a*) that the form of the current wave is peaked and its max. ordinate is greater than the amplitude of its first harmonic, (*b*) that the form of the flux-wave is flat and its max. ordinate is less than the amplitude of its first harmonic.

Thus the max. values of C and F_0 for the different values of T in the above are as follows:—

T..........	∞	·0545	·034	·0234
C_{max}.......	·1844	·1854	·1850	·1866
F_{max}.......	2220	2160	2100	2034

12. Eddy currents must play an important part in the phenomena under consideration, and their effects in this

Fig. 10.



connexion may, in a general way, be explained as follows. If the vector OF (fig. 10) represent the resultant flux down

any cylindrical portion of a rod, the E.M.F. produced by its variation in a circuit in the rod round the outside of this cylinder, being in quadrature with OF and behind it in phase, may be represented by a vector FE where the angle OFE is a right angle. This E.M.F. generates an eddy current which generates a flux down the cylinder that lags behind FE in phase, and hence may be represented by some such vector as FM or IF, where IF=FM. Hence the vector OI will represent in amplitude and phase the flux that would have passed down the cylinder if the eddy current had not been present. Thus we see that the effects of eddy currents at any point of one of our specimens are (*a*) an increase in the phase retardation, *i. e.* in $\theta_x - \theta_0$; (*b*) a diminution in the amplitude of the flux there, which manifests itself as an increase in λ the leakage coefficient.

A further set of experiments was undertaken in which these eddy-current effects were much reduced by replacing the solid rod by a cylindrical bundle of soft iron wires, called specimen D, of which the details are as follows:—

Details of Specimen D.

A cylindrical bundle of soft iron wires.
Number of wires = 185.
Length of each wire = 368 cms.
Diameter " = ·0791 cm.
Area of iron section of specimen = ·909 cm.^{2}
Diameter of bundle = 1·5 cm.
Specific resistance of iron = $1·366 \times 10^4$.
No. of turns on magnetizing coil = 366.
" " search coil central = 5.
" " " " a = 10.
" " " " b = 50.
" " " " c = 100.

Statical Permeability for different inductions.									
B	100	1000	2000	4000	5000	6000	8000	10000	12000
μ	202	475	750	1140	1240	1260	1125	875	575

In Tables XVIII., XIX., and XX. are given the results of three series of experiments with this specimen at one frequency (T = ·0255), but with different initial fluxes, and in Table XXI. those of one series at the frequency for which T = ·0515, the initial flux being approximately the same as in one of the preceding series, while in Table XXII. are given the results for continuous magnetizing force also with approximately the same central flux.

A glance at the columns for $\theta_x - \theta_0$ and λ_1 in Tables XVIII. to XXI. will show that so far as phase retardation and leakage coefficients are concerned, the general character of the results is the same as has been obtained from the other specimens; and, as was expected, not only are these quantities much reduced in magnitude, but also the effect of change of frequency on them is less marked. In addition the apparently coincident point of maximum phase retardation and maximum leakage coefficient is much further removed from the origin than in any of the other specimens.

TABLE XVIII.—Specimen D. T = ·0255.

$$C = \cdot 1697 \sin \omega t + \cdot 0296 \sin 3(\omega t - 58\cdot 6) + \cdot 0091 \sin 5(\omega t - 65\cdot 4).$$
$$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$$

x.	f_1.	f_3/f_1.	f_5/f_1.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	16190	·086	·020	15·77	16·49	29·88	0	·0285
7·3	13150	·097	·021	19·52	12·33	26·98	3·75	·0460
10	11620	·107	·021	21·19	10·56	24·44	5·42	·0414
15	9440	·121	·021	24·27	8·34	20·73	8·50	·0378
20	7816	·132	·022	27·38	6·77	17·72	11·61	·0383
25	6456	·141	·025	29·97	5·68	15·62	14·20	·0396
30	5292	·151	·024	32·50	4·72	11·33	16·73	·0455
40	3360	·166	·033	36·39	4·17	9·86	20·62	·0530
50	1976	·179	·041	39·23	3·61	8·47	23·46	·0614
60	1070	·184	·045	40·97	4·35	8·58	25·20	·0716
80	255·4	·182	·046	41·20	5·52	9·46	25·43	·0651
100	69·5	·160	·041	37·90	6·98	12·83	22·13	·0501
120	25·47	·147	·037	33·38	9·14	15·56	17·61	·0381
140	11·91	·133	·031	30·43	9·04	18·42	14·66	·0270
160	6·94	·145	·027	28·40	11·32	19·95	12·63	·0487
180	2·62	·135	·036	26·83	12·52	19·43	11·06	

TABLE XIX.—Specimen D. T = ·0254.

$C = {\cdot}0846 \sin \omega t + {\cdot}0078 \sin 3(\omega t - 62{\cdot}17) + {\cdot}0013 \sin 5(\omega t - 4{\cdot}49)$.
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5)$.

x.	f_1.	f_3/f_1.	f_5/f_1.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0.........	10030	·057	·005	18·98	26·27	39·69	0	·0304
7·3	8035	·065	·009	22·67	20·12	33·50	3·69	·0450
10.........	7121	·073	·010	25·07	16·81	30·69	6·09	·0433
15.........	5730	·085	·010	29 27	13·05	27·63	10·29	·0432
20.........	4618	·098	·012	32·16	10·86	24·44	13·18	·0437
25.........	3710	·106	·014	34·35	10·07	22·20	15·37	·0545
30.........	2826	·116	·014	37·23	8·71	20·85	18·25	·0545
40.........	1639	·136	·022	41·35	7·60	16·26	22·37	·0652
50.........	854	·142	·025	43 89	7·64	15·63	24·91	·0700
60.........	424	·149	·031	45·05	8·07	15·34	26·07	·0741
80.........	96·4	·136	·030	43·35	10·08	18·86	24·35	·0592
100.........	29·5	·118	·021	39·10	15·05	22 70	20·12	·0469
120.........	11·55	·101	·020	35·73	13·69	20·63	16·75	·0320
140.........	6·09	·092	·020	32·56	16·56	28·71	13 58	

TABLE XX.—Specimen D. T=·0255.

$C = {\cdot}0353 \sin \omega t + {\cdot}0012 \sin 3(\omega t - 58{\cdot}29)$.
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5)$.

x.	f_1.	f_3/f_1.	f_5/f_1.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	2958	·049	006	13·72	27·77	35·36	0	·0469
7·3	2100	·058	·006	17·55	21·02	32·80	3 83	·0761
10.........	1710	·066	·008	19·42	19·47	28·70	5·70	·0751
15.........	1175	·077	·011	22·77	17·29	28·91	9·05	·0745
20.........	809	·084	·014	24 68	15·51	26·75	10·96	·0771
25.........	550	·092	·013	26·03	14·98	25·48	12·31	·0814
30.........	366·4	·096	·016	26·98	14·98	24·07	13·26	·0805
40.........	163·8	·100	·019	27·35	15·60	25·08	13·63	·0824
50........	71·8	·100	·019	27·39	16·49	25·41	13·67	·0756
60.........	33·7	·103	·018	26·40	17·95	28·16	12·68	·0677
70.........	17·13	·089	·015	25·77	18·58	28·79	12·05	·0591
80.........	9·48	·082	·016	24·73	18·51	26 98	11·01	·0458
90.........	6·00	·079	·010	24·35	17·68	30·36	10·63	·0459
100.........	3·79	·083	·013	23·54	16·81	30·66	9·82	·0345
120.........	1·90	·072	·005	20·19	18·16	30·96	6·47	·0287
140.........	1·07	·067	·015	18·85	19·50	31·50	5·13	

Table XXI.—Specimen D. T=·0515.

$$C = \cdot0790 \sin \omega t + \cdot0050 \sin 3(\omega t - 58\cdot46) + \cdot0006 \sin 5(\omega t - 64\cdot86).$$
$$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5).$$

x.	f_1.	f_3/f_1.	f_5/f_1.	θ.	β_3.	β_5.	$\theta_x - \theta_0$.	λ_1.
0	9472	·065	·012	16·92	19·80	33·23	0	·0304
7·3	7592	·077	·013	19·90	14·58	28·61	2·98	·0454
10.........	6700	·083	·014	21·68	13·83	28·83	4·76	·0510
15.........	5193	·098	·017	25·15	10·83	23·91	8·23	·0441
20.........	4167	·115	·016	27·78	9·34	22·38	10·86	·0455
25.........	3319	·123	·019	30·09	8·81	17·68	13·17	·0519
30.........	2560	·136	·022	32·16	8·00	17·50	15·24	·0594
40.........	1414	·153	·033	36·00	7·75	15·05	19·08	·0653
50.........	736	·159	·032	38·16	6·59	13·65	21·24	·0721
60.........	358	·165	·039	39·00	7·42	14·31	22·08	·0789
70.........	162·9	·155	·036	38·38	8·22	14·35	21·46	·0727
80.........	78·7	·152	·038	36·91	9·33	14·39	19·99	·0600
100.........	23·7	·131	·028	33·41	10·42	18·50	16·49	·0443
120.........	9·76	·121	·026	30·06	11·16	22·35	13·14	·0333
140.........	5·01	·107	·014	28·73	12·78	19·78	11·81	·0269
160.........	2·92	·099	·023	28·26	10·25	22·50	11·34	·0354
170.........	2·20	·099	·023	26·27	14·24	28·24	9·35	·0596
180.........	1·13	·093	·020	25·65	14·62	29·63	8·73	

Table XXII.—Specimen D. Static Leak. T=∞.

Magnetizing Current. C=·0915 (obs.).

x...	0	7·3	10	15	20	25	30	40	50	60	75	90	100	110	120
F..	9775	7700	6517	5235	4154	3267	2542	1429	740	355	120	45·6	26·9	15·8	11·1
λ...	·0327	·0617	·0439	·0462	·0481	·0502	·0576	·0658	·0735	·0723	·0644	·0528	·0533	·0531	

13. A much more striking result, however, is the way the upper harmonics of the flux, and hence its wave form, vary as the flux-wave passes along the specimen. In these series the ratios f_3/f_1 and f_5/f_1 of the amplitudes of the third and fifth harmonics to that of the first are much larger than in the

previous ones, where the flux wave was almost sinusoidal. We were therefore able to determine for this specimen these upper harmonics both in amplitude and phase with certainty for greater distances from the origin than for the other specimens; and the results given in the tables show that as we move away from the magnetizing coil we find (*a*) that the ratios f_3/f_1 and f_5/f_1 first increase, then both attain a maximum at approximately the same place, and after this diminish, and this maximum point is the same as that at which the phase retardation and the leakage coefficient of the first harmonic attain their maxima; (*b*) that the retardations in phase β_3, β_5 of these harmonics behind the primary first diminish, become minima, and then increase, and that the point of minimum values of β_3 and β_5 is the same apparently as that of maxima f_3/f_1 and f_5/f_1 as well as that of maxima $\theta_x - \theta_0$ and λ_1. The above may be differently stated as follows:—As the flux-wave moves away from the origin its form gradually changes, becoming flatter and flatter until it reaches the point of maximum lag and maximum leakage coefficient. After this as it continues to move on its flatness steadily diminishes. This change of form is exhibited in fig. 9, in which are plotted the two wave forms for $x=0$ and $x=60$ in the series given in Table XVIII. They are placed inaccurately as regards relative phase, but the ordinates of the one for $x=60$ are increased so that the amplitudes of the first harmonics of the two curves are the same.

Further interesting information as to the behaviour of the upper harmonics of the flux may be obtained by considering their actual amplitudes. Taking the third harmonics for any of the series in this paper, we find that the leakage coefficients are different from what we estimated they would be for a primary or first harmonic of the same amplitude and frequency at the same section of the rod. In some cases, indeed, as will be seen from the tables given, the amplitude of the third harmonic actually increases as the distance from the magnetizing coil increases.

Hence we cannot consider the upper harmonics as being independent of the first harmonic or of each other. Each seems to get continually reinforced by the first harmonic as well as by other harmonics lower in order than itself as it

travels along. Such a transference of energy from the first harmonic to the higher ones must be accompanied either by a reduction of the amplitude of the first harmonic, which in these experiments would manifest itself as leakage in λ_1, or by a retardation of its phase helping to increase $\theta_x-\theta_0$, or by both ; so it is probable that this reinforcement of the higher harmonics by the first has an important bearing on the theory of the experiments described in this paper.

14. In order to investigate the effect of change of diameter we made two smaller bundles with wires taken from specimen D. One of these called d_1 contained 46, and the other called d_2 12 wires, and as D originally consisted of 185 wires the sectional areas of D, d_1, and d_2 were very nearly in the proportions of 16 to 4 to 1 or their diameters as 4 to 2 to 1.

One series of experiments was performed with each of d_1 and d_2 under approximately the same conditions of frequency and initial flux-density as the series already performed on D and recorded in Table XXI.

In Tables XXIII. & XXIV. are given the results of the new series, and it will be seen on comparing them with Table XXI. and with each other, that the effect of change of diameter is if anything more marked than that of permeability which was illustrated by the comparative series on specimens A and B.

Thus we find that the values of $d\theta/dx$ near the origin for the diameters 4, 2, and 1, are 4·75, 9·9, and 15·3 respectively; so that if we deduced a "velocity" from these figures we should get very different values for it, in the same material, with the same frequency and flux-density.

The change of diameter seems to have little or no effect on the max. value of the retardation, though it has a marked effect on the distance from the origin at which this maximum occurs. Thus the coincident points of max. retardation and max. leakage coefficient are at distances from the magnetizing solenoid approximately proportional to the diameters of the specimens in which they occur.

Again we find that reduction of diameter increases the leakage coefficients (for the same conditions of frequency and flux-density) approximately in the same ratio as the diameter is reduced.

TABLE XXIII.—46 wires from Specimen D.
$T = \cdot 050$.

$C = \cdot 0742[\sin \omega t - \cdot 042 \sin 3(\omega t + 0\cdot 28) + \cdot 0035 \sin 5(\omega t - 66)]$.
$T = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5)$.

x.	f_1.	f_3/f_1.	f_5/f_1,	θ.	β_3.	β_5.	$\theta x - \theta_0$.	λ_1.
0.........	2583	·0603	·0095	14·32	17·09	25·24	0	·0857
10.........	1096	·1113	·0158	24·23	7·07	14·75	9·91	·1181
20.........	336·6	·1530	·0302	31·41	5·59	9·76	17·09	·1508
25.........	158·3	·1630	·0345	33·45	4·83	8·93	19·13	·1596
30.........	71·3	·1583	·0374	32·86	5·69	9·35	18·54	·1559
40.........	15·0	·1400	·0321	31·18	8·44	13·64	16·86	·1147
50.........	4·76	·1214	·0225	28·14	10·37	18·80	13·82	·0742
60.........	2·32	·1033	·0205	25·92	10·45	19·94	11·60	

TABLE XXIV.—12 wires from Specimen D.
$T = \cdot 050$.

$C = \cdot 0463[\sin \omega t - \cdot 05 \sin 3(\omega t + 1\cdot 63)]$.
$F = f_1 \sin(\omega t - \theta) + f_3 \sin 3(\omega t - \theta - \beta_3) + f_5 \sin 5(\omega t - \theta - \beta_5)$.

x.	f_1.	f_3/f_1.	f_5/f_1.	θ.	β_3.	β_5.	$\theta x - \theta_0$.	λ_1.
0.........	681	·080	·0164	15·60	11·35	16·24	0	·1690
10.........	125·6	·1534	·0312	30·88	4·88	6·61	15·28	·2820
15.........	30·7	·1637	·0396	34·72	4·89	6·92	19·12	·3142
20.........	6·37	·1485	·0312	32·10	7·61	11·04	16·50	·2230
25.........	2·09	·1210	·0259	25·51	10·73	16·73	9·91	·1641
30.........	0·92	·0776	·0172	25·26	11·71	25·18	9·66	

In particular it is most striking that the max. values of λ_1, viz.:—

$$\cdot 3142, \quad \cdot 1596, \text{ and } \cdot 0789,$$

should be so nearly inversely proportional to the corresponding diameters, 1, 2, and 4 respectively, the corresponding products being

$$\cdot 3142, \quad \cdot 3192, \text{ and } \cdot 3156 \text{ respectively.}$$

In addition we find with the smaller bundles d_1 and d_2 the same change of form of the flux-wave as it passes along them as in the case of specimen D, and also the same peculiarity in the behaviour of the upper harmonics both as regards their amplitudes and phases as has been drawn attention to in the preceding paragraph in relation to the thicker bundle.

15. The fact that so many of the characteristics of the alternating flux attain either their maximum or minimum values at practically the same point in the specimen (particularly in D, d_1, and d_2) is undoubtedly the most striking result of this investigation.

This point might be called the critical point for the specimen under the conditions of initial flux and frequency used, its distance from the magnetizing solenoid the critical distance, and the values of the different characteristics of the flux there the critical values. For the existence of such a critical point we are unable to offer any explanation.

A more fundamental knowledge than any we at present possess of the nature of magnetization and permeability is required. Our want of any exact knowledge of the amplitude and phase relations between induction and magnetizing force and of the relations between the different harmonics of an induction wave precludes the possibility of a satisfactory solution on the basis of our present knowledge.

16. It was thought, however, that a partial explanation of some of the phenomena observed might be obtained if the initial flux was made up of elementary magnetic lines very heterogeneous as regards their phases. As these passed along the specimen there would be a retardation of the phase of each line due to "velocity" as well as a change of the phase of the resultant flux at any section due to groups of lines in special phases leaking out at special distances from the origin. If this were so, the phases of the lines at different distances from the axis of the rod or bundle should be different, as at any section those lines about to leak out would be near its circumference. To test this the wires of specimen D were divided into three groups. The first group of 28 wires was taped to form a circular cylinder, and round it was wound a search-coil of 50 turns insulated. Over this central cylindrical bundle the

second group of 45 wires was uniformly distributed and taped so that the whole formed a circular cylinder coaxial with the first. An insulated search-coil of 50 turns was wound round the middle of the compound bundle formed. The remaining group of 110 wires was now added and taped so as to be coaxial with the others, and a third search-coil of 50 turns was wound round the middle of the whole.

The bulging caused by the search-coils was not very great. The magnetizing coil was wound on a bobbin, and could be placed at any distance on the specimen from the search-coils. By means of a switch the search-coils could be connected in different ways, so that when continued to the wave-tracer the latter determined for us the flux-waves through the central bundle or through either of the surrounding tubular bundles of wires for different distances from the magnetizing coil.

The result of this investigation was to show that while at the centre of the magnetizing coil the flux-density and phase varied over the section—the phase of the first harmonic for the outer layer being 4·56 degrees ahead of that for the central portion and the amplitude varying from 67 lines per wire for the outside portion to 58·5 lines per wire for the central,—yet for distances greater than 15 cms. from the magnetizing coil there was no appreciable difference in the phases of the first harmonics, wave-forms, or densities of the fluxes through the three different portions of the compound bundle. Hence no such explanation as that imagined at the beginning of this paragraph can be tenable.

17. It was also thought that if we determined in amplitude and phase the magnetizing force required to produce the different resultant fluxes at different sections of the specimen in any one series of the transmission experiments, we might obtain some information that would help to explain the phenomena.

This was done for specimen A and for a cylindrical bundle made up of 46 of the iron wires from specimen D.

The specimen was inserted in a long solenoid and by means of the wave-tracer, using it as described in a former paper *

* T. R. Lyle, "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. vol. ix. p. 102 (1905).

by one of us, the amplitudes and phases of the magnetizing forces required to produce uniform oscillating fluxes of different amplitudes and phases were obtained and the results plotted. From these curves and the results given in this paper on specimen A, and similar results obtained from the narrow bundle, we were able to obtain the amplitude and phase of the *resultant* magnetizing force that acted at each section of the specimens in the transmission experiments. When these characteristics of the magnetizing force were plotted against corresponding distances along the specimen from the magnetizing solenoid, the curves obtained were less instructive than those for the characteristics of the flux at different distances which have been given in this paper.

18. To sum up:—When waves of magnetic flux that have been started by alternating currents in a short solenoid placed at the centre of a long iron rod, or bundle of iron wires, are transmitted along the rod or bundle, we find that—

(1) The retardation of phase of the first harmonic of the flux at any point distant x from the centre, behind the first harmonic of the initial flux, first increases with x, attains a maximum and then diminishes, and keeps diminishing until the flux is dissipated if the specimen is sufficiently long for this to be effected.

(2) The leakage coefficient λ_1 of the amp. f_1 of the first harmonic, which we define by the equation $\lambda_1 = -\frac{1}{f_1}\frac{df_1}{dx}$, first increases with x, attains a maximum, and then diminishes and keeps diminishing until the flux is dissipated.

(3) The distances from the magnetizing solenoid at which the phase retardation and the leakage coefficient become maxima are equal (or very nearly so), and the point at which these maxima occur we call the *critical point* of the specimen, and its distance from the magnetizing solenoid the *critical distance*, for the particular initial flux and frequency used.

(4) Previous investigators of this subject, using less sensitive methods, were only able to obtain observations within the critical distance, and from their observations concluded (1) that the leakage coefficients were practically constant at all distances x, thus arriving at a logarithmic decrease of flux

amplitude; (2) that the fairly regular space-rate of phase retardation $d\theta/dx$ observed would be completely accounted for by a velocity of magnetization v, deduced by the formula

$$v=\frac{2\pi}{\mathrm{T}d\theta/dx}.$$

We find that neither of these conclusions is correct, as (*a*) λ, for the same rod and conditions of experiment, varies within wide limits along the rod, and (*b*) the retardation, after increasing as they observed within the critical distance, becomes stationary at the critical point, beyond which the phase of the flux advances: so that if their conclusion (2) were correct, we should have to admit an infinite velocity at the critical point and a negative velocity beyond it.

(5) As the flux passes along the specimens made of wires, the amplitudes f_3, f_5 of its upper harmonics get continually reinforced by a transference of energy from the first harmonic (see § 13, Tables XVIII.–XXI.). The ratios $f_3/f_1, f_5/f_1$ first increase with x, attain maxima, and then diminish; at the some time the differences β_3, β_5 between the phases of these harmonics and the phase of the first diminish first with x, attain minima, and then increase; and the positions of maxima of f_3/f_1, f_5/f_1 and minima of β_3, β_5 are at (or near) the critical point. From this it results that as the flux-wave moves away from the origin its form changes, becoming flatter as x increases (see fig. 9) until the critical point is reached, when its flatness is a maximum; beyond this as x increases its flatness diminishes.

(6) The effects of an end on a flux-wave approaching it are to increase the leakage coefficient and to cause an advance in the phase. (See fig. 6.)

(7) When a rod whose behaviour has been determined is shortened and subjected to the same magnetization as before, no difference is observed in the flux-waves until (for a $\frac{1}{4}$ in. rod) within 30 cms. from the new end. (See § 8 and fig. 6.)

(8) For the same frequency, initial flux, and cross-section, increase of permeability increases the critical distance, increases the critical value of the retardation (see fig. 7), and

diminishes the critical value of the leakage coefficient (see § 9 and fig. 8).

(9). Eddy currents diminish the critical distance, increase the critical value of the retardation, and increase the leakage coefficients. (See § 12.)

(10) For the same frequency, initial flux-density, and material, increase of diameter of the specimen increases the critical distance, has little or no effect on the critical retardation (hence making $d\theta/dx$ near the origin less), and diminishes the leakage coefficients: in fact, we find that the critical distance is approximately proportional to the diameter, and the critical value of the leakage coefficient inversely proportional to the diameter. (See § 14 and Tables XXI., XXIII., XXIV.)

(11) For the same specimen and initial flux, increase of frequency increases the initial value of $d\theta/dx$, increases the critical values of the retardation and the leakage coefficient, and slightly diminishes the critical distance. (See figs. 1, 2, and the Tables.)

(12) For the same specimen and frequency, increase of initial flux increases the critical distance (see fig. 4), increases the critical value of the retardation (fig. 4), and reduces the critical value of the leakage coefficient (see fig. 5): also for low values of the initial flux F_0, $d\theta/dx$ near the origin increases with F_1, but seems to approach an upper limit for high values of F_0 (see fig. 4).

XXIV. *On the Use of Chilled Cast Iron for Permanent Magnets.* *By* ALBERT CAMPBELL, *B.A.* (From the National Physical Laboratory.)*

EARLY in the past year an interesting paper † was published by Mr. B. O. Peirce of Harvard University, drawing attention to the fact that chilled cast-iron is in many

* Read January 26, 1906.

† Amer. Acad. Proc. xl. 22. pp. 701–715, April 1905. Dr. Watson has kindly drawn my attention to Mr. J. R. Ashworth's experiments on chilled cast-iron rods (Proc. Roy. Soc. vol. lxii. p. 210, Dec. 9, 1897); he found that the magnetic quality of these was comparable with that of tungsten steel.

instances a suitable material for permanent magnets. As the subject is of interest to scientific experimenters and of considerable importance to instrument makers, I undertook some time ago a short research upon it, with a view, firstly of obtaining some measurements by standard methods, and secondly of finding, if possible, an easy method of chilling the material so as to give good results.

Form of Test Pieces.—The cast iron tested was of ordinary commercial quality and was obtained in the form of rods and rings. The rods, which were of rectangular section, were shaped to dimensions usual in such tests, viz., 10 cm. × 1 cm. × 1 cm.

By using rods of these dimensions it is easy to compare the results with those for various kinds of steel which we have already tested or with those published by Madame Curie * and other observers.

Two rings were tested; they were of rectangular section, their mean diameters being nearly equal (12·5 and 13·0 cm.). Their cross-sections, however, were very different, being 1·00 sq. cm. and 6·0 sq. cm. respectively. The object of testing a thick and a thin ring was to find if our method of chilling was effective for the hardening of thick castings.

Heat Treatment.—All the test pieces were heated to 1000° C. in a gas muffle furnace, the temperature being measured by a thermo-junction in the usual way. Each piece was removed from the furnace and quickly chilled in water at the temperature of the room. As Mr. Peirce has pointed out, great care is necessary in handling the cast iron at this high temperature (so near its melting-point) for it becomes very brittle. For this reason the thick ring was placed in the furnace on a U-shaped piece of wrought iron, it was lifted out by means of this support, and the two were plunged together into cold water. By this method the brittle material could be handled without risk of breakage.

Tests on Rods.—After the chilling, the rods were magnetized to saturation by means of a very strong magnetic field.

* *Bulletin de la Société d'Encouragement pour l'Industrie Nationale,* Jan. 1898.

This was produced by a solenoid consisting of 70 turns with a length of 16 cm. and having a resistance of about 0·1 ohm. A large current was sent through this coil by connecting it for a very brief interval to a 50-volt circuit (by the process commonly known as "flashing"). Each rod was then tested for

(1) The maximum remanent flux density B (at the medial section of the bar).

(2) The coërcivity H_0, *i.e.* the value of the demagnetizing magnetic force required to annul this remanent magnetism.

(1) In order to measure the medial B, a small square search-coil of 20 turns of very fine wire was used; it was just large enough to slip along the rod. The search-coil was connected to a calibrated ballistic galvanometer, and, when the coil was slipped off the rod, the resulting deflexion gave the required B in the usual manner.

(2) The coërcivity was found by Madame Curie's method, as follows:—The magnetized bar was fixed at the centre of a long solenoid, a search-coil being so arranged that it could be slipped off the bar from the mid position and replaced without removing the bar from the solenoid. By sending a measured current through the solenoid the bar was subjected to a known demagnetizing field. This field was gradually increased until the search-coil when slipped off gave no throw on the galvanometer, thus showing that B had been reduced to zero. The value (H_0) of the magnetic force when this took place is a good indication of the permanence of the remanent magnetism.

If H_0 is large, we should expect the bar to hold its magnetism very obstinately. The following table gives the results for four of the rods (Nos. 1 to 4). For the sake of comparison the corresponding numbers are given for exactly similar rods (M and A) of hardened magnet steel of well known makers, M being supplied by Marchal of Paris, and A coming from the Allevard Forge.

Fig. 1.

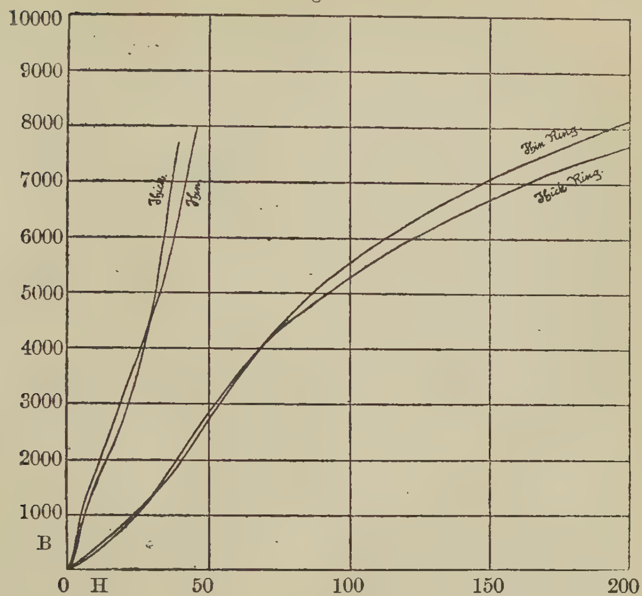

10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
B
0 H
50
100
150
200
Thick
Thin
Thin Ring.
Thick Ring.

Fig. 2.

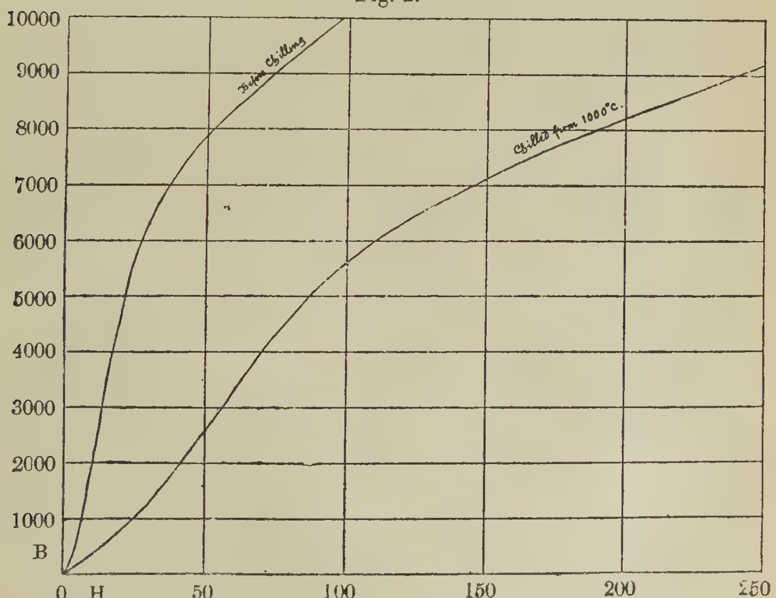

10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
B
0 H
50
100
150
200
250
Before Chilling
Chilled from 1000°C.

TABLE I.

Marks.	Material.	Maximum Remanence.	Coërcivity.
No. 1	Chilled Cast Iron.	1775	52·8
No. 2	„ „	1670	48·9
No. 3	„ „	1690	50·4
No. 4	„ „	1850	52·1
M	Magnet Steel.	2550	55·5
A	„ „	2950	73·0

The above results show that the hardened cast iron rods are not very much inferior to the ordinary magnet steel (M) either in original strength or in power of resisting demagnetization.

Tests on Rings.—Primary and secondary coils were wound on each of the chilled rings, and the (H, B) curves shown in fig. 1 (p. 343) were determined by the ordinary ballistic method. In addition to these the coërcivities (H_0) were found for a number of different values of B, and the (H_0, B) curves are also shown.

Fig. 2 gives the (H, B) curves for the thin ring before and after chilling.

We see from fig. 1 that, after chilling, the thick ring is magnetically very similar to the thin one; and thus it is evident that, by the treatment already described, a quite heavy casting can be satisfactorily hardened throughout. Since the thin ring had the same cross-section as the rods, we may assume that its magnetic hardness was equal to theirs. The experiments here described amply confirm Mr. Peirce's results, and show that large cast iron magnets can be made cheaply and easily; we hope that this short paper will draw the attention of our instrument makers to the matter. In conclusion I would express my thanks to Dr. H. C. H. Carpenter for his kind help in the chilling of the specimens.

XXV. *Fluorescence and Magnetic Rotation Spectra of Sodium Vapour, and their Analysis.* *By* R. W. Wood *.

[Plates I.–V.]

Previous work, which has been recorded in the Philosophical Magazine †, convinced me that a careful study of the remarkable optical properties of the vapour of metallic sodium would, in time, furnish the key to the problem of molecular vibration and radiation. This opinion has been strengthened by the work of the past year, and though much remains to be done, it seems best to place the results already obtained on record. In no other case that I know of is the molecular mechanism so completely under the control of the operator. Its periodicities can be studied in a variety of ways: by absorption, by cathode-ray stimulation, by excitation with light, either white or monochromatic, and lastly by its remarkable selective magnetic rotation of the plane of polarization.

The vapour is, in every case, that obtained by heating metallic sodium in steel or porcelain tubes, usually highly exhausted. From a study of the dispersion of the vapour, it seems probable that we may be dealing with clusters of molecules with which a certain amount of hydrogen may be associated.

As I have shown in a previous paper ‡, if a pool of sodium is heated in a highly exhausted horizontal tube, the top of which is cooler than the bottom, the vapour has an enormous optical density close to the surface of the pool, and a very small density along the roof, the non-homogeneous layer acting as a prism. The only way in which I can reconcile this state of things with the kinetic theory, is to assume that the vapour leaves the metal in the state of molecular clusters, which gradually break up into smaller clusters and eventually into molecules. This is of course only an hypothesis, and I

* Read October 26, 1906.

† "Magneto-Optics of Sodium Vapour," Phil. Mag. Oct. 1905. The "Fluorescence of Sodium Vapour," Phil. Mag. Nov. 1905.

‡ "A Quantitative Determination of the Dispersion of Sodium Vapour," Phil. Mag. vol. viii. p. 293 (Sept. 1904).

mention it in the present paper merely to indicate that our vibrating mechanism may be an aggregate and not a single molecule. It is also possible that hydrogen atoms are associated with the sodium, for the work on the dispersion indicated that there was present always a small trace of some gas other than sodium, which no amount of pumping would remove ; that is, it appeared to be tangled up in the sodium vapour, condensing with it in the cooler parts of the tube. All of this is, however, irrelevant, for we are for the present merely engaged with the study of a certain remarkable vibratory mechanism, and for the present need not concern ourselves whether it is a molecule, a cluster of molecules, or a compound molecule.

We will begin by a description of the various spectra which we shall study and compare in the present paper. The spectrum region with which we are concerned lies between wave-lengths 4600 and 5700, *i. e.* the region of the green-blue channelled absorption spectrum.

The Absorption Spectrum.

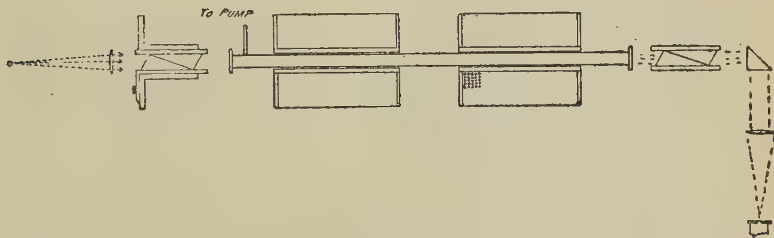
Photographed with the twelve-foot concave grating, the absorption spectrum is found to consist of a multitude of fine lines, to the number of about 1500 in the region specified. Its appearance has been found to be profoundly affected by the presence of hydrogen or any inert gas. It is shown on Pl. I. (fig. 1, *g*), photographed in the second order with a twelve-foot concave grating. In hydrogen at atmospheric pressure its appearance is shown by spectra *f* and *h*, the fluted appearance being very marked. In a high vacuum its appearance is shown by spectrum *b*, the flutings having entirely disappeared. The chief change appears to lie in the increased absorbing power of certain lines or groups of lines. Spectrum *d* is taken under nearly the same conditions, the sodium vapour being less dense, however. A careful study of the effect of the inert gas has not yet been made, and it is mentioned here only on account of its relation with the subject of the paper. Comparatively few of the absorption-lines have any relation with the fluorescent and magnetic rotation spectra, *the ones concerned, however, being those which are affected by the presence of the inert gas.*

Moreover, both the fluorescence and magnetic rotation disappear, *i. e.* cannot be excited, when the sodium vapour is formed in an atmosphere of hydrogen or other inert gas. Of this matter more will be said later on.

The Magnetic Rotation Spectrum.

It was found last year that a number of vapours, showing fine and sharp absorption-lines, when placed in a powerful magnetic field, rotate the plane of polarization for wave-lengths agreeing with that of the absorption-lines, not all of the absorption-lines showing this rotatory power, however. The arrangement of the apparatus for showing the bright-line magnetic spectrum of sodium is shown in fig. 1.

Fig. 1.



A piece of thin seamless steel tubing of such a diameter as to slip easily through the hollow cores of the electromagnet, from which the conical pole-pieces have been removed, is procured. A short piece of small brass tubing is brazed into one end, through which the tube is exhausted.

A lump of sodium the size of a walnut is melted in an iron crucible, and poured out into a V-shaped trough made of thin sheet iron. As soon as the bar is solid it is placed in the iron tube, one end of which has been previously closed with a small piece of plate glass cemented on with sealing-wax. The tube is introduced into the magnet, the sodium bar pushed to a position midway between the helices, and the other end closed with a piece of glass in a similar manner. The ends of the tube should be coated while hot with sealing-wax before the introduction of the sodium. One has then only to wave a Bunsen flame over them and press on the piece of glass, previously heated; the sealing-wax should come into

optical contact with the glass to insure an air-tight joint. The tube is now connected with an air-pump which will produce a vacuum of a millimetre or two. If the air-pump leaks, it is a good plan to place a glass stopcock between the pump and tube to prevent the entrance of traces of air after exhaustion. For purposes of demonstration it is sufficient to heat the tube gradually with a Bunsen burner turned down low. In the present work, however, where constancy of temperature was essential, electrical heating was invariably used.

The light from an arc-lamp, made parallel by a lens, is passed through a Nicol prism, the steel tube, and a second nicol, after which it is brought to a focus by means of a second lens upon the slit of a spectroscope. With the steel tube cooled below the point at which sodium vapour forms, the nicols are set for complete extinction, and the field of the spectroscope becomes dark. The tube is now heated and the magnet turned on, the air-pump being worked occasionally to remove the hydrogen which is given off from the sodium. A bright yellow spot will appear on the slit of the spectroscope, which is seen to be made up of radiations chiefly in the immediate vicinity of the D lines. The phenomena at the D lines have been fully described in the paper already alluded to (Magneto-Optics).

When the vapour acquires a considerable density, a most magnificent bright-line spectrum appears in the red and green-blue region. Each bright line corresponds to a dark line in the absorption spectrum, but only a small percentage of the dark lines appear to exercise a rotatory power. Some of the strongest absorption-lines are absolutely unrepresented in the magnetic-rotation spectrum, which indicates that there is some radical difference in the absorbing mechanism.

It is with the bright-line spectrum in the green-blue region that we are now concerned. This spectrum has been photographed with the large, three-prism long-focus spectrograph, and also with the twelve-foot concave grating. Reproductions of the prism spectrograms are given on Pl. II., *f* and *m*. The magnetic spectrum made with the large grating and the absorption spectrum recorded on the same plate are reproduced on Pl. I., *c* and *d*.

Only about sixty lines appear in this spectrum, in contrast

to the 1500 in the absorption spectrum. The intensities are very variable and apparently bear no relation to the intensities of the corresponding absorption-lines. The rotatory lines in many cases coincide with the heads of the groups of absorption-lines, though the centre of the line appears to be slightly displaced beyond the head of the group of absorption-lines. The displacement is, however, very slight, not more than half the width of the line. A list of the wave-lengths of all the lines visible on the negative follows. The approximate intensities are represented by numerals, 10 indicating the maximum intensity and 1 the minimum.

Green Rotation Spectrum.

1	5225·34	7	5040·65	2	4839·56
1	5218·49	2	5033·54	9	4837·49
1	5212·02	2	5025·66	2	4819·43
1	5186·70	3	5003·12 broad	1	4814·60
1	5179·71	10	5001·57	1	4812·68
1	5172·98	5	4979·34	3	4810·16
1	5171·98	2	4970·85	3	4802·62
1	5169·04	1	4967·10	5	4792·67
1	5165·85	1	4964·39	3	4782·89
1	5147·50	9	4962·85	1	4777·00
1	5140·71	1	4958·62	1	4766·94
3	5133·73	4	4933·93	6	4756·69
3	5126·54	5	4932·64	1	4752·04
2	5119·34	3	4924·32	2	4738·51
1	5095·70	4	4912·10	4	4727·52
2	5094·78	1	4904·67	1	4716·90
7	5087·31	1	4903·38	1	4715·63
7	5079·78	1	4896·65	1	4703·78
2	5071·58	1	4894·58	2	4692·54
1	5052·83	1	4892·77	2	4670·30
1	5049·56	2	4883·81		
5	5048·49	3	4865·59		

At first sight there appears to be no regularity whatever in the distribution of the lines, except perhaps above wave-length 502, where they appear to be about equally spaced in small groups of three or four lines each. Without the aid of the fluorescence spectra of the vapour excited by monochromatic light, it is doubtful whether any regular series of lines could be found in the magnetic spectrum, for, as has been subsequently found, more than half of the lines in the series are absent, and there are six or more series present. The fluorescence spectrum with white-light excitation is shown on Pl. I., *e*, which is from a negative made with the twelve-foot grating. As will be seen, the bright lines coincide

with the bright lines of the magnetic spectrum, though much broader. It will be easier to explain how the series were picked out after we have commenced the study of the fluorescence. I have added to Pl. I. spectrum *i*, the magnetic spectrum with the series indicated. There appear to be five distinct series and a number of lines which thus far have not been brought into any definite relation with one another. These series we will number 1, 2, 3, 4, and 5. All the lines belonging to the first series have one dot under them, those belonging to the second have two dots, &c. These series are shown separate on the chart (Plate V.) at the top. Absent lines are indicated thus : ◊.

The fifth series is at the top, the fourth next, and so on down, the extra lines being indicated in the lower row. This arrangement is considered provisional : it is the best that I can do at the present time, and I believe that it is correct in the main. We shall see presently, however, that photographs of the fluorescence stimulated by monochromatic radiation will have to be made with the large concave grating before we can be absolutely sure of all the lines. We will drop the magnetic spectrum for the present, and consider

The Fluorescence Spectrum.

In the previous paper I have described some of the remarkable changes which take place in the distribution of energy in the fluorescence spectrum of sodium vapour when the wave-length of the exciting light is changed. With white-light stimulation the general appearance of the spectrum is shown in Pl. IV. fig. 1, A. There is, in addition, a broad double band at the position of the D lines, and a red-orange spectrum which, when the vapour is dense, is distinctly banded. In the present paper we shall be concerned chiefly with the portion figured, for it is in this region that most of the remarkable changes occur. As will be seen, it is comprised between wave-lengths 460 and 570, and is devoid of any apparent regularity in the distribution of its lines, except in the region above $\lambda=505$ where we have lines spaced with considerable regularity, the spacing becoming less as the wave-length increases. The distribution of intensity in this portion of the spectrum is such as to give it a fluted

appearance, the flutings being most conspicuous in the region between $\lambda=505$ and $\lambda=535$. With white-light stimulation the flutings cannot be made out above 540, as can be seen from Pl. IV. fig. 1, B, in which the upper limit of this part of the spectrum is shown. The fine lines are present in this region, becoming, however, less and less distinct as the upper limit of the spectrum is approached. If, now, instead of stimulating the vapour with white light, we employ blue light in the region 460–465 obtained from a spectroscope for the excitation the fluorescent spectrum presents a totally different appearance (fig. 1, C). The blue region, corresponding in its range to that of the exciting light (indicated by a double arrow), appears as before, and the upper limit of the spectrum between wave-lengths 540 and 565, *the intermediate portion being entirely absent*, as shown in the lower spectrum of fig. 1, C. Furthermore, at the upper or yellow end, there now appear the flutings which were absent when the fluorescence was stimulated with white light.

If, now, we gradually increase the mean wave-length of the exciting light, the region of maximum intensity in the fluorescence spectrum moves down from the yellow into the green, as is shown by the remaining photographs in fig. 1, C (Pl. IV.). Moreover, as I pointed out in the previous paper, the positions of the fluted bands change slightly, the positions of the individual lines which make up the bands remaining fixed however, the shift resulting from a change in the distribution of intensity among the lines. The reason of this curious phenomenon will appear when we come to the study of the fluorescence spectrum excited by strictly monochromatic radiations.

The spectrum stimulated by white light I have named the "complex fluorescent spectrum," for it has been found that it is a superposition of a number of simpler spectra, any one of which can be independently excited by suitably controlling the wave-length of the stimulating light. Indications of something of this sort were found last year, and were described in the preliminary paper. An insufficient number of photographs were obtained, however, at the close of the university year, to make anything like a complete analysis of the complex spectrum possible.

During the past winter and spring a careful study has been made of the relations existing between the complex fluorescent spectrum, the absorption spectrum, and the bright-line rotation spectrum described in the earlier paper. The fluorescent spectrum has at last been photographed with the twelve-foot concave grating, enabling a study to be made of its more minute structure.

Some very remarkable effects have been observed with monochromatic stimulations obtained by the isolation of certain lines from metallic arcs, which yield comparatively simple fluorescent spectra made up of widely separated sharp lines, placed in many instances at nearly equal intervals along a normal spectrum. A given series of lines can be brought out by stimulating with light of any wave-length corresponding to that of some line in the series, but when the stimulations occur at certain points, some of the lines may be absent, gaps appearing in the series. The most conspicuous example is the case of stimulation with the cadmium line 480, which will be considered in detail presently. It will be remembered that certain lines are absent in each series, in the magnetic spectrum.

The apparatus employed in the experiments was essentially the same as that used in the earlier work. It consisted of a seamless tube of thin steel three inches in diameter and thirty inches long, with a steel retort at its centre in which a large amount of sodium could be stored. The retort was made by fitting two circular disks of steel to a short piece of tubing, just large enough to slip snugly into the larger tube. The circular ends of the retort were provided with oval apertures, as shown in Pl. III., fig. 1. The retort was half filled with sodium, the molten metal being poured in through one of the apertures. It was then introduced into the tube and pushed down to the centre, after which the plate-glass ends were cemented on, as shown in the figure. This arrangement prevented the rapid diffusion of the vapour, and enabled a large supply of metal to be kept at the centre of the tube. The tubes used in the earlier work required re-charging after two hours' continuous operation, while the retort-tube could be operated for several hundred hours on a single charge.

The tube was exhausted with a Fleuss pump and heated

at the centre with a large burner, the ends being kept cool by jackets of absorbent cotton which dipped into pails of water.

The illuminating beam of either white or monochromatic light was focussed just within one of the oval apertures of the retort, falling upon the opposite wall a little to one side of the other aperture. By covering the further end of the tube with a black cloth, the fluorescent spot showed against the dead black background of the second oval aperture, and its spectrum was therefore uncontaminated with the exciting radiations.

A large three-prism spectrograph was constructed for photographing the spectra. The prisms were of clear dense flint four inches in height, and the focal length of the lenses thirty-six inches.

Since only lenses such as are used for telescopes were available, the spectrum lines are not so sharp as one would wish, except near the axis of collimation. By adjusting things so that the centre of the fluorescent spectrum fell at this point, the definition was pretty fair throughout its extent, and wave-lengths could be determined with an error not greater than one or two Ångström units.

The photographs of the complex spectrum of the fluorescence excited by sunlight obtained with this instrument (Pl. IV. fig. 1) showed peculiarities which made it appear of the utmost importance to study the spectrum under higher dispersion. The green fluorescent spot had, after repeated improvements in the apparatus, attained such brilliancy that I felt sure that records could be obtained with the twelve-foot concave grating. An all-day exposure was found to be sufficient, the resulting spectrogram, with the iron comparison lines and the wave-length scale, being reproduced on Pl. I. fig. 2. The scale was printed separately, and slight errors occur, due to stretching and shrinking of the prints. They are not greater in any case than 1·5 A.E. This plate shows the minute structure of the complex spectrum, and enables us to measure the wave-lengths of the bright lines with far greater accuracy than could be done with the plates made with the prism-spectrograph. As much of the detail is lost during the process of reproduction, I have prepared a very careful

drawing of one of the groups of bands at wave-length 5200 (fig. 2). The drawing was made from a print with the aid of a hand-magnifier, and the peculiarities shown are found throughout the entire spectrum.

Fig. 2.

The bright lines are sharp, and quite as narrow as the iron lines of the comparison spectrum. We must remember, however, that the slit was not very narrow, 0·2 mm. perhaps, and it is quite possible that a further contraction would not decrease the width of the fluorescent lines. Each bright line is in general accompanied by two lateral wings, which terminate quite sharply the narrow spaces between, every two adjacent wings appearing as narrow dark lines. These wings do not in general appear with strictly monochromatic stimulation. In the work of last year, when studying the remarkable changes which occur in the spectrum stimulated with the fairly homogeneous light furnished by the monochromatic illuminator with very narrow slits (see previous paper), I observed that as the wave-length of the light was very gradually altered, the fluorescent lines appeared with wings first on one side and then on the other, the change in the appearance of the line reminding one of a flag flying first on one side of the mast and then on the other. With the strictly monochromatic illumination obtained with the isolated metallic arc lines, the fluorescent lines are usually devoid of wings, though in some instances the wings are found, and sometimes the wings appear without the lines. These circumstances appear to indicate that the wings are due to the stimulation of the electron by frequencies slightly greater and slightly less than its own natural frequency.

I have not yet had time to repeat last year's experiments with the monochromatic illuminator, and plan to make a further study of the changes which accompany very gradual changes in the wave-length of the exciting light. The observations are very difficult and uncertain, as the light

furnished by the monochromatic illuminator is not very bright when its slits are made as narrow as possible, and the fluorescence spectrum can only be observed by carefully rested eyes. Probably by using a *large* prism-spectroscope as an illuminator better results can be obtained. What is most desired is a light siren!

Analysis of the Complex Spectrum. Stimulation with Monochromatic Light.

It was found impracticable to use the monochromatic illuminator for the study of the simple spectra which made up the complex spectrum. Even with its slits very narrow, the wave-length range of the emitted light was wide enough to cover several of the absorption-lines of the vapour. The earlier work had shown that the light of the cadmium spark was capable of exciting fluorescence, and experiments were accordingly started with metallic arcs. Just at this time came the very opportune invention of the fused quartz metallic arc-lamps by Stark, working in the Heraeus laboratory at Hanau. Two of these lamps were immediately ordered, one filled with cadmium, the other with zinc. Their form is shown on Plate III. fig. 1. The lamp is kept in communication with a mercury pump during its operation and stands in a dish of water. The cadmium lamp worked well on a circuit of 110 volts direct, but the zinc lamp gave better results on the 220. They are started by a small induction-coil, one terminal of which is connected to the negative pole of the lamp, the other twisted around the quartz U-tube. A blast-lamp is directed against the tube until the portion above the metal electrodes is red-hot, the coil is then started, and the arc usually forms at once. As exposures of eight or ten hours were often necessary, and the lamps have a trick of going out every half-hour or so, an automatic starter was devised, which turned on the coil the moment the lamp went out. As soon as the arc struck again, the coil was stopped. This arrangement is figured on Plate III. fig. 2, and consisted of a small electromagnet in circuit with the lamp, which pulled a steel spring away from a brass screw as long as the lamp burnt. The spring and screw were inserted in the primary circuit of the coil,

The cadmium lamp burns with a greenish-blue light of dazzling brilliancy, the zinc lamp with a curious purple light, which causes all the woodwork in the room to appear blood-red, while most other objects appear bluish white or purple. Both lamps excite a brilliant fluorescence of the sodium vapour when their images are thrown upon the oval aperture of the retort. In this case the fluorescence is excited by several different radiations. Various devices were used for picking out one line at a time. The cadmium radiations which are capable of exciting fluorescence have wave-lengths 5086, 4800, and 4678. Colour screens and the Fuess monochromatic illuminator, as well as the thin crystals of chlorate of potash (described in the Phil. Mag. for June), were tried; also a block of quartz, cut perpendicular to the axis, placed between two nicols. The arrangement which gave the best results, and appeared to be accompanied with the least loss of light, is the one figured on Pl. III. fig. 2. One vertical tube of the lamp is used as the source, the light from which, after collimation, passes through a large bisulphide-of-carbon prism, and is focussed upon the retort by an achromatic telescope objective with an aperture of 12 cms. and a focal length of 2 metres. The dispersion of the prism was sufficient to separate completely the monochromatic images of the lamp, any one of which could be thrown into the aperture, the light passing by the edge of the 90° prism by means of which the fluorescent light was reflected through a lens and thence upon the slit of the spectroscope.

The sources of light which have thus far been successfully employed in stimulating the fluorescence of the vapour are the following: quartz arc-lamps containing cadmium, zinc, and thallium; ordinary arcs between lead, silver, bismuth, and copper electrodes; lithium, sodium, and barium arcs, and vacuum-tubes containing helium and hydrogen.

Unfortunately the quartz lamps are very expensive, and become almost useless after a run of about thirty hours, the surface becoming granular and an opaque black deposit forming on the inner walls. As exposures of eight hours are sometimes necessary, it will be seen that lamps at thirty dollars apiece, with an average life of thirty hours, make the investigation an expensive one.

The photographs of the fluorescent spectra obtained with monochromatic stimulation are reproduced on Plate II.

After each exposure the D lines were recorded on the plate, so that the different spectra could be brought into coincidence for purposes of comparison: the D lines will be found at the extreme right in each spectrum. The photographs have been reproduced as negatives, and the point or points coinciding with the wave-lengths of the stimulating light are indicated by arrows. A large-scale drawing or chart of the most interesting of these spectra, together with others too faint for reproduction, has been made on cross-section paper, the points of excitation being indicated by arrows, as in the photographs. (Folding Plate V.)

Drawings of the complex spectrum and the magnetic-rotation spectrum made from the photographs obtained with the large concave grating will be found at the bottom of the chart: the former is a positive. At the top I have given a composite drawing which represents a superposition of all the simple fluorescent spectra thus far obtained. Immediately below it will be found the spectra excited by the complete radiation of the cadmium and zinc tubes. In each case there are three different exciting radiations simultaneously applied, yet it is almost impossible to find two fluorescent lines which coincide. The other spectra are excited for the most part by a single monochromatic radiation, the wave-length of which is indicated by the arrow. I have not yet obtained photographs of the fluorescent spectra excited by the separated radiations of the copper arc or by the separated zinc lines 468 and 472; consequently these have been drawn together. It is possible, however, by comparing the spectrum excited by copper with the one excited by zinc 4811, to make a guess as to which lines belong together.

We will begin by a study of the spectra excited by the cadmium radiations.

Cadmium Stimulation.

Photographs of the fluorescent spectrum obtained with the cadmium-arc excitation are shown on Pl. II., *h*, *i*, *j*, *k*.

Of these, *i* and *k* are excited by all three cadmium lines; the former taken with a much narrower slit than the latter.

Spectrum *h* was taken when the sodium vapour was excited by the line 480; observed visually, it consists of twelve bright lines, in groups of two and four, as shown on the chart immediately above the complex spectrum at the bottom. Midway between the groups very faint lines can be perceived if the eye is carefully rested. The strong lines are arranged thus: two, absent line, four, absent line, two, absent line, four. The absent lines, or, more correctly speaking, the faint lines, evidently belong to the same series, and taken collectively the lines will be found to be very nearly equidistant, measured along a *normal spectrum*.

The wave-lengths of the lines in this series, as determined from measurements of the plates obtained with the prism-spectrograph, are as follows:—

λ.	λ differences.	λ.	λ differences.
4760		..	$\frac{76}{2}$ 38
4800	34	5095	38
4838	38	5133	
..	$\frac{70}{2}$ 35	..	$\frac{74}{2}$ 37
4908		5207	
4946	38	5245	38
4983	37	5283	38
5019	39	5321	38

These wave-lengths I consider to be accurate to within about 2 A.E. or $\frac{1}{3}$ of the distance between the D lines.

It is, of course, of the utmost importance to determine the law which governs the spacing of the lines in the simple spectra. A criterion may perhaps be obtained by referring to the magnetic rotation spectrum of the vapour, the lines of which correspond in general to the lines of the fluorescence spectrum. This spectrum has been photographed with a large concave grating and the wave-lengths determined certainly to within a tenth of an Ångström unit. The strong lines at the following points of the spectrum form a series analogous to the series obtained with the cadmium 480 series.

λ.	λ differences.
5119·34	
5079·78	39·56 A.E.
5040·65	39·13
5001·57	39·08
4962·85	38·67
4924·32	38·52

This we have called the first series.

The wave-length differences are, in this case, much more nearly constant, and decrease progressively.

The lines of this series are especially conspicuous in the magnetic rotation spectrum (Pl. I., fig. 1, *c*), hence I have mentioned it first; they appear in the fluorescence excited by the lead line 5001, as will be seen by reference to the chart.

If now we try to fit one of the magnetic series to the cadmium 480 fluorescence series, we find that the third magnetic series coincides with it between 5019 and 5134, while in the violet region it coincides with the fourth magnetic set. I do not feel sure whether this peculiarity is due to slight errors in the determination of wave-lengths or not. I think not, however, for I have very carefully superposed the two negatives (cadmium fluorescence and magnetic rotation spectra), both taken with the same instrument, and find the same disagreement. We cannot be sure of anything, however, until the cadmium series has been photographed with the grating.

During the coming year I expect to photograph the fluorescence spectrum excited by cadmium and zinc radiations with the large concave grating. It will then be possible to determine the wave-lengths of the lines to within a tenth of a unit. One of the most remarkable facts connected with the appearance of the lines of a series is that the distribution of energy among the individual lines depends upon the point of excitation.

Unfortunately there are very few arc lights bright enough to excite fluorescence. It was found, however, that the silver line 5207, which coincides with one of the fluorescent lines of the cadmium 480 series, was bright enough for the purpose. The silver was carefully freed from copper, as the neighbouring copper lines are powerful exciters; in their absence, no prismatic separation was necessary, as the rest of the silver lines were inoperative. A photograph of the fluorescence spectrum obtained with silver stimulation is shown on Pl. II., *g*. The series in this case *has no gaps* in it, the line at 5170, which is absent with cadmium excitement, coming out strong (see chart as well). The monochromatic illuminator, with its slits reduced to the width of a hair, was arranged to furnish light of wave-lengths corresponding to

other lines in the series, and photographs obtained which are recorded on the chart. It will be seen that faint or missing lines occur in each case, but that their position varies with the point of excitation. If we consider each line caused by a single electron or vibrator, the phenomena suggest that the vibrators are united in some way, perhaps in a closed ring, and that when the system is set in vibration there are nodal points, the position of which depends upon the point in the chain where the periodic force is applied. Moreover, as has already been pointed out, if the force is applied at the "high frequency" portion of the chain, the regions excited are those of highest and lowest frequency, the intermediate portion appearing to be at rest. This is especially noticeable in the case of the bismuth excitation (Pl. II., *e*; and chart, Pl. V.).

In addition to the lines enumerated above, there are a number of others at the upper end. These do not appear to be distributed with the same regularity, though some of them may form an extension of the series, or more probably may be the beginnings of other series. In general it has been found that in the simple spectra the lines are regularly spaced between the extreme violet end and a point at about $\lambda = 5350$. Above this point the spacing is generally very irregular, and it is difficult to unravel the spectrum. Of this more will be said later.

We will next take the fluorescent spectrum excited by the green cadmium line 5086.

This spectrum is remarkable in that it is made up of eleven pairs of lines regularly spaced (Pl. II., *j*, and chart). The other two cadmium lines appear on the plate, as the spectrograph was not shielded from the diffused light from the lamp.

A series in the magnetic spectrum coincides with the series formed by the shorter λ member of each pair. The wave-lengths and differences are given in the following table:—

	λ differences.	λ.	λ differences.
5165·85			
5126·54	39·31	……	$\frac{77\cdot64}{2}$ 38·82
5087·31	39·23	4970·85	
5048·49	38·82	4932·64	38·21
		4894·58	38·06

I am quite at a loss as to how the series formed by the other members of the doublets is originated. It appears to coincide with the series excited by helium 5014, as will be seen by reference to the chart. It appeared at first as if the exciting line might lie between two adjacent fluorescent lines, and in that way excite a double series; but cadmium 5086 is slightly on the short λ side of the magnetic line 5087·3, while the wave-length of the other line of this pair is 5092, *i. e.* on the long λ side.

This is the only case recorded where a spectrum of doublets is excited by monochromatic stimulation, though I am of the opinion that the copper line 5152 behaves in the same way.

What is still more remarkable is the fact that if the excitation is at a different point we no longer get doublets. The lithium line at 4971 takes hold of one of the more refrangible components of one of the doublets, but only a single series of lines appears in the fluorescence spectrum (see chart, Pl. V.). The other series, *i. e.* the less refrangible components, can be separately excited by stimulation with the helium line 5014 (see chart). If we are dealing with anything in the nature of electron doublets, we should expect both the lithium and helium radiations to excite a fluorescence showing double lines.

If we try to explain the phenomena by assuming two chains of electrons fastened together at the point 5086, we must account for the fact that the 5086 vibrator excites the other chain when it is acted upon by light of its own frequency, but not when it is vibrated by the lithium radiation acting at a different point on the chain. I have adopted this hypothesis of electron chains merely to aid in describing the physical phenomena, and not with much hope that it will explain anything.

It seems much more likely that the different lines represent vibrations of different frequencies of the same system. We must not try to make the molecule too much like a piano. The vibrations may be ripples running over its surface, or they may be unlike anything with which we are familiar. If we had never seen a bell, it would be difficult to work out the theory of its very complicated vibrations from a study of

a set of simple pendulums. Possibly stimulation at some other points might give rise to the double lines.

I attempted to do this with the monochromatic illuminator, but without success. The band of exciting light cannot well be made much narrower than the distance between the components of the doublets. Even with the instrument set at 5086, I could detect no evidence of the doublets. I am planning to investigate this matter further with a larger monochromatic illuminator designed to furnish more nearly monochromatic light.

In addition to the eleven pairs of lines in the fluorescence spectrum excited by cadmium 5086, there are two strong lines at wave-lengths 5305 and 5341. These seem to belong to the same series, and the former has a faint companion, the two forming a doublet. The line 5341 is also accompanied by a companion, which, however, is so faint as to be barely distinguishable.

The spectrum excited by the more refrangible of the two blue cadmium lines 4678 is reproduced only on the chart. It consists of a regular series of five lines in the blue region, and a large number of irregularly spaced lines of widely different intensities in the yellow-green region. None of these lines appear to be represented in the magnetic spectrum.

This spectrum illustrates well the characteristic peculiarity of the sodium fluorescence spectrum, that stimulation at the more refrangible end excites powerful fluorescence at the opposite end. The lines which form the regular series we may call directly excited, the others in the yellow region indirectly excited. The latter in all cases seem to be irregularly spaced. The great problem to solve is to determine the nature of the mechanism and find out how the low-frequency vibrators are set agoing by the stimulation of the high-frequency ones, while they remain quiescent when the stimulation is at the middle of the spectrum. Speculations on these points must be deferred for the present.

Zinc-arc Excitation.

The complete fluorescent spectrum excited by all three of the zinc lines (*i. e.* the total radiation of the lamp) is shown near the top of the folding chart, just below the cadmium

fluorescence. It is scarcely possible to find a coincidence of two lines. The two spectra placed side by side make a striking picture of the variation produced by different excitations of the same fluorescing medium. A photograph of the spectrum is reproduced on Plate II. *c*.

Exciting with the zinc line 4811 alone gives us the spectrum shown on Plate II. *b*. The other two zinc lines are of course present on account of diffused light. Referring now to the chart (Pl. V.), we find that the violet end of the spectrum agrees pretty well with the fifth magnetic series, though other lines are present. The two strong lines at 5188 and 5225 also appear to belong to the same series. The lines on the whole are much less regularly distributed than in the case of the cadmium 480 excitation. The three wide pairs between 523 and 535 are peculiar to this excitation.

The fluorescence excited by the other two zinc lines 4680 and 4722, is also recorded on the chart. These lines are so close together that it was found impossible to illuminate the vapour with light from but one of them, and have at the same time sufficient illumination to excite much fluorescence. I have not yet found much evidence of regularity in the distribution of the lines in this case, though there is undoubted evidence of two series in the immediate vicinity of the exciting lines. I have indicated with the letters A and B the lines which appear to belong together (compare with helium excitation). We have an enormous number of lines in the yellow-green region, since we have a double stimulation at the opposite end of the spectrum. There seems to be some regularity here, but it is difficult to say which lines belong together.

Bismuth Excitation.

The light of the bismuth arc makes a beautiful stimulus for the fluorescence, since it contains but a single operative line, the strong one at 4724. It gives rise to a very regular series in the blue-violet region, the lines appearing to fall midway between the lines of the third and fourth magnetic series (Plate II. *e*, and chart). Though the wave-length of the exciting line is only two Ångström units longer than that of the zinc line 4722, the spacing of the series in the two cases is quite different. The same thing has been noticed in the

case of the shortest cadmium and zinc lines, which makes it seem possible that interesting results may be obtained by altering the wave-length of the exciting line, either by pressure or a powerful magnetic field. Experiments in this direction will be made next winter.

In both of these cases, in each of which we have excitation by lines of nearly the same wave-length, the wider-spaced series is produced when the stimulation is by the longer wave-length. It remains to be determined whether we take hold of different absorption-bands and excite entirely different series, or whether we stimulate the same vibrator in each case, the spacing of the resulting lines depending upon how nearly we approach its natural period in our exciting vibrations.

In addition to the regularly spaced lines in the violet, we have a complex assortment of lines in the yellow-green region, the intervening portion being totally devoid of lines. One of these lines, $\lambda=5300$, has a broad diffused wing, and it is perhaps worthy of remark that in the spectrum excited by the two zinc radiations we have a hazy doublet at this point, in the spectrum excited by zinc 4811, a single line, and in the spectrum excited by cadmium 480, two faint lines. Some of the other lines have wings, as will be seen from the chart, and at wave-length 546 we find a broad hazy band. All of these peculiarities complicate things; and I have drawn attention to them merely to show that we must not expect to explain matters by too simple a mechanism.

A word or two about the bismuth arc may not be out of place. Various plans were tried, such as immersion of the electrodes in water, burning in the carbon arc, &c. The best arrangement was found to be a shallow iron dish about 4 cms. in diameter (pounded from a piece of thin sheet iron), filled nearly full of molten bismuth, and kept hot over a small burner. The dish of metal formed the positive electrode, the negative being a bar of iron which could be raised or lowered by a rack and pinion. The arc required constant attention, fresh metal being put into the dish every ten or fifteen minutes, and as exposures of eight hours were necessary, it will be seen that an enormous amount of very fatiguing work was necessary in all cases where open-air arcs were used.

Copper Excitation.

I have been unable thus far to obtain photographs of the fluorescence excited by the separated copper radiations. The lines are close together, and the arc climbs about over the electrodes. I hope next year to improve matters in this respect. On Plate II. *a*, is seen the fluorescence excited by the total copper radiation. Only the three green lines are operative in stimulating the vapour. The lines in the fluorescence spectrum appear to bear no very definite relation to the lines of the magnetic spectrum, as will be seen by the chart. There are many coincidences, however, with lines in the spectrum excited by zinc 4811, and by zinc 468 and 472. By comparing the three spectra I have made a provisional determination of the lines which belong together in the spectrum excited by the copper radiation. These lines are indicated by crosses and vertical dashes placed above them; other lines, which do not appear in the zinc spectra, have not been marked. I suspect that excitation with copper 5152 will produce a doublet at this point, and probably at other points, just as does cadmium 5086. An attempt will be made to verify this surmise. I have on one or two occasions, when trying to stimulate the vapour with this isolated line, been of the impression that I saw doublets distinctly, but at the time I attributed it to incomplete separation of the exciting lines.

Lead Excitation.

A lead arc, operated in a manner similar to the one described for bismuth, was used for exciting the vapour. The only line operative was the one at 5001, and it gave rise to a well-marked series of fluorescence lines, which coincided exactly with the first series of the magnetic rotation spectrum. It is worthy of remark that one of the extra lines of the magnetic spectrum lies very close to the exciting line, yet none of these lines appeared in the fluorescence spectrum. The fluorescence excited by lead was very feeble; and even with an eight-hour exposure the lines were very faint.

Helium Excitation.

A large "end on" helium tube with a 3 mm. bore was constructed for the investigation. This tube could be run continuously with an induction-coil yielding a heavy 10-inch spark. An exposure of about twelve hours was given. Two of the helium lines are operative: line 5015 gives a well-marked series, the lines of which fall exactly midway between the lines of the second and third magnetic series; line 4713 gives a good series in the blue and at least six distinct lines in the yellow-green, the wave-lengths of which can be seen from the chart. The exciting line in this case coincides with one of the lines in the spectrum excited by zinc 468; and there is perfect agreement in position between the fluorescent lines in both cases, not only in the blue, but also in the yellow-green region. The lines in the spectrum excited by line 468, or at least as many as could be identified, have been marked. The identification was of course made by comparison with the spectrum excited by helium.

Lithium Excitation.

An arc was caused to play between a carbon rod and a large carbon block on which the lithium salt was placed. The image of the red flame was projected upon the window of the retort and excited a bright fluorescence. Two of the lithium lines were operative,—one at 4601, the shortest monochromatic stimulation thus far found, which gives the series in the violet (see chart, Pl. V.), and a large number of lines in the yellow-green; and another at 4971, which gives a beautiful series in the green, coinciding exactly with the second magnetic series. The 4971 stimulation should be especially interesting, since there are several lines in the magnetic spectrum very close to it. The line is unfortunately not very bright, and the fluorescence lines were so feeble that they could only be measured with difficulty.

The lines in the yellow-green region are also of considerable interest, since they result from a single monochromatic stimulation applied practically at the extreme lower end of the fluorescence spectrum. In fact, this line is considerably below the limit of the fluorescence spectrum as usually seen with

white-light stimulation: this limit is not far from wave-length 4670, which is the shortest thus far detected in the magnetic spectrum photographed with the grating, though faint lines are visible even below 4600 on negatives made with the prism spectrograph. There seems to be evidence of a number of series in the yellow-green region, the spacing, however, decreasing with increasing wave-length, just the opposite of the state of things which holds in the blue-green region. It is more likely, however, that the apparent decreasing of the spacing as the yellow end of the spectrum is approached is due to other series similar to those which are found in the green and blue, the nearness of the lines resulting from the large number of superposed series. With white-light stimulation no trace of the lines can be seen above wave-length 555, and they are so faint as to be almost indistinguishable for a considerable distance below this point. The broad flutings seen in the spectrum stimulated with white light are doubtless to be referred in some way to the circumstance that the lines of the different series get into and out of step periodically: they may thus be considered analogous to the bands seen when two diffraction-gratings of slightly different spacing are superposed.

Barium Excitation.

The fluorescence excited by the barium arc appears to be due chiefly to the line 4934, which coincides with one of the extra lines in the magnetic spectrum. Line 4932 of the second magnetic spectrum is also very near it; and we find that the fluorescence spectrum contains lines which coincide with the magnetic lines of the second series, as well as lines which coincide with some of the extra magnetic lines. The barium arc contains a good many other fainter lines which may give rise to some of the fluorescent lines. It will be necessary to repeat the experiment with the 4934 line isolated.

Sodium Excitation.

As I showed in the earlier paper referred to, if we stimulate the vapour with intense sodium light, we obtain a yellow fluorescence which the spectroscope shows to be made up of two lines in the position of the D lines. We have here a

re-emission of light of the same wave-length as the exciting light, and nothing else. This I have called resonance radiation, as we may find that it is different from fluorescence, though the two are doubtless intimately related. As there are a number of pairs of lines in the ultra-violet which belong to the same series as the D lines, it seemed of great importance to determine whether these appeared in the spectrum of the fluorescence excited by the sodium flame. The sodium tube was provided with a quartz window, and the light of the oxyhydrogen flame, heavily charged with sodium, focussed upon the aperture of the retort with a glass lens. White light from the arc was also used, as this excites the D-line vibrations in the fluorescence. The spectrum was photographed with a small quartz spectrograph, and though the D lines were greatly over-exposed, no traces of any of the ultra-violet doublets were found on the plate. Conversely, illumination with ultra-violet light in the region of the first ultra-violet pair of lines failed to produce any visible fluorescence. It was hoped that a faint yellow fluorescence might be produced in this way, due to emission in the region of the D lines. I have not yet tried stimulation with D_1 and D_2, alone, to see whether both D lines appear in the fluorescence. This will be a very difficult experiment, and I am saving it for the last. It will settle the question as to whether the principal series of sodium is a series of doublets or two series of single lines.

Cathode-ray Excitation.

The cathode rays, I find, excite a fluorescence similar to white light. The lines of the principal and subordinate series appear as well, some of them of overpowering intensity. The apparatus for the electrical excitation is shown on Plate III. fig. 3.

It consisted of a steel tube 3 cms. in diameter and 35 cms. in length, one end closed with a glass plate, the other cemented with sealing-wax to a glass tube carrying the cathode. The mercury-pump was kept in continuous operation to remove the hydrogen liberated from the sodium. On looking into the tube through the glass window a blazing spot of yellow light 2 cms. in diameter was seen at the point where the

cathode rays entered the vapour. Its spectrum was photographed with the prism spectrograph, and is reproduced on Plate II., n. In addition to the fluorescent spectrum, and the sodium lines of all three series, the hydrogen lines come out strong. I have never been able to eliminate them entirely. Very few experiments have been made on the electrical excitation, but some very curious phenomena have been observed. In some cases, by looking into the tube in an oblique direction, it was seen that at the point where the cathode rays entered the mass of vapour there was a bright green spot of fluorescent light, while at the point of exit there was an orange-yellow spot, *the intervening space being non-luminous.* Seen in a direction oblique to the direction of the rays, the two spots were seen completely separated. This I consider a very remarkable circumstance, and a spectroscopic study of the two spots of light will undoubtedly yield very fruitful results. Unfortunately the condition is a difficult one to keep fixed, for the phenomenon only appears when the density of the sodium vapour is just right and the surrounding vacuum high. As I have shown in the paper on the dispersion of sodium vapour, we can have a dense mass of the metal vapour, bounded on each side by a very high vacuum, a very anomalous condition from the point of view of the kinetic theory of gases. My impression is that the green spot will show the fluorescent spectrum, and the yellow spot the lines of the principal and subordinate series, as found in the sodium arc, but as yet I have not found time to make even a visual examination. Several attempts have been made, but by the time the image of the spot was thrown upon the slit in the proper direction to pass the light through the prisms, and the eye brought to the instrument, the conditions in the tube changed.

It is difficult to account for the absence of luminosity of the centre of the mass and the two bright spots. Perhaps the condition under which the rays excite fluorescence exists only where the vapour mass is in contact with the vacuum, *i. e.* in the region where the hypothetical clusters of molecules are breaking up and flying to the cooler walls of the tube. Even assuming this to be the fact, the difference in the colour of the two spots is still to be accounted for. Possibly the

cathode rays excite the green spectrum, while the canal rays travelling towards the cathode excite the orange-yellow luminescence. I have made one experiment with a similar tube arranged so as to deliver a stream of canal rays against the vapour. The luminescence was bright yellow, but the tube cracked before a spectroscopic examination was made.

On the other hand, it may be that whatever causes the green luminescence is removed from the ray-bundle by absorption, the residue exciting the yellow luminescence at the point of exit. If this is the case, we should expect the same amount of yellow light in each spot; and I am of opinion that the green light is much too pure for this to be the case. Further experimenting will be necessary before it is possible to draw any very definite conclusions.

In the spectrum excited by the cathode rays the D lines are of immense brilliancy, running together into a single band of light. On each side of this are seen three or four symmetrically placed bands, decreasing in brilliancy as they recede from the D lines in each direction. No trace of these bands appears in the magnetic spectrum, which in this region shows only fine lines arranged in narrow groups, which do not coincide with the bright bands of the cathode luminescence.

A photograph of these bands is reproduced in Pl. IV. fig. 2. They have some connexion with the D lines, I feel sure, for they are symmetrically arranged on each side of them. If the photograph had been made with a grating, we should of course call them ghosts. It may be that they are analogous to satellite lines; but if they are, we are certainly dealing with the phenomenon on a grand scale, for the fourth one is not far from the sodium doublet at 5688! All of these points will be more fully investigated during the coming year.

Other Possible Excitations.

It has occurred to me during the preparation of this paper that very interesting results would be obtained by exciting the fluorescence with the light selectively rotated by the vapour in a magnetic field, *i. e.* by the magnetic bright-line spectrum. This light is fairly intense; and it would be interesting to see whether the intensity distribution among

the excited fluorescence lines was the same as in the magnetic spectrum.

What I most need, however, is a set of screens which will enable me to separate lines such as those of copper without resorting to the systems of prisms and lenses. A good collection of solutions of the rare earths would probably be very useful in the work. Erbium, praseodymium, and neodymium I have, but I should feel very grateful for the loan of any others which might prove serviceable, or for any suggestions regarding other possible sources of monochromatic light. As I have said before, the instrument most needed is a light siren!

Composite Excitation.

At the top of the chart just below the magnetic series will be found a spectrum containing about two hundred lines. This is a composite drawing made by superposing all of the drawings made of the simple spectra excited by monochromatic stimulation. It contains many lines not found in the complex spectrum excited by white light. In the latter, between wave-lengths 5000 and 5100, we find but ten or a dozen lines, while in the composite spectrum there are at least twenty. This circumstance is of interest in connexion with the periodic dark regions of the complex spectrum, which give it a fluted appearance. The formation of these flutings requires further study, as their position shifts as we alter the wave-length of the exciting light, which in this case is a rather broad band isolated from the continuous spectrum with the monochromatic illuminator. The phenomenon was more fully described in the earlier paper, but requires further study.

The Series in the Magnetic Spectrum.

As we have seen, the complex fluorescent spectrum is made up of six or more series of lines, the individual lines of each series being about 38 Ångström units apart, the spacing becoming less as we pass from yellow towards violet. The fact that the lines in the magnetic spectrum coincide with the lines of the complex spectrum makes it seem certain that the same series will be found there. By comparing the

various fluorescent spectra with the magnetic spectrum, and by measuring carefully the distances between the lines of the latter, I have made a provisional assignment of the magnetic lines thus far observed into five series.

The wave-lengths and wave-length differences are given in the following tables :—

FIRST SERIES.

λ.	λ differences.
5119·34	
	39·56
5079·78	
	39.13
5040·65	
	39·08
5001·57	
	38·67
4962·85	
	38·52
4924·32	

SECOND SERIES.

λ.	λ differences.
5165·85	
	39·31
5126·54	
	39·23
5087·31	
	38·82
5048·49	
......	$\frac{77·64}{2} = 38·83$
4970·85	
	38·21
4932·64	
	38·06
4894·58	
......	$\frac{75·15}{2} = 37·57$
4819·43	
	36·54
4782·89	
......	
	$\frac{112·89}{3} = 37·63$
......	
4670	

THIRD SERIES.

λ.	λ differences.
5211·71	
	38·89
5172·82	
	39·09
5133·73	
	38·95
5094·78	
......	
	$\frac{115·34}{3} = 38·45$
......	
4979·34	
......	$\frac{75·96}{2} = 37·98$
4903·38	
	37·79
4865·59	
......	$\frac{113·55}{3} = 37·85$
4752·04	
	36·41
4715·63	

FOURTH SERIES.

λ.	λ differences.
5219·00	
	39·29
5179·71	
	39·00
5140·71	
......	
	$\frac{115·05}{3} = 38·35$
......	
5025·66	
......	
	$\frac{113·56}{3} = 37·85$
......	
4912·10	
......	$\frac{74·61}{2} = 37·30$
4837·49	
......	
	$\frac{109·97}{3} = 36·68$
......	
4727·52	
	34·98
? 4692·54	

FIFTH SERIES.

λ.	λ differences.
5225·34	
	38·64
5186·70	
	39·20
5147·50	
......	$\frac{76}{2} = 38·00$
5071·50	
	38·00
5033·54	
......	$\frac{74·92}{2} = 37·46$
4958·62	
......	$\frac{74·81}{2} = 37·40$
4883·81	
......	$\frac{73·65}{2} = 36·82$
4810·16	
......	$\frac{71·66}{2} = 35·83$
4738·5	

EXTRA LINES.

5096·00
5052·83
5049·56
5003·12
4964·39
4967·10

As will be seen by reference to the chart, the first series has the largest *average* spacing, and the fifth the smallest, the "scale," if the term is allowed, decreasing gradually from the first to the fifth, the two coming into coincidence at about wave-length 4860.

Doubtless these series could be extended to wave-length 5500 or thereabouts by making use of the grating-photograph of the complex fluorescence obtained by white-light stimulation. The lines are, however, so diffuse in their nature, with overlapping wings and other peculiarities, that I have not yet attempted any further extension. I think that by employing a much denser vapour the magnetic spectrum can be considerably extended; and as the lines are much sharper in this case, an extension of the series will be an easy matter. The other series necessary to give the close spacing found in the yellow may be discovered in this manner.

A theoretical discussion of the results will have to be deferred to a subsequent paper. In fact, I prefer to have this side of the subject attacked by those who have given especial attention to the theory of molecular radiation. The absence of many lines in each series in the magnetic spectrum, and the absence of certain lines in the fluorescent spectrum, are especially suggestive. We have similar conditions in other series of lines, as is well known; but the present case is, so far as I know, the only one in which we can, by varying the exciting conditions, bring about a change in the position of the absent lines. It appears to me that the data furnished us by sodium vapour ought in the end to enable us to choose between the various theories proposed to account for spectrum series.

The investigations recorded in the present paper have been made possible through generous aid given from the Rumford Fund by the American Academy of Arts and Sciences.

I feel also under great obligation to my assistant Mr. F. W. Cooper for the many hours of very fatiguing work which he devoted to the research.

XXVI. *The Strength and Behaviour of Ductile Materials under Combined Stress.* *By* WALTER A. SCOBLE, *A.R.C.Sc., B.Sc., Whitworth Scholar* *.

1. *Previous Tests and their Differences from those given.*

THIS branch of the testing of materials, although of considerable theoretical and practical importance, has been seriously neglected until quite a recent date. Experiments on wires under combined tension and torsion are mentioned in Lord Kelvin's article on Elasticity in the *Encyclopædia Britannica*, but numerical results are not given. Excellent work was done by Mr. J. Guest † using thin tubes under tension, torsion, and internal pressure. A preliminary report has also recently appeared of tests on the effect of combined stresses on the elastic properties of steel, by Mr. E. L. Hancock ‡. In addition to the above there are the experiments by Tresca upon the ultimate strength under combined stress.

The ultimate practical value of experiments of the kind under consideration will greatly depend on the limiting condition of the material which is selected as the basis for comparison. Reasons are given below for selecting the yield-point. This point was taken by Mr. Guest, the choice being justified in his paper. This being so, it may appear that the present results will only be a confirmation of previous tests, but the following points should be noted :—Mr. Guest used thin tubes, and although this course appears to be justified, experiments on solid bars are desirable, the distribution of stress being different. The tensions were applied either directly or by internal pressure. Perhaps the most important practical instance of combined stresses is that due to torsion and bending. Due to bending the stress varies gradually from a maximum tension to a maximum compression, the shear stress due to torsion being zero at the centre and a maximum at the surface. This is a further difference in the distribution of the stress. Several tests with different loadings were made on each of the tubes mentioned, and the results

* Read October 26, 1906.

† Proc. Phys. Soc. vol. xvii. p. 202, and Phil. Mag. [5] vol. 1. p. 69.

‡ Phil. Mag. Feb. 1906.

show that in certain cases at least the properties of the material changed appreciably, making accurate comparison impossible, or at least difficult. It was impossible to allow the specimen to yield very much, so that the value found for the critical stress was open to error. The writer preferred to use separate specimens, with possible slight differences in properties, and to allow the yield to be quite definite and of considerable magnitude.

2. *Separation of Metals into Ductile and Brittle.*

Before discussing the theories of elastic strength, it will be well to separate the materials into two classes—ductile and brittle. It is unnecessary to discuss these at length as the difference in properties is sharply defined. Ductile materials yield considerably before fracture, drawing out at and near a particular section when under tension, whereas those of the brittle class yield very little, or are very "short." The planes of fracture vary with the two classes when tested under the same conditions, and it is significant that with ductile specimens the planes of fracture approximate to those of greatest shear. This indicates that after yield a ductile material behaves like a viscous fluid, the large yield supporting this view.

3. *The Behaviour of Brittle Materials should not be judged by Tests on Ductile Specimens.*

Perhaps the most common examples of combined stresses in practice occur in vessels under internal pressure and shafts subjected to bending and twisting. The material used for the first may be ductile or brittle, but with ductile the thickness will be comparatively small and the maximum stress or maximum shear theories will lead to little difference when calculating the dimensions. From the distinction drawn above, it will be noticed that it is not justifiable to apply the results of tests on ductile materials when dealing with brittle specimens. The critical case in which internal pressure is applied is that of a thick cylinder of cast iron, and it would not be safe to treat this in the same way as a comparatively thin boiler-shell. In the latter case the two theories held in England give very similar results, but in the former the material has properties different from those of the specimens

tested. This distinction should be remembered when reading all that follows.

4. *The Theories of Elastic Strength.*

The real need is a determination of the true complete theory of elastic strength. The theory of elasticity and all engineering applications are based on Hooke's law, that strain is proportional to stress. When this law fails the assumptions are no longer true, consequently the formulæ deduced are incorrect and it is impossible to reason further with certainty. In certain cases there will be a redistribution of stress, usually varying from that satisfying Hooke's law to a state of uniform stress, reached after considerable yield, through all the intermediate conditions. By the theory of elastic strength is meant the conditions which determine when the law fails and the uncertain stage is reached, the body considered being also seriously deformed. Three theories have claimed much support. That adopted by English engineers assumes that the material yields when the maximum principal stress reaches a certain value. The view generally held by Continental elasticians is that a definite maximum strain determines the failure of the specimen. A third theory states that yielding occurs when the maximum shearing stress becomes a specific amount. Mr. Guest has shown that neither the maximum stress nor maximum strain is even approximately constant. The maximum shear is constant within close limits; and it has been suggested that the variation is probably due to something analogous to friction, related to the force perpendicular to the plane of maximum shear. This point will be considered later when the results are discussed.

5. *The Yield-Point selected as the Criterion of Strength.*

Remembering that our accepted formulæ are based on Hooke's law, which holds to the elastic limit, it would at first appear that this point should be selected as a basis for comparison. The yield-point has here been used throughout. If the material is satisfactory, the stresses at the elastic limit and yield-point are nearly proportional and it makes little difference which is taken. Faulty specimens will usually have a low elastic limit whereas the yield-point is little affected, and

the same applies to changes in the metal due to any special treatment to which it may have been subjected. Taking these facts together, it is evident that the yield-point is much more nearly constant than the elastic limit, and in making a simple test it is correct to consider both points in relation to each other. The yield-point, unfortunately, is not quite definite, but that taken for the purpose of these tests will be precisely stated. Having decided exactly how the yield-point is to be found, it is more easily obtained accurately than the elastic limit. The error in proportionality between the stress and strain is small to the yield-point. The view is now generally held that the lack of proportionality immediately after the elastic limit is due to local yielding, parts of the material being either weak or under internal stresses. The considerable lowering of the elastic limit with poor specimens supports this view very strongly; and if it is accepted, the yield-point is more important than the elastic limit, being less dependent on unique conditions. It will be seen later that when a specimen is tested to yield under combined stresses, there is yield in both ways. We therefore have definite loads of both kinds. Mr. Hancock has shown that when the elastic limit is reached under one loading, a large stress is needed to reach it with the other. A bar tested to the elastic limit in tension needs a considerable torque before this point is reached in torsion. For comparative work a difficulty arises. The first load reaches the elastic limit and the second can do no more except reach it with the other yield, therefore the first load can be taken with any fraction of the second to give the combined loads. No doubt the first kind of stress affects the elastic limit under the other kind, but the conditions are not so definite as when the yield-point is taken. Unfortunately more than one position can be ascribed to the yield-point, the point at which the curvature of the stress-strain curve becomes large, or the stress causing considerable yield. As the distribution of stress due to bending varies from a maximum tension to a maximum compression, and due to the torque from zero to a maximum shear, elements will yield in succession, this being especially the case under combined stresses with irregular distribution, the maximum shear stress only occurring at certain places. On this account Mr. Guest used thin tubes to obtain nearly uniform stresses.

Since the elastic strain is so small a little yield at certain points causes a complete redistribution of stress across the section, and the yield-point stress extends to other areas. For this reason the stress at the first sign of yield should be taken. When a simple tension test is considered, there is still the rounded portion, due to local yielding and eccentricity of loading, although the stress is finally uniformly distributed over the section. When the stress-strain curve becomes approximately horizontal there is general yielding, and the corresponding stress should be taken as the yield-point stress. In the present case there are both causes for the rounded portion of the curve, local yielding and redistribution of stress: one operates to cause the first sign of yield, and the other complete yield, to be taken as the limiting condition. As the object of the tests is comparative, it matters very little which is taken, the results being in a sense proportional within the limits of experimental error and those arising from taking curve-readings. It is quite impossible to accurately locate the point at which yield first occurs, therefore the selection has been made as described below. This seems quite satisfactory when considering a suitable formula for bending and twisting. The same moments have been tabulated and the corresponding stresses calculated, making the usual assumptions. These have been found to bear a definite relation to the stresses corresponding to the first sign of yield, so nearly as could be read from the curves, when the usual assumptions hold good, and stresses can easily be changed to those calculated on the assumption of a uniform distribution of stress.

6. *Determination of the Yield-Point.*

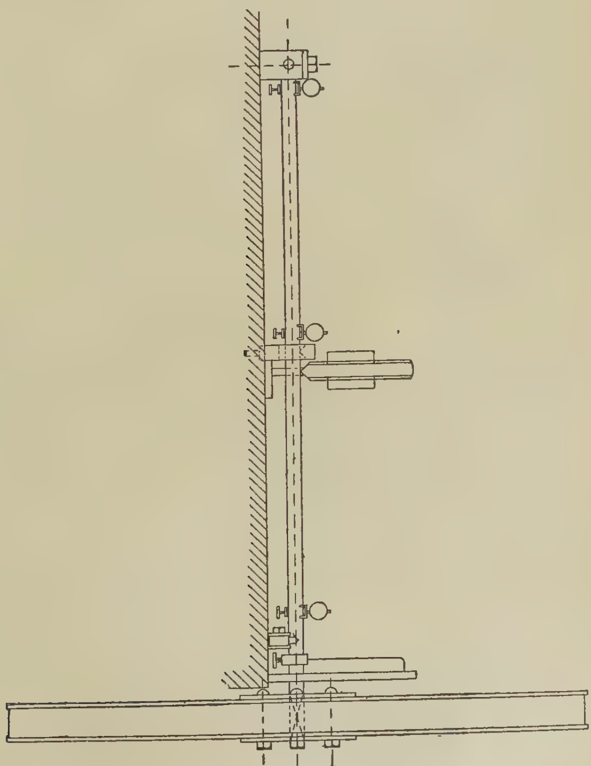
The yield-point was determined as indicated in fig. 4, test 11. As the intermediate state between perfect elasticity and great flow, or the approximately vertical and horizontal portions on a stress-strain curve, is due to successive yield at various planes of weakness, most elasticians neglect this part of the curve altogether, and there is considerable justification for this course. This has practically been done here, the lines representing the two states being produced to meet in a point. Supposing this course was not justified, at least this is a definite, easily determined point to deal with, and any

probable error would not be greater than that which is likely to arise when taking a point less closely defined.

7. *The Nature of the Loading.*

It has already been stated that the most important case of combined stresses with ductile materials is that of bending and twisting. Fortunately, tests of this kind could be made with the limited apparatus at the disposal of the writer, and they have the advantage that the distribution of tensile and compressive stress is similar to that in the practical case, whereas it is usual in tests under combined stresses to apply the tension directly.

Fig. 1.

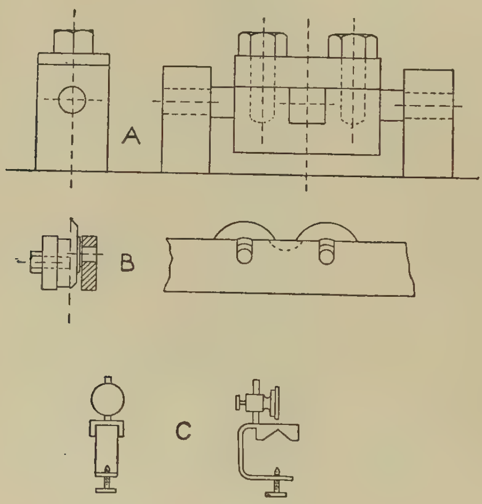


8. *The Apparatus and Specimens.*

A general view of the apparatus is given in fig. 1. The bars tested were 36 inches long and turned to $\frac{3}{4}$-inch

diameter, with $\frac{3}{4}$-inch squares at each end. They were of steel, all taken from the same batch, and two were prepared from each length supplied. One end was held so that it could not turn but was not constrained against bending. The grip is given in fig. 2. The squared part of the rod fits into a suitable slot and the cover-plate grips it. There are two pins branching from the central portion which fit into bearings, these steel pins being carefully ground in place. Drawings of the support at the other end are also given in fig. 2. The bar rests on two rollers of large diameter which are in turn supported on their side pins by the metal strips.

Fig. 2.



This arrangement reduces the resistance to twisting at this end to a minimum. The torque was applied through a light wooden pulley having metal plates bolted to it, these having $\frac{3}{4}$-inch square holes cut in them in line. Suitable wire ropes were attached to this pulley, and each passing over another pulley, weights could then be attached and considerable motion was possible. The bending load was applied directly by placing weights on a platform attached to a knife-edge resting on the bar. Deflexions were measured by means of a scaled strip which rested on the specimen and fitted easily in a slide

with a vernier cut on the side. This arrangement was quite satisfactory, although it does not appear very sensitive, but it should be remembered that the deflexion is large even before the yield-point is reached, and it was possible to read accurately to 0·005 of an inch. The twist was measured at four points. At three of these, mirrors (see fig. 2) were attached to the bar. Telescopes were fixed at approximately a metre from the mirrors, and there were vertical scales; the arrangement being similar to that in common use in connexion with galvanometers. Outside the support, on the pulley side, a pointer was fixed to the bar and moved over a fixed scale as the rod twisted. This served to give fairly accurate readings of the twist directly. The length of the beam, or the distance from the centre line of the pins to the rollers, was 30 inches. The load was applied at the centre and the deflexion measured at 14 inches from the roller support. The end mirror was $1\frac{1}{2}$ inches from the roller support. The pin joints at the one bearing, and the combined roller and pulley arrangement, were tested to determine the amount of constraint due to friction. This was in each case small, but has been allowed for in working out the results.

9. *The Tests Grouped.*

The tests may conveniently be divided into three classes. The first includes the tests with one kind of loading only, these giving the loads necessary to cause yield either by bending or twisting. For the second class a definite fraction of the critical bending load was put on and the specimen then twisted to yield. In the third set a given part of the yield twisting moment was applied and then sufficient bending load put on to cause the bar to give. Each kind of load was applied directly by weights, so that it increased by stages and not gradually. When the elastic limit was reached the load added was considerably smaller than those put on at first, it being of the order of one pound. The pulley was lightly tapped after the addition of each twisting load to minimize the effect of friction or sticking. If a result appeared doubtful, or if the increments of load had been too large at the yield-point, preventing its accurate measurement, the test was repeated on a new bar.

10. *Table of Results.*

The results of the tests are tabulated below:—

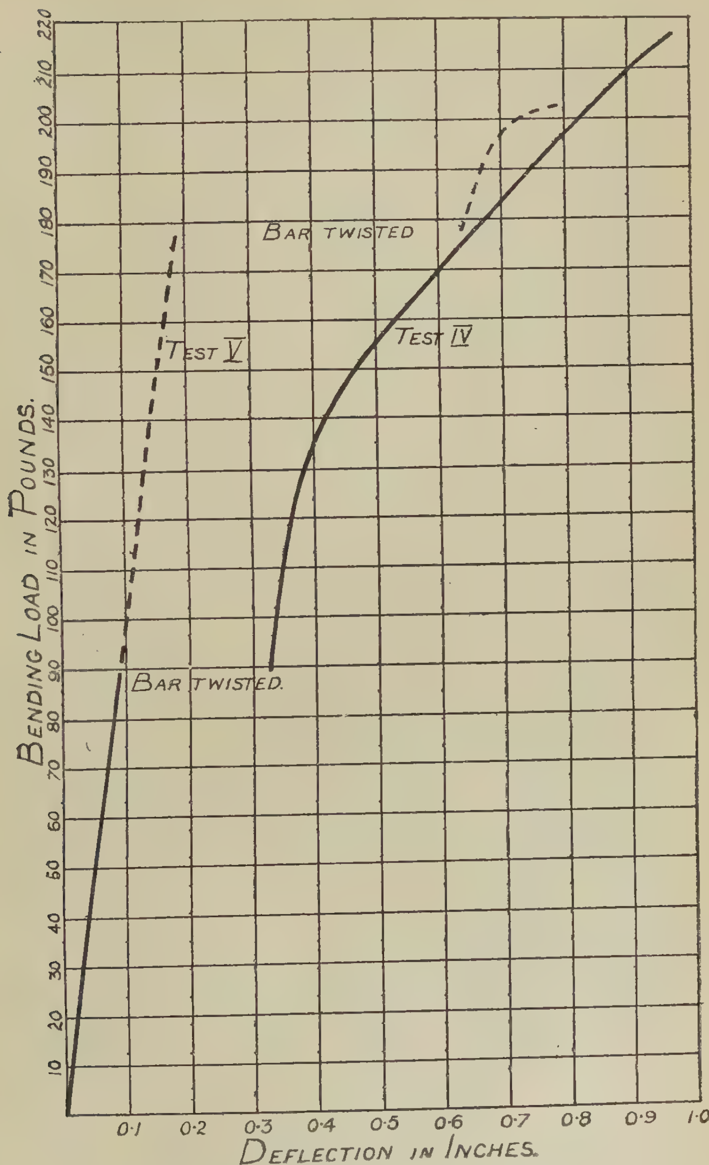
Group of Test.	Number of Test.	Bending Moment.	Twisting Moment.	Tensile Stress due to Bending.	Shear Stress due to Twisting.	Max. Principal Stress.	Min. Principal Stress.	Max. Shear Stress.
1.	I. ...	2600	0	64600	0	64600	0	32300
	III. ...	0	2400	0	29170	29170	−29170	29170
2.	IV. ...	667·5	2280	16220	28250	37500	−21300	29400
	V. ...	1331	2120	32350	25750	48200	−15800	32000
	VI. ...	2000	1899	48600	23050	57800	− 9200	33500
	VII. ...	2420	1171	58750	14240	61980	− 3220	32600
	XII. ...	2000	1720	48600	20900	56740	− 8140	32440
3.	VIII. ...	2558	645	62100	7840	63080	− 980	32030
	IX. ...	2310	1335	56100	16220	60450	− 4350	32400
	XI. ...	1454	2033	35330	24700	48060	−12740	30400

The principal stresses are $T/2 \pm \sqrt{S^2+T^2/4}$, and the shear stress, equal to half the difference between the greatest and least principal stresses, is $\sqrt{S^2+T^2/4}$.

11. *Facts not included in the Table.*

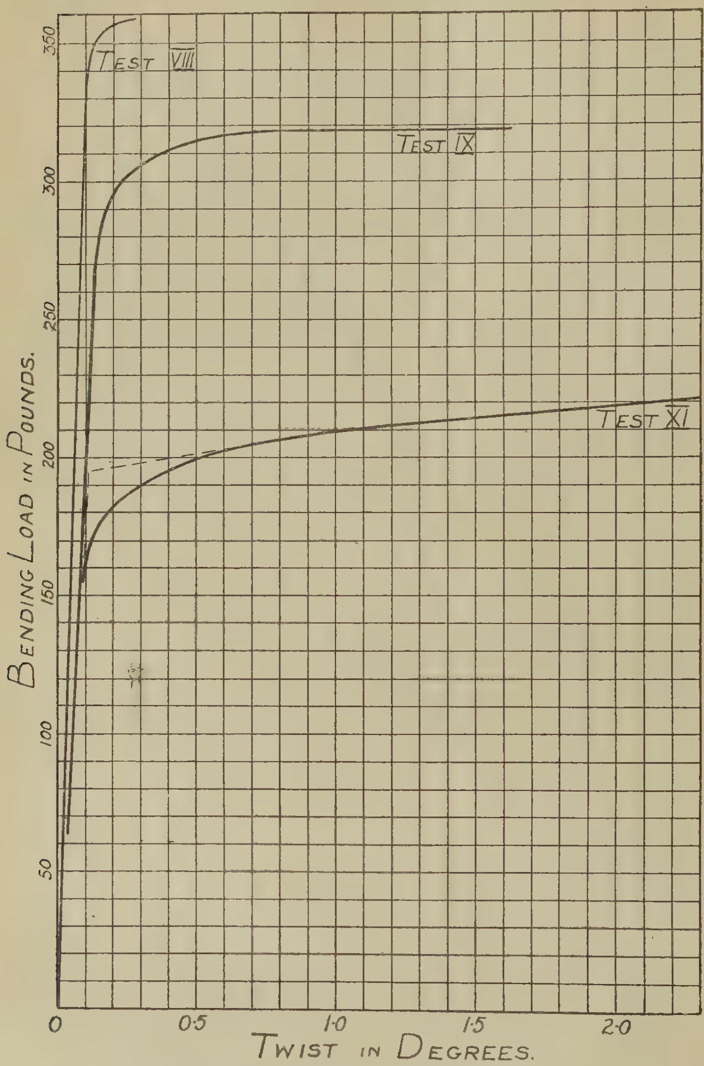
The following interesting facts may be noticed in connexion with the tests. Considering Test IV., a quarter of the bending load was put on and torque increased to yield. There was simultaneous yielding by bending and twisting (fig. 3), but a large addition to the bending load was needed to cause further appreciable yield. When half, or more, of the bending load had been put on originally, the bar gave in both ways simultaneously, and a small addition to the bending load caused further yield (fig. 3). When the rod could stand a further load without much deflexion, the bending yield was almost sudden at the critical load, there being none preceding it that could be measured (Test IV.). With larger fractions of the bending load, the yield could be detected before the critical load was reached and deflexion was plotted against twisting moment, in addition to the twist; in fact the specimens gave first by bending in Tests V., VI., and VII. The yield by twisting, due to the bending load, was small in Test VIII., fig. 4, but the sensitive method of measuring readily detected it.

Fig. 3.—Tests IV. & V. Bending Load and Deflexion.



Test V. ½ bending load on originally.
Test IV. ¼ bending load on originally.

Fig. 4.—TESTS VIII., IX., & XI. Bending Load and Twist.



Test VIII. $\frac{1}{4}$ Twisting load on originally.
Test IX. $\frac{1}{2}$ Twisting load on originally.
Test XI. $\frac{3}{4}$ Twisting load on originally.

The bar could stand an increase of torque with little more than normal twist. In Tests IX. and XI. the yield was large, and a small addition to the twisting load produced a large twist, the increase of load being unnecessary as the bar was badly deformed in this way without it. The critical loads causing yield in the two ways are given below :—

Test.	Original Load.	Yielded by Bending at	Yielded by Twisting at	Nature of Load being added and stated.
IV. ...	$\frac{1}{4}$ bending load.	77 lbs.	77 lbs.	Torque.
V. ...	$\frac{1}{2}$,, ,,	71·5 ,,	72·25 ,,	,,
VI. ...	$\frac{3}{4}$,, ,,	64 ,,	65·25 ,,	,,
VII. ...	$\frac{9}{10}$,, ,,	39·5 ,,	41·5 ,,	,,
XII. ...	$\frac{3}{4}$,, ,,	58 ,,	58·5 ,,	,,
VIII. ..	$\frac{1}{4}$ torque.	356 ,,	341 ,,	Bending.
IX. ...	$\frac{1}{2}$,,	308 ,,	310 ,,	,,
X. ...	$\frac{3}{4}$,,	223 ,,	194 ,,	,,

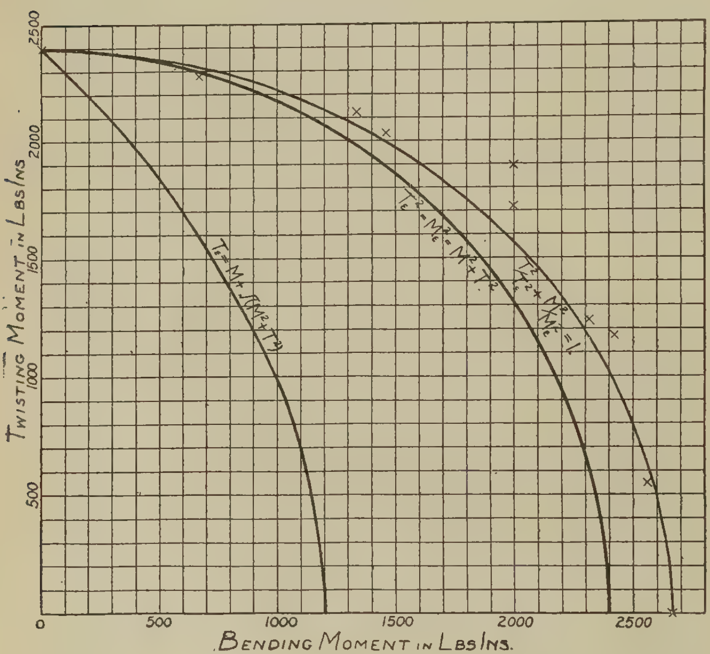
12. *A Formula for Combined Bending and Twisting.*

The corresponding bending and twisting moments required to cause yield are plotted in fig. 5. The curves are an ellipse through the points representing the limiting cases with load of one kind only, a circle with the smaller moment as radius, and a curve representing the common formula $T_e = M + \sqrt{M^2 + T^2}$, T_e being the equivalent twisting moment, M the actual bending moment, T the actual twisting moment. The last is ridiculously wrong, and it is difficult to understand why it has found so much favour. Putting $T=0$, $T_e = 2M_e$. This is only true if the tensile and torsional shear strengths are equal, whereas all experimental results indicate that this is far from correct. The ellipse fits the results best, but the circle is recommended on account of an explanation of the difference between the limiting bending and twisting moments which is given later. Thus the formula for combined bending and twisting becomes $T_e = M_e = \sqrt{M^2 + T^2}$.

T_e = the equivalent twisting moment.
M_e = ,, ,, bending ,,
T = the actual torque to which the shaft is subjected.
M = ,, bending moment to which the shaft is subjected.

When determining the size of a shaft under both loadings it should be calculated from the working tensile strength, by using the equivalent bending moment if this is less than twice the working torsional shear strength, or from the shear strength if this is less than one half the tensile strength. In other words, the tensile strength should be taken as twice the shear strength, and the smaller value taken when obtaining

Fig. 5.—Twisting and Bending Moments at Yield.



these from the values given by tests, or used as working stresses. A simple case will make this quite clear. Unwin gives 13200 lbs/in.^{2} and 5400 lbs/in.^{2} as the tensile and torsional strengths of mild steel under the same conditions of loading. These should be taken as 10800 and 5400 lbs/in.^{2} when working with the formula given, thus taking the smaller values. If the twisting moment was small, there would be no danger in using 13200 and 6600 lbs/in.^{2}, thus approximating to the ellipse given in fig. 5. The formula

based on the ellipse is slightly more complicated and may be given as

$$\frac{T^2}{4f^2_{\text{torsion}}} + \frac{M^2}{f^2_{\text{tension}}} = \frac{T_e^2}{4f^2_{\text{torsion}}}$$

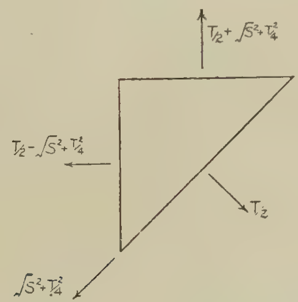
or
$$T^2 + \left(\frac{2f_{\text{torsion}}}{f_{\text{tension}}}\right)^2 M^2 = T_e^2;$$

the letters having the same meanings as before.

13. *An Explanation of the Variation of the Maximum Shear Stress.*

The complete theory of elastic strength has yet to be determined. The table, section 10, gives the principal stresses, the maximum in one case being almost twice that in another. The maximum shear stress is nearly constant and may be

Fig. 6.



taken as the determinant to a first approximation. After allowing for experimental errors, it is difficult to explain the differences in the values. Mr. Guest suggested a force corresponding to friction due to the force normal to the plane of greatest shear, the same having been mentioned by other writers. It is now necessary to examine the results more closely to determine whether they justify this explanation. When the principal stresses are $T/2 \pm \sqrt{S^2 + T^2/4}$, the distribution is that indicated in fig. 6. Assuming a force similar to friction this would assist the shearing force, so that the yield in tension would take place with a lower stress. With a force the reverse of friction, opposing the shear stress if there is tension across the plane, the tensile strength should be more than twice the shear strength. In the case of bending, there

being both tension and compression, either kind of additional force would reduce the strength, causing the bending moment to be less than the torque. This is contrary to the results of the experiments, so that the idea must be abandoned. Unwin gives the working tensile stresses as more than twice the torsional shear stresses for nearly all materials, in certain cases the proportion being exact, thus opposing the idea of friction. Referring to Mr. Guest's results, friction would cause the maximum shear stress at yield to diminish with increase of tension, and anti-friction would have the reverse effect. In the case of tube 1, the tension was greater than twice the shear stress. With tubes 4, 6, 7, 8 and 9, the tension was less than twice the shear stress. All these tubes were steel. With the copper tubes the shear strength was more than half the tensile stress. The brass tubes gave no definite evidence either way. The bulk of these results are contrary to the working stress proportion, and there is an exception amongst them. To trace the matter more completely, the results were analysed to trace the connexion between the maximum shear and the tensile stress across the plane. It was found that there was absolutely no connexion between them. As an example, taking the twenty tests on tube 8, the maximum shear stress and the tensile stress across the plane have been tabulated :—

Max. Shear.	Tension.	Max. Shear.	Tension.	Max. Shear.	Tension.
23735	22320	21760	20290	20975	19220
23175	22325	21600	19600	20800	19000
22810	21590	21460	20040	20600	20600
22500	0	21370	19630	20535	18765
22260	20340	21250	0	20480	18720
22100	0	21125	17000	20100	20100
22000	13400	...	...	20050	18450

There is no sign of a regular increase or decrease of the maximum shear corresponding to an increase of the tensile stress across the plane of shear. The present tests have a rough indication of increase of maximum shear stress with increase of bending moment, and therefore of both tension and compression, but it is by no means conclusive. It is opposed to the idea of either positive or negative friction.

The true reason of the variation in the maximum shear stress, after allowing for errors in measurement, appears to be that the shearing resistance varies in different directions. Bauschinger tested the shear strength of various specimens in six different directions. For annealed puddled plate it varied from 8 89 to 19·68 tons/in.^{2}, for rolled iron bar from 10·15 to 22·55 tons/in.^{2} With steel the variation was less, for Bessemer plate from 21·45 to 27·35 tons/in.^{2}, and for another specimen of the same from 25·05 to 29·2 tons/in.^{2} Remembering these results, assuming the maximum shear stress to be the criterion determining yield, and also that with tension and torsion the planes of shear are different, it cannot be expected that the tensile strength of steel will be exactly twice the shear strength. The fact that sometimes it appears to be more, and at others less, points to this being the true explanation of the variation of the maximum shear stress at the yield-point under combined loading. The relation noticed above between the stresses in the present tests is just the kind of connexion that would be expected if the assumption is correct, remembering that different specimens were used, no doubt differing slightly in properties and strength.

14. *Final Conclusions.*

It must be concluded that the maximum shear stress determines when yield takes place, but this will vary slightly on account of the difference in the shearing resistance in various directions, and any idea of a force analogous to friction must be abandoned. For the common case of combined bending and twisting either the circular or elliptic formula may be safely used, or to approximate to the elliptic formula $\sqrt{M^2+T^2}$ may be equated to the equivalent moment, taken as being of the same kind as the larger which is actually operating, and the corresponding true working stress used. It is not admissible to extend the conclusions to brittle materials until further evidence is available, and the author hopes to make experiments in this direction shortly. The writer wishes to acknowledge his debt to Mr. Guest for the results borrowed from his paper, and to thank Lord Blythswood for affording the opportunity of making the tests.

Discussion.

Dr. Morrow, in a letter to the Secretary, remarked that he had read Mr. Scoble's paper with great interest, and hoped it would help in removing the obstacles to the application of some theoretical formulæ in design. It was well known that many of the applications of the theory of bending were worse than useless in practice. In cases of combined tension and bending of solid bars (as in crane-hooks, coupling-rods of locomotives, &c.) the usual methods were misleading. In Proc. Roy. Soc. vol. 73, he (Dr. Morrow) described experiments showing that the strains, and hence probably also the stresses, caused by bending rectangular beams were considerably smaller than those given by calculation. Since then the method has been extended to higher stresses in wrought iron and steel bars, but still within the elastic limit. Direct measurements of the longitudinal tensile and compressive strains bear out the previous conclusions. Until it is realized that the ordinary theory of bending is inapplicable to materials of construction (except as a useful approximation when the stresses are low) it will be difficult to arrive at correct conclusions in any problems of combined stress in which bending occurs. Taking some of the results from Mr. Scoble's paper in which failure was by bending, and assuming that the tensile stress was only $\frac{5}{8}$ of that given by the Euler-Bernoulli theory, Dr. Morrow has drawn up a table which shows that there is nothing to disprove the maximum principal stress theory when the specimen fails by bending. The method adopted is exceedingly rough and does not apply to the very different cases in which failure is by torsion or shearing.

The Chairman said the paper was a most valuable one, and more scientific experiments upon the subject were required. He could not agree with the notion put forward in many English text-books that the maximum principal stress determined fracture.

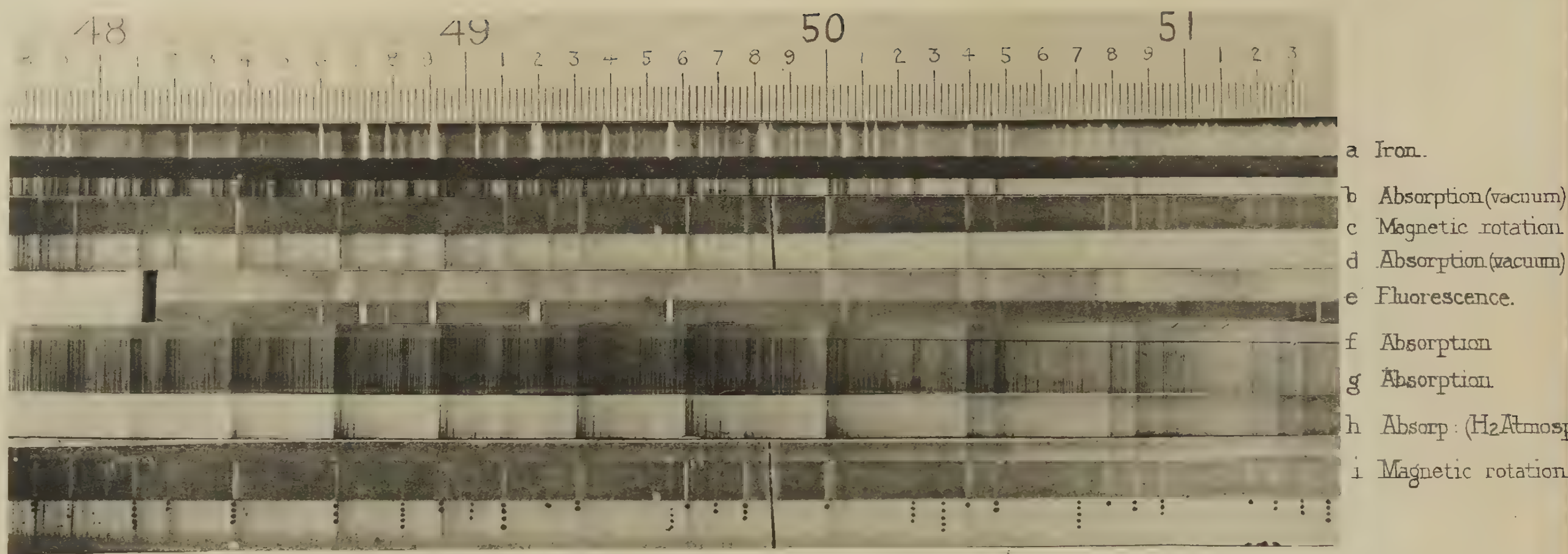
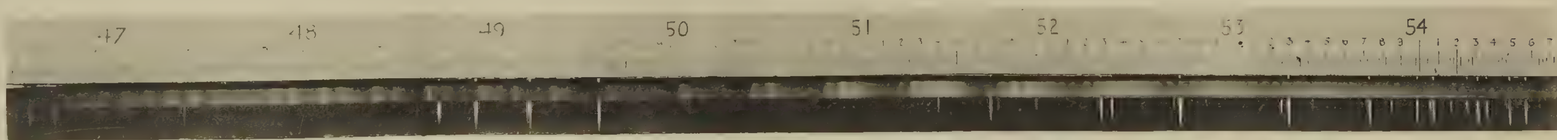
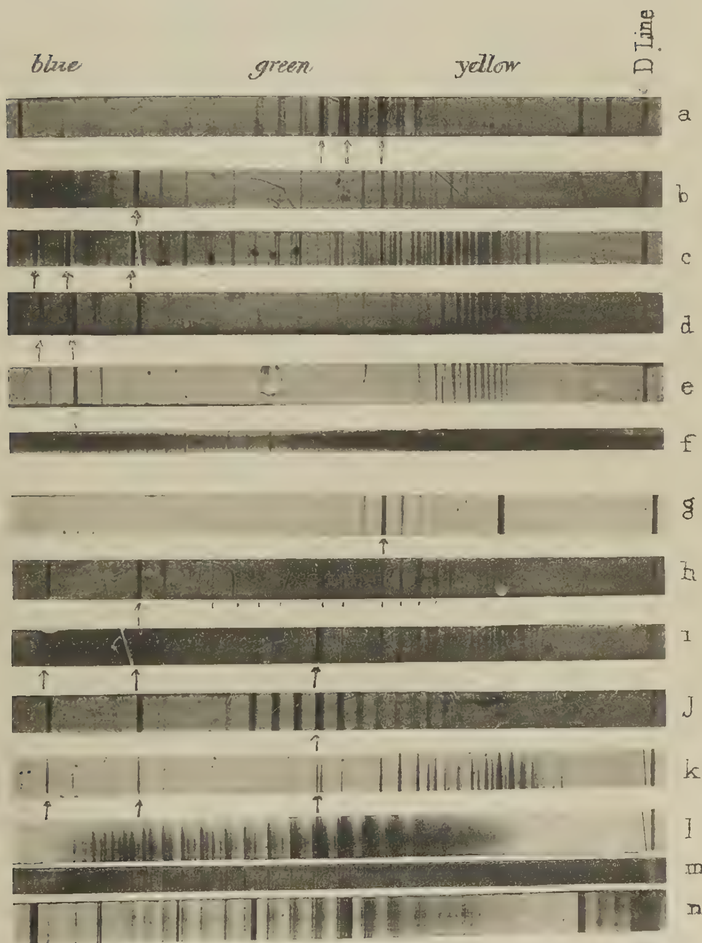


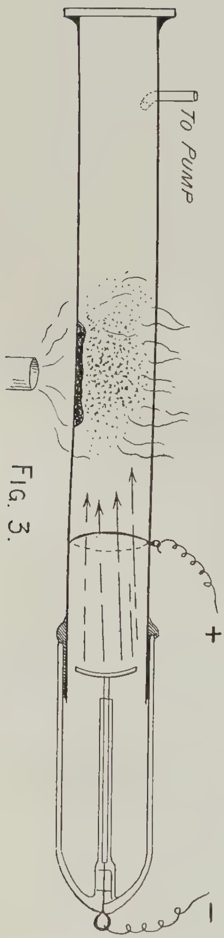
Fig. I.



Fluorescence of Na. Fe Comparison.

Fig 2.


blue
green
yellow
D Line
a
b
c
d
e
f
g
h
i
J
k
l
m
n


TO PUMP
+
−
FIG. 3.

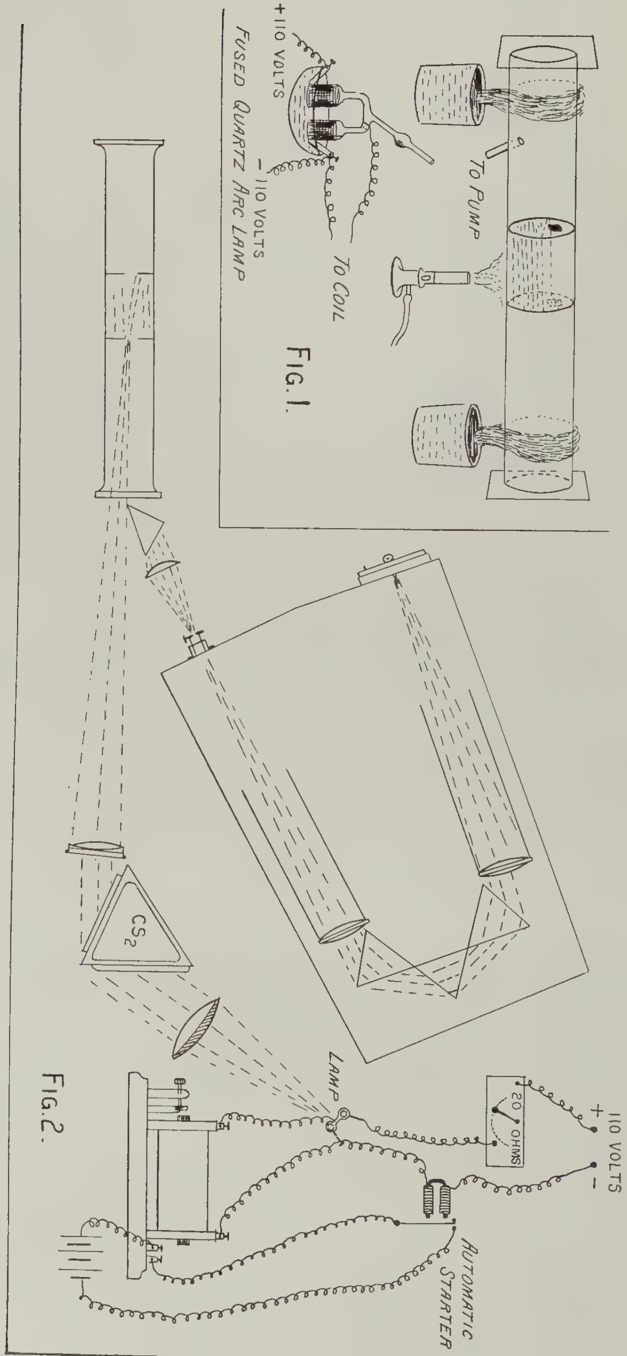

TO PUMP
+110 VOLTS
−110 VOLTS
TO COIL
FUSED QUARTZ ARC LAMP
FIG. 1.
CS2
LAMP
20 OHMS
110 VOLTS
+
−
AUTOMATIC STARTER
FIG. 2.

FIG. 1.

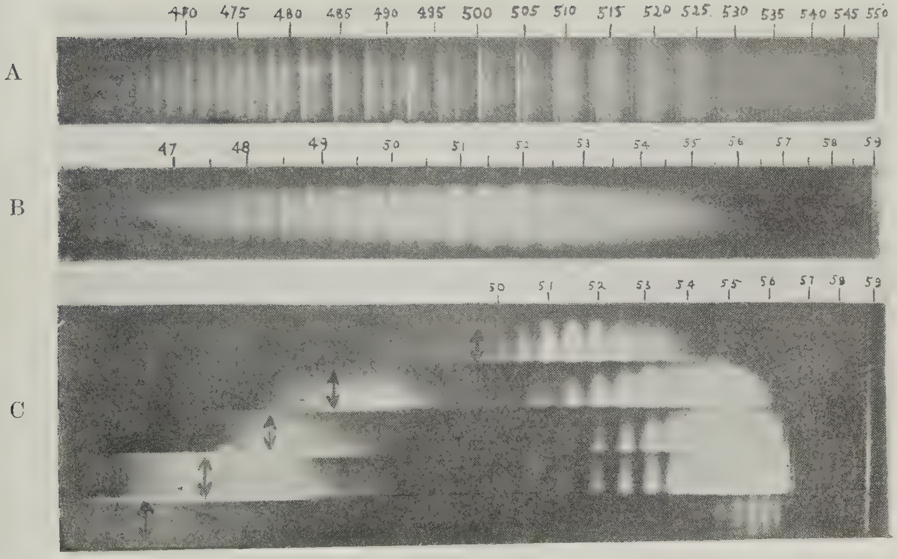
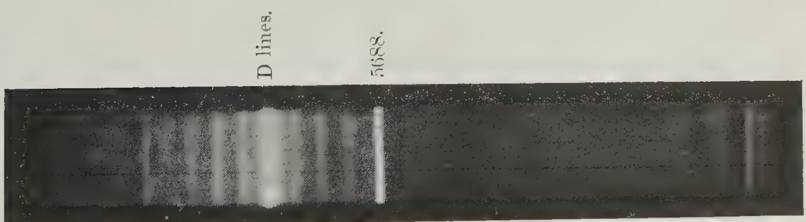
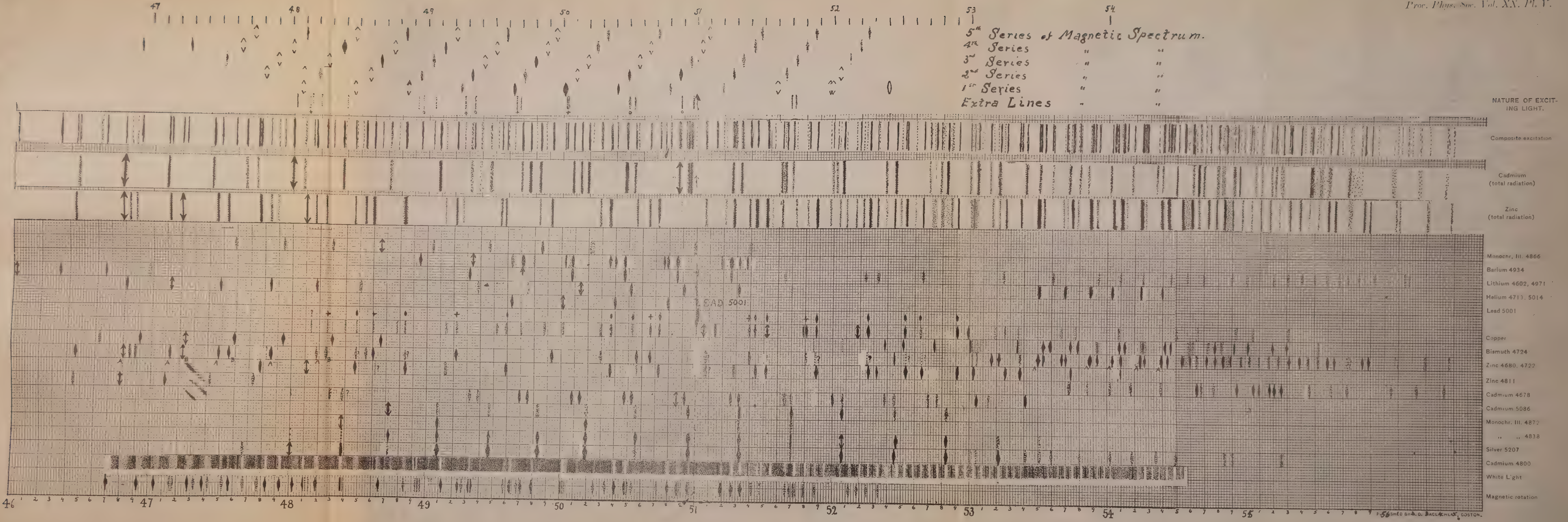

A
470 475 480 485 490 495 500 505 510 515 520 525 530 535 540 545 550
B
47 48 49 50 51 52 53 54 55 56 57 58 59
C
50 51 52 53 54 55 56 57 58 59

FIG. 2.


D lines.
5688.


5th Series of Magnetic Spectrum.
4th Series " "
3rd Series " "
2nd Series " "
1st Series " "
Extra Lines " "
NATURE OF EXCITING LIGHT.
Composite excitation
Cadmium (total radiation)
Zinc (total radiation)
Monochr. Ill. 4866
Barium 4934
Lithium 4602, 4971
Helium 4713, 5014
Lead 5001
LEAD 5001
Copper
Bismuth 4724
Zinc 4680, 4722
Zinc 4811
Cadmium 4678
Cadmium 5086
Monochr. Ill. 4872
" " 4838
Silver 5207
Cadmium 4800
White Light
Magnetic rotation
46
47
48
49
50
51
52
53
54
55
56

XXVII. *The Behaviour of Iron under Small Periodic Magnetizing Forces.* *By* J. M. Baldwin, *M.A.*, *B.Sc.**

[Plate VI.]

1. The object of the present paper, which was undertaken at the suggestion of Professor Lyle, is to investigate the relations which obtain under actual working conditions between the amplitudes and phases of the different harmonics of the magnetic induction produced in iron, and those of the magnetic intensity producing it, for low values of the induction; and from the harmonic expressions obtained for these quantities to deduce the loss of energy in the iron. The work which has been done previously on this subject has been performed under statical conditions, which are seldom realized in practice.

The induction produced in iron in weak magnetic fields has been measured by Baur † using the ballistic method (statical). For the range H=0·38 to H=0·016 (with corresponding limits of the induction B=283 and B=3·3) he found that the permeability μ could be represented closely by an expression of the form

$$\mu = a + b\mathrm{H},$$

where a and b are constants for the specimen. By extrapolation, μ would tend to a definite limit in exceedingly weak fields. This limit for the specimen examined was 185.

Later, Lord Rayleigh ‡, working with the compensated magnetometer method (statical), showed that, for unannealed iron, the permeability over the range H=0·04 to H=0·00004 was appreciably constant, while for values of H greater than 0·04 the expression

$$\mu = a + b\mathrm{H}$$

agreed well with experiment up to H=1·2. The value of a for hard iron was 90 or 100. Similar results held for steel; but for soft iron a complication arose due to the time-lag of

* Read October 26, 1906.

† C. Baur, Wied. *Ann.* xi. p. 394 (1880).

‡ Lord Rayleigh, Phil. Mag. [5] vol. xxiii. p. 225 (1887).

the magnetization. The effect of this time-lag in soft iron has been further investigated by Ewing *, who found that up to H=0·1 any change of H instantaneously produced an induction proportional to the change in H, and that then the induction gradually changed (in the same direction as the instantaneous effect) for some seconds afterwards. In the experiments, a description of which is given later, it will be found that when a bundle of wires (in which the eddy-currents are small) is subjected to cyclic variations of magnetic intensity, the hysteresis loop is practically a straight line at the lowest inductions over the whole range of frequencies used (50 to 8·5). This would show that the amount of creeping of the induction that had time to occur was negligible for these frequencies. The iron, however, was not very soft.

2. The relation between the amplitude and phase of the harmonics of the intensity of an alternating magnetic field and of the corresponding magnetic induction produced in iron have been investigated for different frequencies by Lyle †; the amplitude of the first harmonic of the intensity varying from 0·4 to 5·67 for approximately sinusoidal waves, and to 14 for waves not sinusoidal. The total amount of energy lost in the iron per c.c. per cycle was also calculated from the Fourier expressions for the allied current and flux waves, and was found to be given with considerable accuracy within the limits B=1000 and B=12,000 (when the magnetizing current wave was approximately sinusoidal) by the formula

$$\mathrm{I} = (a+bn)\mathfrak{B}^x,$$

where I is the total iron loss,

a, b, x are constants for any particular specimen,

n the frequency,

and $\mathfrak{B}$ the "effective induction," which is $\dfrac{\sqrt{2}}{2\pi n}$ times the square root of mean square of $\dfrac{d\mathrm{B}}{dt}$.

From the total iron loss I, the sum of the static hysteresis and the calculated eddy-current loss was subtracted, and thus

* J. A. Ewing, Proc. Roy. Soc. vol. xlvi. p. 269 (1889); Magnetic Induction in Iron and other Metals, 3rd edition, p. 127.

† T. R. Lyle, Phil. Mag. [6] vol. ix. p. 104 (1905).

the kinetic hysteresis was obtained. This kinetic hysteresis increased with the induction and with the frequency.

3. In the present series of experiments, in which the magnetic intensity ranged from H=1·5 to H=0·01 and B from 600 to 2·5, the hysteresis areas could have been determined only for the higher inductions by the statical methods, but the wave-tracer of Professor Lyle * affords a convenient method of obtaining the total iron loss, when the magnetic field is produced by an alternating current of any desired frequency. The portion of this loss of energy due to eddy-currents set up in the iron can be approximately calculated, and subtracting this from the total iron loss, the loss due to hysteresis is obtained.

Two specimens were experimented upon, in one of which (a bundle of iron wires) the eddy-current loss was comparatively small, while in the other (a rod of Lowmoor iron 0·3 sq. cm. in cross section) the eddy-current loss was considerable. For the former specimen it was found that for very low inductions the lag of phase of the induction behind the magnetizing force was very small, and also that the permeability was practically uninfluenced by the frequency of the alternations. In the latter, where the eddy-currents are considerable, the resultant intensity inside the rod will no longer be in phase with the current in the solenoid, and so the resultant induction will always lag behind the magnetizing current.

Assuming that the induction at any point is in phase with the magnetic intensity at that point, as the experiments with the wire bundle show, and also that the permeability has a constant value for low inductions, equations can be obtained connecting the apparent permeability and the lag in phase of the induction behind the magnetizing current with the frequency.

Let OF (see figure) represent the resultant flux across the central section of the rod. The variation of the flux will produce an E.M.F. in a circuit round the rod in quadrature with OF and behind it in phase. This E.M.F. produces a current, which in turn produces a flux FM or IF (where IF=FM) along the rod. This flux will also, on the assumptions which have been made, be at right angles to OF. Then

* T. R. Lyle, Phil. Mag. [6] vol. vi. p. 549 (1903).

OI is the flux which would have passed along the rod if there had been no eddy-currents.

If $OI=F$, $OF=F_r$, $FI=F_e$ it follows that

$F=\mu H\alpha$, $F_r=\mu_r H\alpha$, $F_e=kF_r\omega$,

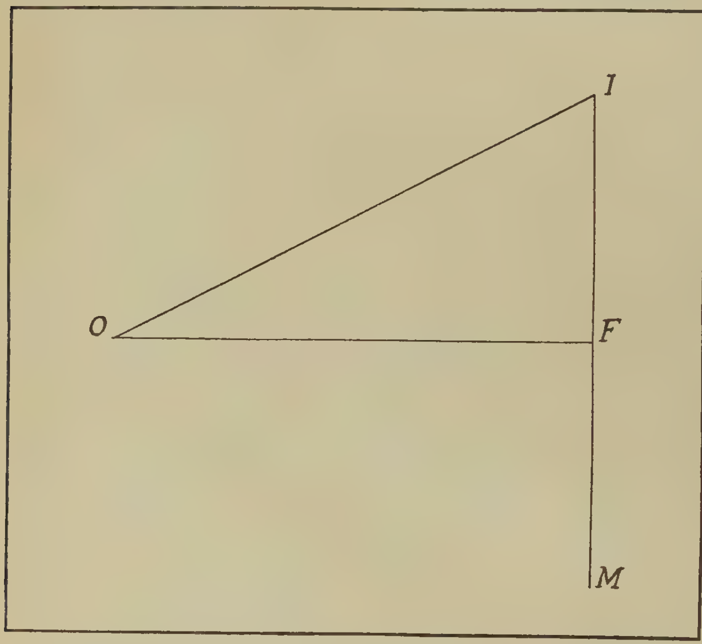
where μ is what the permeability would be if there were no eddy currents,

μ_r is the observed apparent value of the permeability,

α is the area of cross section of the rod,

ω is the frequency multiplied by 2π,

and k is a constant.



Then writing θ for the angle FOI, the lag in phase of the induction behind the magnetizing current,

$$\cos\theta=\frac{F_r}{F}=\frac{\mu_r H\alpha}{\mu H\alpha}=\frac{\mu_r}{\mu},$$

and

$$\tan\theta=\frac{F_e}{F_r}=\frac{kF_r\omega}{F_r}=k\omega.$$

In these equations ω, μ_r, θ are observed variable quantities,

while μ and k are constants for the specimen on the assumptions given above, and these assumptions can be tested by noting the agreement between the values of the quantities found from the equations.

4. The arrangement of apparatus was practically identical with that already described by Lyle *, the iron ring and the coils on it being replaced by a long solenoid, through which the specimen (a long straight rod or bundle of wires) passed. Round the centre of the specimen a small secondary coil was wound, by means of which the flux was determined. The speed was regulated by altering the number of cells in the armature circuit of the rotary converter and by altering slightly the resistance in the field-magnet circuit. The reducing factor of the galvanometer was determined by connecting a Clark cell to the galvanometer, with a megohm in series, the current balance being used only to adjust the current to the required value.

The procedure in any one experiment has also been described. In every case a full wave was taken, the ordinates being 12° in phase apart. In the reductions, no notice was taken of the seventh harmonic, which was always small and could not have affected the results, its influence on the amplitude being quite negligible, and its influence on the phase being practically eliminated by taking the phase of the first harmonic from four ordinates on each side of the zero ordinate of the curve. In every case the third component † was plotted, and from it the ninth harmonic was taken, thus leaving the third harmonic.

The wave forms having been analysed, the results were reduced to absolute measure by applying to them the proper factors ‡ to reduce them to magnetic intensity and induction respectively in the form

$$\mathrm{H} = \mathrm{H}_1 \left[\sin \omega t + h_3 \sin_3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5)\right]$$

$$\mathrm{B} = \mathrm{B}_1 \left[\sin(\omega t - \theta_1) + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5)\right]$$

the harmonics of order higher than the fifth being neglected.

* T. R. Lyle, Phil. Mag. [6] vol. ix. p. 104 (1905).

† T. R. Lyle, Proc. Roy. Soc. Victoria, vol. xvii. pt. 2, p. 394 (Feb. 1905); Phil. Mag. [6] vol. xi. p. 25 (1906).

‡ T. R. Lyle, Phil. Mag. [6] vol. vi. p. 549 (1903).

From these the value of the iron loss per c.c. per cycle $\left(I=\frac{1}{4\pi}\int\dot{H}dB\right)$ is

$$I=\frac{H_1B_1}{4}\{\sin\theta_1+3h_3b_3\sin 3(\theta_3-\phi_3)+5h_5b_5\sin 5(\theta_5-\phi_5)\}.$$

To obtain the amount of this energy loss which is due to hysteresis, the eddy-current loss has been calculated from the approximate formula given by Searle and Bedford * on the assumptions that μ is constant and that $\frac{dH}{dt}$ does not vary over the section

$$\frac{dX}{dt}=\frac{A}{8\pi\rho}\cdot\left(\frac{dB}{dt}\right)^2;$$

where X is the eddy-current loss per c.c.,
A the area of section of the rod or wire,
and ρ the specific resistance of the iron ;

$$\therefore\quad E=\frac{A}{8\pi\rho}\int_0^T\left(\frac{dB}{dt}\right)^2dt,$$

where T is the period of the alternations,
and E is the eddy-current loss per c.c. per cycle.

Substitute for B and perform the integration,

$$E=\frac{A}{8\pi\rho}B_1^2\omega\,.\,2\pi^2[1+9b_3^2+25b_5^2]$$

$$=\frac{\pi A}{8\rho T}\mathfrak{B}^2,$$

where $\mathfrak{B}^2=B_1^2[1+9b_3^2+25b_5^2]$. ($\mathfrak{B}$ is the quantity which has been referred to before as the "effective induction.")

E having been calculated, the hysteresis U is given by

$$U=I-D.$$

5. The first specimen experimented upon was a bundle of iron wires of low permeability, which were taped together to form a long circular cylinder, the details of which are given below. A coil was wound on the centre of the bundle for

* G. F. C. Searle and T. G. Bedford, Phil. Trans. cxcviii. p. 33, 1902, App. 1

measuring the induction produced in the iron. Four different frequencies were employed, namely, 50, 28, 15·5, and 8·5.

Details of Iron Wire Bundle.

Number of wires	46.
Mean diameter of wires	0·0789 cm.
Sectional area of iron	0·2250 sq. cm.
Length of wires	307 cm.
Specific resistance at 12°·5 C.	$1·366 \times 10^4$.
Turns on central coil	400.

Details of Magnetizing Solenoid.

Length of solenoid	170 cm.
Internal diameter of windings.	1·6 cm. q.p.
Constant of solenoid	H=53·37 C.

The results obtained with this specimen are given in the following tables (I. to IV.). The meanings of the different symbols in the tables are given by

$$T = \text{period},$$

$$H = H_1 [\sin \omega t \quad + h_3 \sin 3 (\omega t - \phi_3) \quad + h_5 \sin 5 (\omega t - \phi_5)],$$
$$B = B_1 [\sin (\omega t - \theta) + b_3 \sin 3 (\omega t - \theta - \psi_3) + b_5 \sin 5 (\omega t - \theta - \psi_5)],$$

$$\mu_0 = \frac{B_1}{H_1}, \qquad \mu = \frac{B_{max.}}{H_{max.}},$$

$$\omega \mathfrak{B} = \sqrt{2} \,.\, \text{R.M.S.} \left(\frac{dB}{dt}\right),$$

I=Iron loss per c.c. per cycle,

E=Calculated eddy-current loss per c.c. per cycle,

$$I = \sigma \mathfrak{B}^{1·6}.$$

Statical Permeability for different Inductions.

B	11·3	44·8	111·7	171·3	281·8	421·2	603·0
H	·0747	·2452	·5324	·7474	1·0664	1·3903	1·7119
μ	152·0	182·8	209·5	229·2	264·3	303·0	352·2

A number of the results contained in the tables are shown graphically by means of curves. The wave-form of the magnetizing current is in every case approximately sinusoidal, and so the results obtained may be taken as typical of

Table I.—Wire Bundle. $T = \cdot 020$.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	$B_{max.}$	$H_{max.}$	μ.	$\mathfrak{B}$.	I.	E.	I–E.	σ.
·02017	1·659	·0344	59°·63	·0009	61°·66	528·3	14°·25	·0532	30°·20	·0078	40°·43	318·5	533·2	1·720	310·0	535·4	53·09	3·76	49·33	·002286
·02019	1·289	·0384	63·54	·0000	......	362·9	12·44	·0467	36·24	·0100	41·42	281·5	369·5	1·340	275·8	366·9	24·73	1·76	22·97	·001950
·02015	·998	·0436	65·01	·0020	59·64	253·6	10·94	·0442	42·80	·0097	43·57	254·1	260·6	1·040	250·6	256·1	11·80	·86	10·94	·001654
·02016	·679	·0489	66·61	·0026	61·17	150·7	8·16	·0437	50·62	·0067	47·95	221·8	156·4	·7060	221·5	152·1	3·571	·303	3·268	·001152
·02016	·4532	·0514	67·73	·0014	70·73	92·02	6·22	·0459	56·26	·0104	51·14	203·0	95·62	·4724	202·4	93·01	1·104	·113	·991	·000782
·02025	·2216	·0547	68·07	·0082	73·38	39·36	3·80	·0487	60·90	·0094	63·78	177·6	41·40	·2314	178·9	39·82	·1410	·0207	·1203	·000388
·02008	·0736	·0523	68·44	·0109	82·32	12·18	1·59	·0482	64·10	·0103	75·90	165·4	12·67	·0764	165·8	12·32	·00590	·00200	·00390	·000106
·02007	·0148	·0576	69·41	·0113	86·85	2·36	1·27	·0481	65·84	·0134	82·77	158·8	2·46	·0155	153·8	2·39	·00018	·00007	·00011	·000045

Table II.—Wire Bundle. $T = \cdot 036$.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	$B_{max.}$	$H_{max.}$	μ.	$\mathfrak{B}$.	I.	E.	I–E.	σ.
·03575	1·666	·0352	61°·37	·0026	48°·28	558 9	14°·97	·0616	28°·51	·0137	39°·07	335·5	563·2	1·724	326·7	569·7	58·94	2·45	56·49	·002298
·03595	1·287	·0373	60·62	·0048	63·39	374·8	12·79	·0477	33·42	·0124	40·59	291·2	377·5	1·347	280·2	379·4	26·22	1·08	25·14	·001959
·03614	·9943	·0369	62·17	·0044	62·01	257·2	10·67	·0426	37·50	·0127	44·29	258·6	261·6	1·041	251·3	259·8	11·62	·50	11·12	·001591
·03592	·6847	0386	61·83	·0029	61·16	158·4	8·35	·0438	41·46	·0093	48·63	231·3	163·0	·7130	228·6	159·9	3·856	·192	3·664	·001148
·03576	·4745	·0389	60·81	·0050	57·30	97·23	6·29	·0412	47·45	·0054	51·01	204·9	101·4	·4975	203·8	98·01	1·242	·073	1·169	·000809
·03567	·2550	·0385	61·38	·0019	54·72	47·10	3·99	·0389	51·70	·0061	50·31	184·7	48·98	·2647	185·0	47·45	·2048	·0170	·1878	·000426
·03558	·0858	·0379	61·94	·0030	22·01	14·31	2·60	·0362	57·18	·0040	61·99	166·7	14·93	·0893	167·2	14·40	·01379	·00157	·01222	·000193
·03559	·0179	·0395	60·54	·0038	65·02	2·81	0·35	·0296	57·37	·0043	60·38	157·0	2·92	·0184	158·8	2·82	·00007	·00006	·00001	·000013

Table III.—Wire Bundle. T = ·065.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	B_{max}.	H_{max}.	μ.	𝔅.	I.	E.	I–E.	σ.
·0656	1·681	·0327	58°·97	·0001	55°·30	575·2	14°·43	·0619	27°·18	·0101	38°·47	342·2	575·2	1·733	331·9	585·8	59·10	1·38	57·72	·002204
·0656	1·290	·0375	59·71	·0026	67·32	375·2	12·36	·0531	34·35	·0096	43·02	290·9	382·6	1·337	286·2	380·4	25·45	·58	24·87	·001894
·0655	·9804	·0373	62·03	·0009	64·15	254·4	10·42	·0476	38·47	·0091	45·70	259·5	260·8	1·019	255·9	257·2	11·06	·27	10·79	·001539
·0657	·7056	·0419	61·80	·0008	22·11	164·6	8·40	·0454	42·59	·0041	43·35	233·2	170·4	·7360	231·5	166·1	4·152	·110	4·042	·001163
·0656	·4597	·0431	62·18	·0027	9·71	94·20	6·09	·0410	47·65	·0035	38·26	204·9	98·17	·4823	203·6	94·93	1·123	·036	1·087	·000770
·0656	·2608	·0399	62·63	·0071	67·04	48·54	4·07	·0392	56·68	·0065	58·57	186·1	50·24	·2742	183·2	48·90	·2225	·0096	·2129	·000441
·0651	·0962	·0364	60·44	·0107	68·30	16·12	1·94	·0336	57·39	·0097	65·38	167·6	16·82	·1006	167·2	16·22	·01299	·00106	·01193	·000151
·0648	·0229	·0380	60·66	·0122	68·83	3·64	1·09	·0295	58·18	·0125	68·67	158·8	3·78	·0241	157·2	3·66	·00039	·00005	·00034	·000049

Table IV.—Wire Bundle. T = ·117.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	B_{max}.	H_{max}.	μ.	𝔅.	I.	E.	I–E.	σ.
·1150	1·985	·0416	61°·75	·0027	68°·80	852·8	18°·20	·0734	26°·45	·0175	33°·60	429·7	843·1	2·072	406·9	876·5	129·1	1·8	127·3	·002527
·1146	1·005	·0371	60·73	·0034	63·48	261·4	10·06	·0421	37·73	·0073	40·60	260·1	265·8	1·044	254·6	263·6	11·28	·16	11·12	·001509
·1163	·7130	·0417	61·95	·0020	58·83	163·7	8·07	·0409	43·32	·0073	62·76	229·6	168·3	·7418	226·9	165·0	4·009	·062	3·947	·001133
·1173	·4656	·0438	61·06	·0033	68·74	95·40	5·92	·0418	48·57	·0086	51·37	204·9	99·72	·4871	204·7	96·24	1·124	·021	1·103	·000754
·1174	·2576	·0448	61·40	·0067	70·00	47·41	3·94	·0390	52·91	·0081	66·61	184·2	49·65	·2718	182·7	47·81	·2059	·0051	·2028	·000423
·1196	·1014	·0359	61·46	·0090	70·60	16·88	1·56	·0392	56·82	·0023	66·93	166·5	17·50	·1063	164·6	17·00	·01138	·00063	·01075	·000122
·1206	·0216	·0368	63·36	·0074	67·78	3·47	−0·62	·0309	55·28	·0064	45·33	160·5	3·61	·0226	159·7	3·49	−·00023	·00003	−·00026	……

a series of induction-waves produced by magnetizing currents of similar wave-form.

The most important of the characteristics are μ_0 and θ, which connect the first harmonic of the induction with that of the magnetic intensity. If the magnetizing current and the flux-waves are assumed sinusoidal, then they will be given by expressions of the form

$$H = H_0 \sin \omega t,$$
$$B = B_0 \sin (\omega t - \theta),$$
$$B_0 = \mu_0 H_0 ;$$

and from these, since the total iron loss per c.c. per cycle is

$$I = \frac{1}{4\pi} \int H \, dB,$$

it follows that

$$I = \frac{H_0 B_0}{4} \sin \theta.$$

Thus when μ_0 and θ are given in terms of B_0, the behaviour of the iron is determined.

The relation between μ_0 and B_1 for the different series is shown in Pl. VI. fig. 1 (upper curve). It is seen that the value of μ_0 is very little affected by the frequency, the four sets of points lying on practically the same curve for values of B_1 up to 300 or so ; but beyond this the curves seem to separate, the value of μ_0 becoming smaller as the frequency is increased. On the same diagram are plotted the points for the statical permeability, and these also lie on the same curve.

μ_0 decreases as the induction is lowered, this decrease becoming more rapid as the value of B_1 becomes smaller, and the value of μ_0 appears to tend to a definite limit of about 157 for very low values of B_1. This limit is better shown by plotting μ_0 against H_1. If this be done, it appears that up to about $H = 1 \cdot 0$ the relation between μ_0 and H_1 can be well represented by a straight line, as Baur and others have already shown for the statical permeability.

The values of θ for the four series are plotted against B_1 in fig. 1 (lower curve), and here, although there is not so close an agreement between the four series as there was for

the $\mu_0 - B_1$ curves, yet the points all lie near a mean curve. The value of θ is seen to decrease rapidly as B_1 diminishes, and the rate of increase becomes more and more rapid as the induction becomes lower, the value of θ being about 1° when B_1 is as low as 2, and there seems no reasonable doubt that in the absence of eddy-currents θ would become exceedingly small for very low values of B_1.

The curves for b_3 (the ratio of the amplitude of the third harmonic of the flux-wave to that of the first) seem to show that this quantity decreases slowly as B_1 decreases, and finally takes up a value of about ·030 for the weakest fields used. The behaviour of b_3 for the period $T = {\cdot}020$ is somewhat different, for there, in the weaker fields, b_3 increases. It will be seen from Table I. that for this series h_3 also increases in the weaker fields, and has a considerably higher value than in the other series, though whether this is the cause or an effect of the increase of b_3 cannot be said. ψ_3, the phase-lag of the third harmonic of the induction-wave behind the first, shows the same general behaviour for all four series; gradually increasing as the induction decreases, it appears to tend to a value of about 60° for exceedingly low values of B_1.

Of b_5, all that can be said is that it has a value somewhere below ·01, and of ψ_5 that it increases as the induction diminishes; but little dependence can be placed on the values of b_5 and ψ_5 for the lower inductions. This arises from the small value of b_5. The amplitude of the galvanometer deflexion was seldom more than 20 mm., so that if b_5 were ·01, the amplitude of the fifth harmonic would be only ·2 mm., and a slight unsteadiness of the readings from any cause might greatly affect the values of both b_5 and ψ_5.

In fig. 2 (lower curve) the hysteresis loss is plotted against the maximum value of B for the different experiments of the four series. To avoid confusion, several points at the lower inductions have been omitted, but in every case all points that have been left out lie on the curve. It is seen that here again the four series of points lie on practically the same curve; that is, the hysteresis loss for the specimen under

examination is practically independent of the frequency, provided the maximum induction does not exceed 600. This being the case, it would be impossible with the present apparatus to detect with certainty a loss due to kinetic hysteresis at these low inductions.

On the same diagram (fig. 2, upper curve) is shown the variation of Steinmetz's coefficient σ* ($I=\sigma\mathfrak{B}^{1\cdot6}$), with the maximum value of the induction. The curves show that σ decreases rapidly as the induction diminishes, tending towards the value zero at very low inductions. Further, σ increases with frequency, the increase being due to the increased eddy-current loss, for if the eddy-current loss is deducted, the points all lie on one curve.

6. A further series of experiments was performed on a long rod of Lowmoor iron 0·3 sq. cm. in cross section, in which the eddy-current effect would be considerable. As for the former specimen, four different frequencies were employed, namely, 55, 28, 15, and 8·5.

Details of Lowmoor Iron Rod.

A straight cylindrical rod of circular section.

Area of section	·2999 sq. cm.
Length	360 cm.
Specific resistance at 12°·5 C...	$1\cdot119\times10^4$.
Number of turns on central coil.	400.
Magnetizing solenoid as before.	

The results obtained with this specimen are given in Tables V. to VIII.

Statical Permeability for different Inductions.

B	12·9	52·1	144·2	228·0	389·3	608·5	920·0
H	·0747	·2450	·5334	·7467	1·0671	1·3903	1·7097
μ	172·6	212·5	270·3	305·4	364·8	437·7	538·1

* In calculating σ, the effective induction $\mathfrak{B}$ has been used, not the maximum induction B_{max}.

TABLE V.—Lowmoor Iron Rod. T = ·018.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	B_{max}.	H_{max}.	μ.	$\mathfrak{B}$.	I.	E.	σ.
·01822	1·687	·0261	76°·21	·0029	13°·32	312·0	47·97	·0108	18°·13	·0033	29°·46	184·9	308·5	1·724	178·9	312·2	97·70	106·8	·009972
·01791	·9890	·0346	81·12	·0066	18·75	160·4	45·81	·0146	38·46	·0030	50·61	162·2	161·5	1·007	160·4	160·6	28·45	28·73	·008412
·01804	·4604	·0332	79·41	·0078	78·72	67·12	40·65	·0157	48·06	·0029	54·66	145·8	67·68	·4717	143·5	67·20	5·039	4·998	·006006
·01802	·0763	·0327	80·76	·0128	24·99	9·51	33·92	·0182	61·51	·0032	63·07	124·7	9·68	·0784	123·4	9·53	·1014	·1006	·002751

TABLE VI.—Lowmoor Iron Rod. T = ·036.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_5.	μ_0.	B_{max}.	H_{max}.	μ.	$\mathfrak{B}$.	I.	E.	σ.
·03598	1·666	·0442	77°·69	·0043	10°·37	436·5	43°·62	·0230	41°·30	·0039	40°·41	262·0	440·5	1·730	254·6	473·6	125·85	106·22	·007484
·03607	1·295	·0442	79·23	·0051	10·82	315·3	41·06	·0209	49·94	·0024	48·98	243·5	318·4	1·329	239·6	316·0	67·23	55·23	·006731
·03609	·9935	·0499	79·54	·0071	18·99	229·2	39·16	·0235	52·87	·0039	43·56	230·7	232·6	1·026	226·7	229·8	36·07	29·21	·006012
·03616	·6930	·0541	79·84	·0097	18·76	148·0	35·47	·0262	54·12	·0030	57·18	213·6	151·7	·7204	210·6	148·5	14·93	12·17	·005004
·03625	·4780	·0551	79·35	·0108	19·60	94·72	33·00	·0284	59·89	·0043	60·19	198·2	97·50	·4968	196·3	95·08	6·200	4·978	·004241
·03632	·2412	·0577	80·15	·0121	21·47	42·89	27·79	·0340	63·71	·0072	68·03	177·8	44·33	·2483	178·5	43·15	1·214	1·023	·002939
·03653	·0803	·0569	80·69	·0129	18·95	12·40	21·86	·0353	71·04	·0067	72·52	154·5	12·88	·0825	156·0	12·48	·0935	·0850	·001648
·03642	·0237	·0602	82·20	·0134	18·29	3·47	20·19	·0385	75·37	·0086	82·49	146·6	3·63	·0245	147·8	3·49	·00718	·00669	·000972

TABLE VII.—Lowmoor Iron Rod. T = ·065.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5.	ψ_3.	μ_0.	B_{max}.	H_{max}.	μ.	$\mathfrak{B}$.	I.	E.	σ.
·0640	1·682	·0295	61°·73	·0022	44°·77	580·5	38°·34	·0318	27°·72	·0033	20°·97	345·1	576·6	1·731	333·1	583·3	151·55	105·05	·005690
·0654	·9839	·0398	62·93	·0026	59·75	279·9	32·04	·0348	36·10	·0030	47·18	284·5	282·2	1·018	277·2	281·4	36·60	23·96	·004411
·0662	·4548	·0410	61·49	·0012	44·57	107·0	24·50	·0386	42·48	·0017	60·15	235·3	110·0	·4721	233·0	107·7	5·061	3·467	·002836
·0652	·0799	·0396	62·36	·0040	62·87	13·58	14·16	·0377	54·66	·0020	19·63	170·0	14·02	·0833	168·4	13·67	·0668	·0567	·001017

TABLE VIII.—Lowmoor Iron Rod. T = ·118.

T.	H_1.	h_3.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	b_3.	ψ_3.	b_5,	ψ_5.	μ_0.	B_{max}.	H_{max}.	μ.	$\mathfrak{B}$.	I.	E.	σ.
·1166	1·601	·0428	60°·70	·0030	49°·79	660·8	32°·66	·0463	32°·72	·0056	31°·16	412·7	668·9	1·677	398·9	667·3	143·2	75·88	·004335
·1178	1·218	·0408	60·83	·0041	47·26	444·5	28·46	·0455	36·30	·0063	37·51	365·0	453·0	1·265	358·1	448·8	64·68	33·97	·003694
·1176	·9902	·0419	61·21	·0026	49·64	332·4	25·47	·0417	36·29	·0057	37·59	335·7	339·3	1·029	329·7	335·1	35·41	18·97	·003228
·1183	·6943	·0459	61·25	·0044	43·22	202·9	22·11	·0459	40·80	·0058	48·47	292·2	208·4	·7244	287·7	204·9	13·22	7·05	·002647
·1182	·4995	·0464	60·00	·0037	25·20	131·4	19·42	·0459	41·19	·0036	57·07	263·1	134·7	·5207	258·7	132·6	5·457	2·956	·002192
·1180	·2292	·0480	58·32	·0058	42·66	48·79	13·65	·0464	49·76	·0012	12·41	212·9	50·48	·2406	209 8	49·27	·6645	·4085	·001301
·1178	·0755	·0501	57·98	·0048	56·13	13·18	7·23	·0450	53·03	·0023	45·51	174·6	13·78	·0797	173·0	13·30	·0315	·0298	·000502
·1182	·0149	·0474	57·84	·0067	64·44	2·37	5·64	·0525	62·37	·0074	13·64	158·8	2·47	·0157	157·5	2·40	·00090	·00097	·000222

From the tables it appears that the wave form of the magnetizing current is nearly sinusoidal, and so the induction may be considered as that produced by magnetizing currents of similar wave-form.

The relation between μ_0 and B_1 is exhibited graphically in fig. 3 (upper curves). These curves show that for any one frequency the value of μ_0 diminishes as the induction is decreased, the decrease becoming more and more rapid as the induction get slower. If μ_0 is plotted against H_1, however, it is found that, as for the bundle of wires, the portions of the curves for low values of H_1 are approximately straight lines, but the range over which this is true is much more limited for the rod than for the wire bundle, extending in the former only to about $H = 0{\cdot}3$. The curves appear to approach a definite limit for each particular frequency when the value of the induction becomes very low. It is seen also that the frequency has a marked effect on the value of μ_0, an increase in frequency being accompanied by a decrease in μ_0, the decrease being more marked for the higher frequencies.

On the same diagram is shown the relation between the statical permeability μ and the induction.

θ also decreases as the induction is decreased, the rate of decrease becoming more and more rapid as the induction becomes lower (fig. 4). For each frequency, θ seems to approach a definite value when the induction is very low. Also an increase in the frequency is accompanied by an increase in θ.

The limiting values of θ and μ_0 obtained from the curves satisfy fairly well the equations obtained in § 3. The values used have been

θ	5°	$11{\cdot}8^\circ$	$19{\cdot}6^\circ$	32°
μ_r...........	155	150	142	119
ω	53·2	96·4	172·6	349·1

Substituting these values in the equations

$$\mu = \mu_r \sec\theta, \qquad k = \frac{1}{\omega}\tan\theta,$$

the values obtained for μ are

	155·3	153·2	150·7	140·3,
and for k				
	·00165	·00217	·00206	·00179,

which must be considered a fairly satisfactory agreement.

It appears then that the experiments agree fairly well with the assumptions that at very low inductions the permeability tends to a definite limiting value, and that the lag in the phase of the induction at any point behind that of the magnetic intensity at that point is zero.

As regards the third harmonics, b_3 appears to have an approximately constant value for each series, its value decreasing as the frequency is increased, while ψ_3 increases as the induction decreases, tending to a value of about 60° in all cases for very low inductions.

In fig. 3 (lower curves) is shown the relation between the total iron loss and the induction, and the curves here show the marked effect of the eddy currents in increasing the loss of energy in such a rod as that at present being discussed. It will be noticed in Table V. that the calculated eddy-current loss is greater than the total iron loss. This eddy-current loss was calculated from a formula (§ 4) obtained on the assumptions that μ was constant and that $\frac{d\text{H}}{dt}$ was constant over the section. It appears then that for a thick rod such as the present one these assumptions depart considerably from the truth, and the question arises how far the results calculated on these assumptions for any particular specimen are in error. Owing to the uncertainty of the eddy-current loss, no attempt has been made to recognize kinetic hysteresis in the rod.

In fig. 5, Steinmetz's coefficient σ is plotted against B_{max}. σ is seen to decrease rapidly as the induction is diminished and for very low inductions takes a very low value not very different from zero.

In fig. 6 the hysteresis loops are shown for four of the experiments, the upper ones being the 1st and 7th of Table II. and the lower ones the 1st and 7th of Table VI.,

the frequency for all four being very nearly 28. Each pair shows strikingly the effect of the value of θ on the iron loss. In the upper figure the one curve is for the highest induction ($B_{max} = 563$), where the value of θ was $14^\circ{\cdot}97$; the other (for which each of the scales is 20 times that of the first) is for $B_{max.} = 14{\cdot}9$, the corresponding value of θ being $2{\cdot}60$. In the lower figure the values are $B_{max.} = 440$, $\theta = 43^\circ{\cdot}62$, and $B_{max.} = 12{\cdot}9$, $\theta = 12^\circ{\cdot}40$. The marked effect of the eddy currents is seen by comparing the two curves A (for which the value of $H_{max.}$ is the same), and also the two curves B, with one another.

7. The main results obtained in this paper are as follows, the value of the induction being below 600.

In a specimen in which eddy-currents were very small :—

(1) μ_0 diminishes with B_1, and tends to a finite value for very small values of B_1, which for the specimen was about 157. For a considerable range at the lower inductions μ_0 can be represented closely as a linear function of H_1.

(2) θ diminishes rapidly as B_1 diminishes, and tends to a very small limiting value for very low values of the induction.

(3) b_3 decreases slowly as B_1 diminishes, and approaches a value about ·030 for the weakest fields used.

(4) ψ_3 gradually increases as the induction decreases and tends to a value of about 60° for exceeding low values of B_1.

(5) Change of frequency has practically no influence on μ_0 and θ.

(6) The hysteresis loss was practically independent of the frequency, and so the loss due to kinetic hysteresis was too small to detect with certainty.

(7) Steinmetz's coefficient σ decreases rapidly as the induction is decreased, and has a value not very different from zero for very low values of B_1.

In a specimen in which eddy currents were considerable,

(1) μ_0 diminishes with B_1 and tends to a finite value for very small values of B_1 for any particular frequency. The value for slow speeds was about 156.

(2) For a limited range at low inductions, μ_0 can be represented closely as a linear function of H_1.

(3) θ diminishes as B_1 diminishes, and tends to a definite limiting value at very low values of the induction for each particular frequency.

(4) The limiting values of μ_0 and θ are consistent with the assumption that the induction at any point is in phase with the magnetic intensity at that point.

(5) b_3 is fairly constant for any particular frequency, its value diminishing as the frequency is increased.

(6) ψ_3 increases as the induction decreases, and tends to a limiting value of about 60° when B_1 is zero.

(7) Increase of frequency largely increases the total iron loss, but, owing to the uncertainty as to the eddy-current loss, it could not be said what part of this increased iron loss was due to an increase of hysteresis.

(8) For such a rod, the formula for the eddy-current loss

$$E = \frac{\pi A}{4\rho} \frac{\mathfrak{B}^2}{T}$$

calculated on the assumptions that μ is constant and $\frac{dH}{dt}$ constant over the section gives results which in some cases must be too large.

The work described in this paper has been performed in the Physical Laboratory of the Melbourne University, and I have to record my thanks to Professor Lyle for his kindness in placing the resources of his laboratory, and in particular his wave-tracer, at my disposal, and for his help in preparing this paper for publication.

Discussion.

The Chairman expressed his interest in the paper and said that 12 or 14 years ago he worked at the subject in connexion with choking-coils, not kinetically, but using the ordinary hysteresis curves. Taking a periodic voltage applied to a choking-coil, it was interesting to note that the iron manufactured harmonics whose frequencies were the odd multiples of the frequency of the fundamental. He contrasted this with the case of trains of mechanism where the frequencies manufactured were always octaves or double octaves of the original frequency.

XXVIII. *On the Electric Radiation from Bent Antennæ. By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* *.

[Plates VII. & VIII.]

In March 1906, Mr. Marconi described in a Paper presented to the Royal Society his inventions and interesting investigations with wireless telegraph antennæ consisting partly of vertical and partly of horizontal wires for the purposes of directive electric-wave telegraphy. At the conclusion he says † :—

"I have found the results (*i. e.*, the unsymmetrical radiation and reception) to be well marked for wave-lengths of 150 metres and over, but have not been able to obtain as well defined results when employing much shorter waves, the effect following some law I have not had time to investigate."

At the same meeting the present writer gave a brief explanation of the nature of the electric radiation from such a bent antenna, and subsequently expanded it into a more detailed mathematical theory ‡.

In this last Paper it was analytically proved (as previously discovered experimentally by Mr. Marconi) that a bent antenna radiates unsymmetrically, that is, not equally in all directions but most energetically in its own plane and in a direction opposite to that in which the free ends point. The theory showed that this inequality in fore and aft radiation depends upon the ratio of the lengths of the vertical and horizontal portions of the antenna and upon the ratio of the wave-length employed to the distance between the transmitter and receiver. Hence it appeared clear that with a suitable

* Read November 23, 1906.

† See G. Marconi, "On Methods whereby the Radiation of Electric Waves may be mainly confined to certain Directions, and whereby the receptivity of a Receiver may be restricted to Electric Waves emanating from certain Directions," Proc. Roy. Soc. Lond. A. vol. lxxvii. p. 413, 1906.

‡ See J. A. Fleming, "A Note on the Theory of Directive Antennæ or Unsymmetrical Hertzian Oscillators," Proc. Roy. Soc. A. vol. lxxviii. p. 1 (1906).

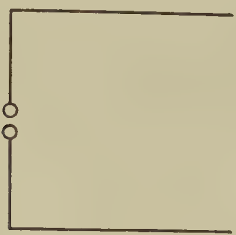
bent antenna the effect should be obtained just as well at comparatively short distances when using relatively small wave-lengths as at long distances with long wave-lengths.

During the past summer numerous experiments and measurements have been made for me by Mr. G. B. Dyke in the quadrangle of University College, London, employing short bent antennæ 10 or 20 feet in length receiving at distances varying from a few yards up to 140 feet or so, with the object of confirming the general theory of the action given by the writer.

It may be convenient in the first place to explain generally and briefly the reasons for the unsymmetrical radiation of such a bent oscillator.

Suppose a pair of rods terminating in spark-balls to be placed with the balls in apposition and the rods in one straight line. It is obvious by reason of symmetry that this radiator must radiate equally in all directions which have the line of the rods for an axis. If, however, the rods each have a bend made in them so as to make an oscillator resembling

Fig. 1.

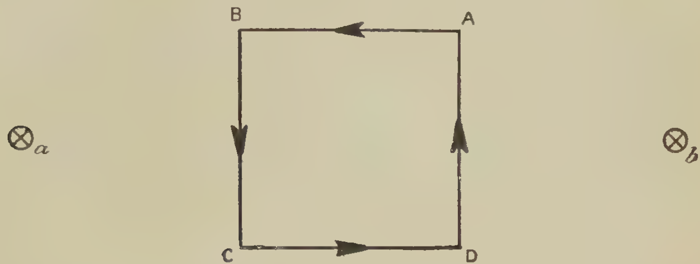


in shape three sides of a rectangle (see fig. 1), then the symmetry of radiation is destroyed.

We may consider the oscillator as constructed in the following manner. Imagine a rectangular circuit ABCD of wire (see fig. 2) placed perpendicularly to the earth's surface, and let it be traversed by a high frequency oscillation. Then if a horizontal line is drawn through the centre of the rectangle and two points b and a chosen at equal distances on the right and left side, the magnetic force at those points will be equal and normal to the plane of the rectangle. If at any

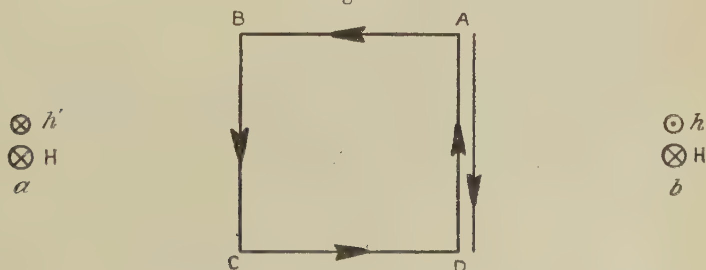
instant the current is flowing counter clockwise round the circuit, the magnetic forces H at these selected points will be directed away from the spectator *. Next suppose a wire to

Fig. 2.



be placed in contiguity to one side of the rectangle and to be traversed by an equal current the same at all points of its length and always in opposite phase to that in the side of the rectangle adjacent to it (see fig. 3).

Fig. 3.

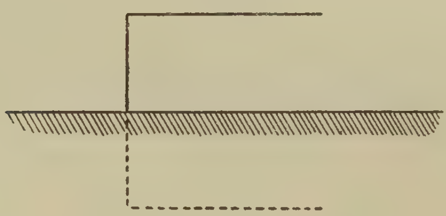


The field due to this open circuit current at the two selected points a and b will be towards the spectator at the right-hand point b and away from it at the left-hand point a. Hence if h and h' are the magnetic forces due to the open circuit current at the points in question, the resultant fields

* It is convenient sometimes to represent an end-on view of a tube or line of electric or magnetic force. We then must indicate in some way the direction of the force whether to or from the spectator. This may be done by representing the section of the tube by a small circle and putting a dot in the circle if the force is towards the reader, and a cross on it if it is away from the reader, as is done to denote current directions in similar diagrams in other cases.

are $H-h$ at the right-hand point and $H+h'$ at the left. But since the open circuit and the adjacent side of the rectangle are traversed at any instant by equal and opposite currents, we may consider them both annihilated, and we have as a consequence that the magnetic field due to a bent oscillator similar to that in fig. 1 at two symmetrical and equidistant points on its axis is greater at that side away from which the free ends point than it is at the other. If we suppose a doubly bent oscillator of the above kind to be half buried in the earth, then we have a singly bent earthed oscillator of the kind used by Mr. Marconi (see fig. 4) *. The antennæ used in

Fig. 4.



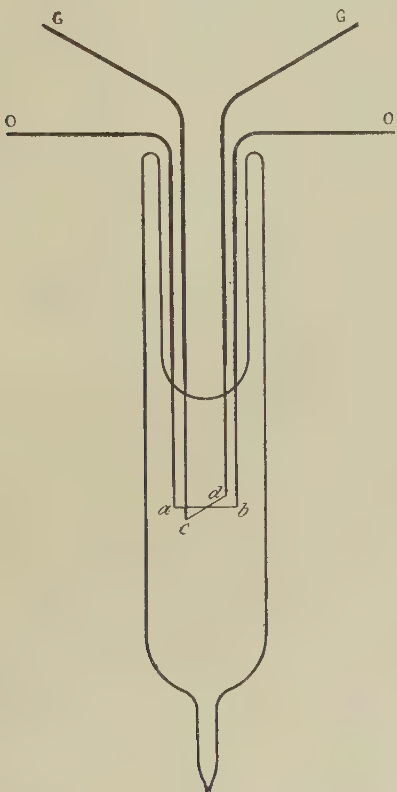
the experiments here described consisted of a couple of No. 16 bare copper wires loosely twisted together and from 10 to 20 feet in length. On the grass of the Courtyard some large sheets of zinc were laid down to form earth-plates. At one of these posts used as a transmitting station, a bent radiating antenna was constructed by connecting one of a pair of spark-balls to the earth-plate and the other to a length of the above said copper wire which was so arranged and upheld by insulators that any fraction of its length could be placed vertical and the remainder horizontal as in fig. 4. The free insulated farther end of the horizontal portion generally carried a plate of zinc 18 inches square which served as an additional capacity. The receiving antenna consisted of a vertical copper wire upheld by insulators from a light bamboo rod.

* In the discussion on Mr. Marconi's paper at the Royal Society on March 22nd, 1906, Prof. J. Larmor, Sec.R.S., pointed out that a bent oscillator as in fig. 4 is equivalent to a magnetic oscillator *plus* an electric oscillator.

The wave detector used was of the thermoelectric type, and made as follows :—

A glass test-tube had four platinum wires sealed through the bottom (see fig. 5). These were soldered to four copper strips spaced apart on the inside of the test-tube. On the external side at the base one pair of platinum wires were connected by a fine constantan wire *a b* about 1 cm. long and

Fig. 5.



Tellurium-Bismuth vacuum thermal detector for electric oscillations. *ab*. Constantan wire. *cd*. Thermojunction. GG. Galvanometer terminals. OO. Antenna and earth terminals.

0·02 mm. in diameter. The other pair of platinum wires were connected with small pieces of bismuth wire and tellurium which were soldered to each other and to the centre

of the constantan wire, thus making a thermoelectric junction *cd* with one junction on the constantan wire. This test-tube was then sealed into another vessel which was highly exhausted, thus providing a tellurium-bismuth thermojunction in vacuum, one junction in contact with a fine resistance wire which could be heated by electric oscillations sent through it (see fig. 5).

The thermojunction was connected by long wires with a sensitive movable-coil galvanometer placed in a room about 40 feet from the receiving antenna. The fine wire circuit was included in the receiving antenna between the base and the earth-plate. This thermal receiver could be calibrated by passing a continuous current through the heating wire, and in the case of the instrument mostly used in these experiments the sensitiveness was such that 1 milliampere passed through the heating wire produced a deflexion of 1 cm. of the spot of light on the galvanometer scale.

Observation showed that the deflexion of the spot of light on the scale was almost exactly proportional to the square of the current through the heating wire. Hence an oscillatory current, the root-mean-square value of which was as small as 300 microamperes, could be read by the spot deflexion. This sensitiveness cannot be called great, but it was sufficient for the purpose, and such resistance-wire combined with a vacuum tellurium-bismuth couple is not only more easily calibrated but less troublesome to use than a bolometer bridge arrangement.

No attempt was made to tune the receiving antenna to the transmitting antenna with any great exactness. The actual experiment consisted in placing the bent transmitting antenna at a certain distance from the fixed vertical receiver antenna and then swivelling round the horizontal part of the transmitter into various directions and taking readings of the current in the receiving antenna in each case. To avoid errors due to variation in the sending antenna current, a comparison measurement was always made in each case between the current in the receiver with the free end of the bent transmitting antenna pointing directly away from the receiver and the current when the transmitting antenna had any other assigned azimuth.

In the first set of experiments the sending antenna consisted of a double copper wire 20 feet in length having at the

free end a terminal capacity plate, the wire being bent over at various heights from the ground or earthed end, so as to make a bent antenna with 1 foot vertical and 19 feet horizontal, 2 feet vertical and 18 feet horizontal, 3 feet vertical and 17 feet horizontal, 4 feet vertical 16 feet horizontal, 5 feet vertical 15 feet horizontal, &c. &c.; and in each case observations were taken of the current in a receiving antenna at the same distance but at various angular positions round the transmitter. The currents produced in the receiving antenna, as calculated from the galvanometer deflexion, were plotted as radii vectores of a polar curve; and these radii were therefore proportional to the magnetic and electric fields of the radiator taken at equal distances round its centre, but in various azimuths. The reduced observations are given in Table I., (p. 416) and from them the polar curves (see Plate VII. figs. 1 to 5) have been plotted. These show the results for five such experiments when the transmitting antenna had its bend made at various heights as indicated in the diagrams.

The azimuths are reckoned in degrees counting from the position in which the free end of the bent transmitting antenna points directly away from the receiving wire. It will be seen from the curves that the greatest fore and aft inequality is obtained with this particular antenna bent so that 2 feet of it is vertical and 18 feet horizontal, and that at the distance chosen, viz., 138 feet, the field in the direction in which the free end points is then about 60 per cent. of that in the opposite direction (see fig. 4, Plate VII.). The polar diagrams moreover confirm Mr. Marconi's observation and also my mathematical theory that the minimum radiation is not in the direction 90° from the maximum but more nearly 105°, to 110°. It will be seen that the effect of making the horizontal part of the antenna longer is not so much to increase the fore and aft inequality as to squeeze in the sides of the polar diagram and make it narrower.

The wave-lengths emitted in each case were measured with a special form of cymometer consisting of a condenser of known capacity and a variable inductance of calculable magnitude. This cymometer circuit was placed near to the horizontal portion of the transmitting antenna and tuned to it until the current in the cymometer circuit was found to be

TABLE I.

Radiation from Bent Earthed Transmitting Antenna 20 feet in total length.
Receiving antenna vertical and 20 feet high.
Distance between receiver and transmitter, 138 feet.

Length in feet of Vertical part of Transmitter ...	5	4	3	2	1
Length in feet of Horizontal part of Transmitter.	15	16	17	18	19
Radiated wave-length in feet...	100	92 (?)	105	106	110
Azimuth of horizontal part of transmitter.	Current in the Receiving Antenna in arbitrary units.				
0°	100	100	100	100	100
15	98	97	94	92	93
30	92	85	96	83	75
45	82	79	79	77	67
60	78	74	70	71	58
75	77	67	59	56	45
90	72	66	57	52	48
105	71	65	57	46	41
120	70	66	62	53	49
135	72	64	60	54	48
150	73	80	58	67	59
165	70	74	56	69	60
180	82	69	64	63	68
Plate VII.......	Fig. 1.	Fig. 2.	Fig. 3.	Fig. 4.	Fig. 5.

a maximum, as determined by a thermoelectric detector. The frequency in the antenna thus being known from the constants of the cymometer circuit, the radiated wave-length could be calculated. On account of the capacity plate at the end of the antenna, the emitted wave had a length rather greater than 5 times the total length of the antenna, the exception being the 4 : 16 case in which the wave-length found by several measurements was 92 feet.

It is clear then that when using electric waves of about 100 feet in length and receiving at a distance of 138 feet, or not much greater than one and one-third of a wave-length, the inequality in radiation in different azimuths is well marked,

and is greatest for a certain ratio of horizontal to vertical portions of the antenna. This fore and aft inequality moreover, as well as the form of the polar curve, varies with the distance of the receiving antenna as shown by the following experiment. A bent transmitting antenna 10 feet long was used of which 3 feet of the length was vertical and 7 feet horizontal. The receiver was placed successively at 138 feet and 78 feet distance and the polar curves taken. The values of the current in the receiving antenna in different azimuths were as shown in Table II. It will be seen that at the shorter distance the inequality in fore and aft radiation is less evident. These values are plotted in the polar curves in figs. 7 and 8 (Plate VII.).

Table II.

Bent transmitting antenna 3 feet vertical, 7 feet horizontal. Straight receiving antenna (vertical) 10 feet high.

Azimuth of horizontal part of Antenna.	Current in Receiving Antenna in arbitrary units.	
	At 78 feet distance.	At 138 feet distance.
0°	100	100
15	91	94
30	96	87
45	91	76
60	81	77
75	83	67
90	81	65
105	74	60
120	72	52
135	70	56
150	71	57
165	71	43
180	64	50

In comparing these results with the theory given we must notice that the theory presupposes that the current in the sending antenna or circuit is everywhere the same. The mathematical investigation of the problem becomes immensely difficult unless we make this assumption.

On the other hand, in the actual antennæ the current varies from a maximum at the earthed end up to zero at the

free end. The effect of this is to make the actual magnetic moment of the bent oscillator less than would be the case on the assumption of a uniform current in it.

Suppose, however, we consider a small closed rectangular circuit to be traversed by an oscillatory current of maximum value I. Through the centre of the rectangle draw a line parallel to one side and produce it both ways. Take points a and b at equal distances r from the centre both on the right and left-hand sides of the rectangle. Let H be the maximum magnetic force at a and b perpendicular to the plane of the rectangle, and E the electric force in the plane of the rectangle and perpendicular to the bisecting line. Then it can be shown that

$$\mathrm{H}=\frac{\mathrm{I}\delta y\delta z}{r^3}\sqrt{m^4r^4-m^2r^2+1},$$

$$\mathrm{E}=\frac{\mathrm{I}\delta y\delta z}{r^3}\sqrt{m^4r^4+m^2r^2},$$

where δy, δz are the sides of the rectangle and $m=2\pi/\lambda$, where λ is the wave-length of the radiation.

These formulæ can be arrived at on the principles explained in the writer's Paper to the Royal Society ("A Note on the Theory of Directive Antennæ or Unsymmetrical Hertzian Oscillators," Proc. Roy. Soc. Lond., ser. A, vol. lxxviii. p. 1).

If δz be the length of the side perpendicular to the bisecting line on which r is measured, and if we consider the oscillation in this one side only, then on the same principles it can be shown that the maximum magnetic (h) and electric (e) forces at distances r from the oscillation taken in the same directions as H and E are given by

$$h=\frac{\mathrm{I}\delta z}{r^2}\sqrt{m^2r^2+1},$$

$$e=\frac{\mathrm{I}\delta z}{r^2}\sqrt{m^2r^2-1+\frac{1}{m^2r^2}}.$$

At distances such that mr is large compared with unity, all these forces vary inversely as the distance, and $\mathrm{E}=\mathrm{H}$ and $e=h$.

Also $$\mathrm{H}/h=m\delta y=2\pi\delta y/\lambda.$$

Hence when using the bent earthed oscillator as in the experiments here described, it is clear that the field at a distance r from the centre, not small compared with the wave-length, taken in the direction in which the free end of the oscillator points, is to the field in the opposite direction at an equal distance in the ratio of $h-\mathrm{H}$ to $h+\mathrm{H}$, that is in the ratio of $\lambda-2\pi\delta y$ to $\lambda+2\pi\delta y$.

Under the assumption here made as to the equality of the current at all points of the oscillator, the quantity δy is the length of the horizontal part of the oscillator. In comparing the theory with the practical results, we must notice that the value to be assigned to δy will always be very much less than the actual length of the horizontal part, because the actual magnetic moment of the bent oscillator is always much less than the product of the lengths of its vertical and horizontal portions and the maximum current at the earthed end. A consideration of all the circumstances shows that the δy in the formula may, as a first approximation, be taken to be half the length of the horizontal portion of the antenna, whilst the wave-length in the present case was about five times the total length. Accordingly, in the case of the antenna 10 feet long bent over so that 7 feet were horizontal, the theory would predict that the ratio of the two magnetic forces H and h should be in the ratio of $2\pi \times 3\frac{1}{2}$ to 5×10 or 22 to 50, and the ratio of the fore and aft fields or radiation in the ratio of 40 to 100. Experiment shows this to be approximately the case.

In a further set of experiments a number of measurements were made by erecting a vertical antenna having a lateral wire terminated in a capacity plate, attached to some point on its height (see figs. 9 to 13, Pl. VII.). In all cases such a transmitting antenna was found to radiate more equally in all azimuths than if the vertical part above the lateral projection was removed. Hence the polar curves are made more nearly circular by adding this vertical part. This showed that strengthening the electric moment of such a bent antenna makes it a less unsymmetrical radiator, and that what is required to improve the simple bent antenna and make it radiate more unequally in a fore and aft direction is an increase in the magnetic moment.

To investigate this point a little more, the whole of the observations taken with the 20-foot antenna bent over at various heights were plotted to the same scale in a family of polar diagrams together with the results obtained from the same antenna in a vertical position (see Plate VIII.). Thus if the 20-foot antenna is placed in a vertical position it radiates equally in all azimuths, and the polar curve obtained by plotting its field at a constant distance is a circle. If the antenna is bent over, say, 5 feet from the top so that 15 feet are vertical and 5 feet horizontal, the radiation is still nearly equal in all directions, but the field is not so great as when the antenna is wholly vertical. Hence the polar curve is smaller than the above circle and not quite circular. If it is bent over at the height of 10 feet, then the polar curve is smaller still, also decidedly non-circular and has little depressions in it. If bent over at 15 feet so that 5 feet are vertical, the polar curve is still less circular. Hence as the antenna is bent over so that less and less of it is vertical and more and more horizontal, the radiation in various azimuths becomes more and more unequal, and has in an increased degree a decidedly minimum value at about 105° reckoning 180° in the direction in which the free top end points. The family of polar curves so obtained and depicted in Plate VIII. show very well the gradual diminution in the current in the 20-foot receiving antenna as the 20-foot bent-over transmitting antenna is swivelled round, and they show the gradual increase in the asymmetry of radiation in various azimuths, for constant distance between radiator and receiver.

The curves here given, however, never reach the well defined figure-of-8 form of those given by Mr. Marconi, since he employed a transmitting antenna the vertical part of which was 1·5 metres and the horizontal part 60 metres long. In other words, his ratio of horizontal to vertical was as much as 40 : 1, and the greatest ratio in my experiments was only 19 : 1. This entirely confirms Mr. Marconi's remark, *loc. cit.* that "in order that the effects should be well marked it is necessary that the length of the horizontal conductors should be great in proportion to their height above the ground." This is only another way of saying that the magnetic moment of the oscillator must not be small

compared with the electric moment. This is achieved by making the length of the horizontal portion of the antenna large compared with that of the vertical portion.

One point of great interest in the family of polar curves above mentioned is that the minimum radius always lies near to 105°, reckoned from a zero opposite to the direction in which the free end of the antenna points. In the mathematical theory given by the writer (Proc. Roy. Soc. vol. lxxviii. p. 7, 1906) it is shown that for a doubly bent insulated antenna azimuthal angle (θ) of minimum force is given by the expression

$$\cos\theta = \frac{2\phi v}{\mathrm{M}} \cdot \frac{1}{mr},$$

where ϕ is the electric moment of the oscillator, M the magnetic moment, v the velocity of radiation, $m=2\pi/\lambda$, and r is the distance between radiator and receiver.

Now $\quad \phi=\mathrm{Q}\delta z \quad$ and $\quad \mathrm{M}=\mathrm{I}\delta y\,\delta z=\mathrm{Q}n\,\delta y\,\delta z,$

where Q is the maximum end charge of the oscillator and $n=2\pi/\mathrm{T}$.

But $v=n/m$. Hence we have

$$2\frac{\phi v}{\mathrm{M}}=\frac{2}{m\delta y},$$

and
$$\cos\theta=\frac{2}{m^2 r\,\delta y}=\frac{\lambda^2}{2\pi^2 r\,\delta y}=\frac{1}{20}\frac{\lambda}{r}\cdot\frac{\lambda}{\delta y}.$$

Mr. Macdonald has shown (see 'Electric Waves') that in the case of a linear oscillator the emitted wave-length is 2·5 times the total length. Hence for our case

$$\lambda=2{\cdot}5(2\delta y+\delta z)=7{\cdot}5\delta y.$$

Therefore we have for the bent oscillator with three equal branches

$$\cos\theta=\frac{3}{8}\frac{\lambda}{r}=0{\cdot}375\,\lambda/r.$$

The experiments here detailed were made at distances equal to about one and one-third of a wave-length between the radiator and receiver. Hence for this case $\lambda/r=0{\cdot}72$, and therefore $\cos\theta=0{\cdot}27$ or $\theta=74°\ 20'$ or $\theta=105°\ 40'$, according

to the way of reckoning the angle. It is not a little remarkable that in the case both of Mr. Marconi's measurements and mine, made with bent oscillators having the ratio of the horizontal to the vertical parts varying from 40 : 1 to 3, the angle of minimum radiation always lies near to 110°, as shown by fig. 2 in Mr. Marconi's paper (*loc. cit.*) and the diagrams in Plate VIII. in this present paper.

It is also very interesting that the actual results obtained from bent antennæ of such different proportions as those mentioned should agree so well with the theory evolved for a doubly bent antenna with three equal branches.

One other interesting point remains to be considered. The experiments above mentioned were all made with antennæ bent at a right angle. The question arises, what would be the effect of inclining the upper part of the bent antennæ at various angles? A series of experiments was accordingly made with a radiating antenna 20 feet in total length, the vertical part of which was kept always at 5 feet and the remainder tilted at various angles from 20° above the horizontal to 20° below the horizontal. The receiving wire was 21 feet vertical and at a distance of 138 feet. The current in the receiving wire as read by the thermoelectric receiver was taken for various inclinations of the upper part of the sending antenna, both with the free end pointing away from the receiving wire and with the free end pointing towards it. The ratio of these two currents is given in the following Table III. and plotted out as ordinates in terms of the inclination angle as abscissæ in fig. 6.

The figures of Table III. when plotted out into a curve (see fig. 6) show a gradual increase in ordinate as the free end of the antenna is elevated above the ground, but no decided minimum value of the ratio of the fore and aft radiation. One very striking difference was, however, found between a transmitting antenna with the upper part horizontal and one with the upper part placed in a down-sloping position when the complete polar diagram was taken in the two cases. The following experiment shows this difference. An antenna was formed with the 20-foot wire and capacity-plate above described, and was fixed to a wooden frame so that 5 feet of it was vertical and contained the spark-gap, and 15 feet was

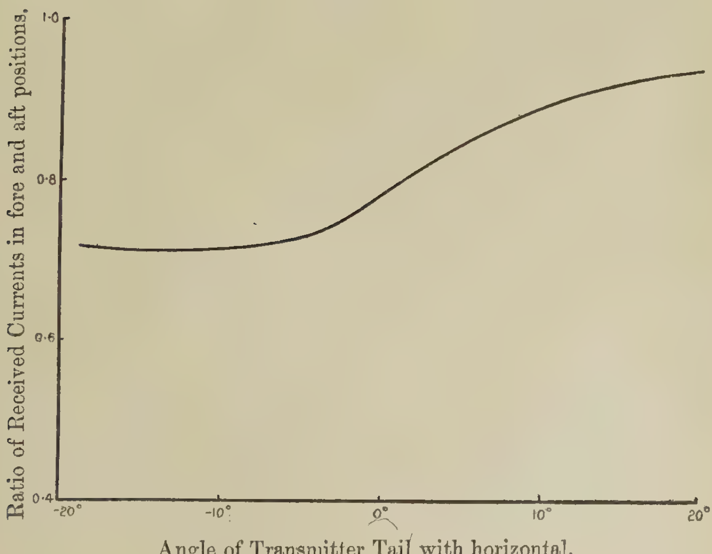
Table III.

Observations made with bent Transmitting Antenna 20 feet in total length, 5 feet vertical, and 15 feet inclined at various angles to the horizontal line.

Receiving wire vertical and 21 feet high at 138 feet distance.

Height of free end of sending wire above ground in feet.	Angle of inclination of upper part of sending wire to horizontal.	Ratio of currents in receiving wire with free end of sending wire towards and away from receiver.
0·5	−17·5°	0·72
1	−15·5	0·71
2	−11·55	0·72
3	− 7·65	0·71
4	− 3·85	0·73
5	0	0·77
6	+ 3·85	0·84
7	+ 7·65	0·87
8	+11·55	0·94
9	+15·5	0·92
10	+19·45	0·91

Fig. 6.



bent in a down-slanting direction so that the free end was only 2 feet from the ground. This was then swivelled round into various azimuths, and the current in a vertical receiving wire 21 feet high and at 138 feet distance was measured and plotted as a polar diagram (see fig. 14, Plate VII.). When this polar diagram is compared with the polar diagram (fig. 1, Plate VII.) for the antenna of same length but having its upper part 15 feet long kept in a horizontal position whilst being swivelled round, a very considerable difference is seen in their form.

The polar diagram (fig. 14, Plate VII.) for the antenna with down-sloping upper part has two deep indentations in it. The ratio of the radiation in the fore and aft directions in the two cases is not very different, that in the direction of the free end being 72 to 76 per cent. of that in the opposite, but whereas the diagram for the horizontal antenna is a smooth oval curve, the other is of a distorted figure-of-8 form.

Moreover, the absolute value of the current in the receiving wire in the two cases for the maximum readings at 0° are in the ratio of about 100 : 70, so that the maximum value of the current in the receiver is reduced by bending down the free end so that it lies below the horizontal.

The explanation of the distortion is not difficult. By bending down the free end of the antenna we increase the magnetic moment and reduce the electric moment of the antenna. The polar curve for an antenna with magnetic but no electric moment, viz., for a closed circuit, is a figure-of-8-shaped curve. The polar diagram for an antenna with electric moment but no magnetic moment, viz., for a vertical wire, is a circle.

In proportion as we lower the free end of the antenna by bending down the upper part into a down-sloping position we make the magnetic moment of that antenna more pronounced in its effect on the polar curve.

We do not alter much the ratio of the fore and aft radiation in the plane of the oscillator, but we cut it down immensely in two regions lying between 30° and 85° reckoned from the direction in which the free end points.

These differences in the polar curves are perfectly accounted

for on the theory given by the writer (see Proc. Roy. Soc. Lond. vol. lxxviii. A. p. 1, 1906). It has there been shown that for the bent oscillator, as used in the above experiments, the magnetic force H parallel to the earth's surface, which cuts across the vertical receiving wire, at the distance r, is given by the expression

$$H = \frac{1}{r^3}\sqrt{(\phi v m^2 r^2)^2 + \left(\phi v m r - \frac{M}{2} m^2 r^2 \cos\theta\right)^2},$$

where ϕ is the electric moment and M the magnetic moment of the transmitting antenna. It is obvious that if $\phi=0$, then H varies as $\cos\theta$, assuming a constant value for the distance r, and since the current in the receiving antenna varies as H, it follows that the plotting of the current for various azimuths of the sending antenna gives us a polar cosine curve consisting of two circles with perimeters in contact at the origin. If, however, $M=0$, then H is constant for constant values of r and the polar curve is a circle with centre at the origin. For various intermediate ratios of M to ϕ the polar curve takes some irregular figure-of-8-shape as depicted.

The similarly shaped polar curves which Mr. Marconi has given (*loc. cit.*) for the current in the receiving antenna when a *bent receiving antenna* is swivelled round its earthed end are, however, to be explained in a slightly different manner. If a receiving antenna is employed which is partly vertical and partly horizontal, then, when acted upon by a similar or even a vertical transmitting antenna, there are three sources of electromotive force in the receiving wire :—

1st. That due to the action on the vertical part of the receiving wire of the electric force of the incident wave which is perpendicular to the earth's surface.

2nd. That due to the cutting of the vertical part of the receiving wire by the magnetic force of the incident wave which is parallel to the earth's surface.

3rd. That due to the magnetic-force lines passing under the horizontal part of the receiving wire being alternately reversed in direction, aided by the wave-length being roughly equal to 4 times the length of that wire.

The total E.M.F. is the vector sum of these three separate

E.M.F.s. The numerical values of (1) and (2) are equal, and (1) is proportional to the minimum radius of the polar curve in the direction 105° to 110°.

It is not difficult to show that when the receiving antenna has its free end pointing anywhere in the semicircle which lies nearest to the sending station, the E.M.F. (3) is opposed in direction to E.M.F. (2), and for a certain azimuth these nullify each other. Hence the minimum radius of the polar curve is proportional to E.M.F. (1). Again, since (1) and (2) differ in phase by 90°, it follows that the receiver current, when the free end of the receiving wires bears 90° from the shortest line connecting the stations, is proportional to the vector sum of E.M.F. (1) and E.M.F. (2).

On the other hand, when the free end of the receiving wire points away from the sending station, the current in it is proportional to the vector sum of E.M.F. (1) and the scalar sums of (2) and (3).

An analysis of Mr. Marconi's numerous polar curves in the light of this explanation shows that we have E.M.F. (1) =E.M.F. (2), as it should be, and that in the case of the various bent receiving antennæ used by him E.M.F. (3) is from one to four times E.M.F. (2).

In conclusion I wish to mention my obligations to my Assistant, Mr. G. B. Dyke, B.Sc., for his willing and energetic as well as intelligent assistance in the above described work. The measurements not only required accurate observations and care to avoid various sources of error, but necessitated a large amount of active exercise in going to and fro between the stations mostly taken in very hot weather.

XXIX. *Electrical Resistance of Alloys.*
By R. S. Willows, *M.A.*, *D.Sc.**

The experiments of Fleming and Dewar † and others have shown clearly the wide differences between the electrical properties of alloys and of the pure metals composing them. The well-known investigations of Matthiessen show that the resistance of alloys containing two of the metals lead, tin, cadmium, or zinc, and no other substance, can be calculated from the resistances of their constituents, when the proportions present of each metal are known. The specific resistance of all other alloys is greater than would be given by a calculation based on the assumption that the components conduct proportionally to the volume of each present. This gives rise to the curious fact that the addition of pure silver (a good conductor) to gold (a worse conductor) decreases the conductivity of the latter. Fleming and Dewar have also shown that the resistance of pure metals decreases very greatly as the absolute zero is approached, while alloys still retain a great part of their resistance.

Lord Rayleigh ‡ has advanced a theory, intended to account for these differences, which is based on the thermo-electric properties of a mixture of two metals. Liebenow §, in various publications in Germany, but at a later date, has advanced a theory which, physically, is identical with that of Rayleigh. (I might mention that in German publications the theory is always referred to as Liebenow's ; Lord Rayleigh's contribution seems to have escaped notice.)

The theory is as follows :—When electricity flows from one metal to another there is an absorption or development of heat at the junction—the Peltier effect. The temperature disturbance thus created increases until the conduction of heat through the metals balances the Peltier effect at the junctions, and it sets up a back electromotive force. The

* Read November 23, 1906.
† Phil. Mag. [5] vol. xxxvi. p. 271 (1893).
‡ Scientific Papers, vol. iv. p. 232.
§ *Encyklopädie der Elektrochemie*, Band 10.

difference of temperature at the alternate junctions is proportional to the current, so is also the back E.M.F. called into play. But a reverse E.M.F. proportional to current is indistinguishable experimentally from a resistance, so that an alloy should on these grounds possess a spurious resistance, differing in nature from that of a pure metal. Rayleigh's calculation shows that the false resistance, R, per unit length is given by

$$\mathrm{R}=273e^2/(\kappa/p+\kappa'/p');$$

where e=thermo-electric force of a couple for 1° difference of temperature between the junctions; κ and κ' are the heat conductivities of the metals in ergs; p and p' are the proportions by volume in which the two metals are taken. The temperature is supposed to be near 0° C.

It will be noticed that n, the number of couples per unit length, does not enter into the above expression. The number co-operating is indeed increased by finer subdivision, but the efficiency of each is decreased owing to the readier conduction of heat between the junctions. An alloy of equal volumes of copper and iron should have a false resistance amounting to 1·5 per cent. of that of copper, as is readily shown by substitution of the appropriate numbers in the above formula.

It may be noted in passing that this expression shows that it is possible always to choose the proportions of the metals present so that the resulting alloy shall have a maximum resistance. For $p+p'=1$,

$$\therefore \qquad \mathrm{R}=\frac{273e^2}{\dfrac{\kappa}{p}+\dfrac{\kappa'}{1-p}},$$

and for a maximum or minimum $d\mathrm{R}/dp=0$;

whence $$\kappa/p^2-\kappa'/(1-p)^2=0;$$

or $$p=\frac{1}{1+\sqrt{\dfrac{\kappa'}{\kappa}}}.$$

The positive sign is to be taken with the square root, since

the other sign would make p come outside the limits 0–1, and this value it is readily seen corresponds to a maximum R.

It appeared possible to put this false resistance in evidence in two ways, one direct, the other not so. It was only after working for a considerable time that I became acquainted with Liebenow's work, which is entirely concerned with the indirect method.

This paper is a description of the attempts I have made to separate the true and false resistances directly. Suppose a current runs through an alloy and that it sets up a back E.M.F.: if it is now quickly reversed, this inverse E.M.F. will at first assist its passage, and more will flow in the second direction than in the first, or, what comes to the same thing, the resistance will appear to be less for the quickly reversed current than for the steady direct one. As the temperature of the junctions will be equalized rapidly on account of their small distance apart, the current reversals must be rapid. I therefore used an alternating current.

The alloy to be tested formed one of the arms of a Wheatstone's bridge, the adjacent arm being a simple metal such as copper or lead, generally the latter on account of its greater resistivity. The resistances were first balanced for alternating and then for direct current. The pure metal possesses no spurious resistance, and hence the apparent resistance of the alloy should decrease when the alternating current is used.

The figure (p. 430) shows the arrangement of apparatus.

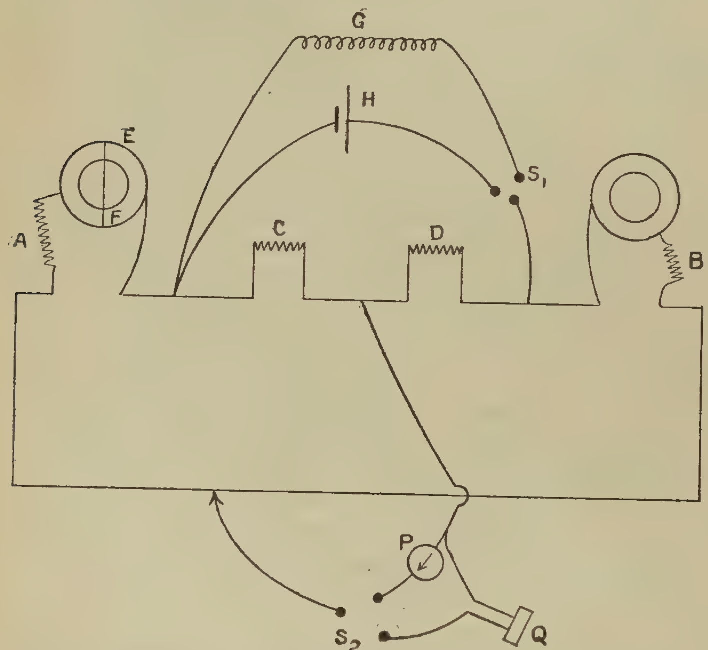
A is the resistance-coil of alloy; B the copper or lead resistance made from specially purified material obtained from Johnson and Matthey; C, D, are two coils of manganin or eureka wire wound on the same bobbin; G is the secondary of a small induction-coil; H, a Leclanché cell; P, a galvanometer; Q, a telephone specially wound for bridge work, or a vibration galvanometer; S_1, S_2, mercury-paraffin switches by means of which the source of current or the detector could be changed.

There is, in addition to the false resistance of the alloy, the skin effect, arising from the concentration of current in the outer layers, to be taken into account. If R is the true resistance of a wire of length l to steady currents, then the

resistance to currents of moderate frequency and of simple periodic form is

$$R' = R\left(1 + \frac{1}{12}\frac{\mu^2 p^2 l^2}{R^2}\right),$$

where μ is the permeability and $p = 2\pi$ (frequency). The wave-form of the current used was not known, but the second term in the bracket was made small by using wires of small diameter. Were any skin effect present, only the difference between that in lead and in the alloy would be given by the experiment.



For the same frequency the value of this term depends on l/R, and this is greatest for the pure metal, and hence if $\mu = 1$, the skin effect, when it first becomes appreciable, would make the alloy appear to decrease in resistance when referred to copper or lead as standard.

All the coils in the bridge were made of wire 0·33 mm. in diameter, formed after the manner given by Chaperon, by first winding a definite number of turns in the right-handed

direction and then an equal number the opposite way. The resistance of each was 3 ohms approximately; the induction in the arms by this method was rendered very small, but was still large enough, with these low resistances, to give a flat minimum in the telephone. With an ordinary set of coils out of a resistance-box it was impossible to obtain a balance. In order to render the minimum sharper the method of Rayleigh was adopted *. E is a wooden ring of 4 inches diameter, F one of 2 inches; the smaller can be rotated round a common diameter of the two. Each ring carries three turns of copper wire, 0·33 mm. diameter, joined in series, and the whole arrangement is in series with A. By rotating the coil F, the induction in the arm could be varied within narrow limits. A similar pair of coils is in series with B. By this means the balance point on the wire of the bridge with alternating current could be found readily to 1 mm., generally much nearer, and as the resistance of 1 mm. of the bridge wire was 0·0006 ohm, this corresponds to a minimum accuracy of about 0·02 per cent.

Various current interrupters were used. In the preliminary experiments a secohmmeter and the galvanometer were tried, but soon abandoned, because thermo-currents were troublesome, and also because I desired to get a greater range of frequency. Finally a vibrating wire with mercury contact was used in the low-frequency experiments. This broke the primary circuit of a small induction-coil, the secondary of which was joined to the bridge, and it possessed the advantage that it could readily be tuned to unison with the vibration-galvanometer used as detector. For most of the other observations a wheel interrupter was used. This was made by letting into the circumference of a disk of beech-wood, 1 foot in diameter, about 120 pieces of brass whose width along the circumference was about $\frac{1}{10}$ inch, the whole being carefully turned. Pressing on the circumference were two springs which completed the circuit through the brass pieces as the wheel revolved when driven by a motor. A small condenser was connected to the springs to prevent sparking.

The bridge was balanced first with alternating current by shifting the movable contact and adjusting the coils E, F

* Phil. Mag. Dec. 1886.

until the inductances as well as the resistances of the arms were equal; then with direct current. The coils A and B were then generally interchanged and the observations repeated. Mr. F. G. Bratt helped me considerably by taking readings alternately with me.

The alloys used were eureka, brass, platinoid, German silver, platinum-iridium, and platinum-silver. The frequency of the current varied between 10 and 980 per second.

No certain differences could be detected between the resistance of an alloy to direct and alternating current at temperatures 20° and 100° C.

As already mentioned, Fleming and Dewar found the resistance of a pure metal very small at low temperatures, while that of alloys was not greatly changed. It seemed possible, therefore, as the temperature was reduced, that the true resistance of an alloy might decrease and the false resistance become relatively great. Experiments similar to those described above were therefore made with alloys at the temperature of solid CO_2 and of liquid air, but in no case could any false resistance be found.

Other experiments with a 20-ohm coil of German silver balanced against one of lead gave similarly negative results. Whether this is due to the frequency being too small to overtake the equalisation of temperature of the junctions, future experiments, which I hope to carry out, will perhaps determine.

The experiments of Hagen and Rubens * on the reflecting and emissive powers of metals for infra-red rays do not help us. They show that the electrical conductivity can be found by measuring the reflecting or emissive powers to within a few per cent. Alloys do not occupy an exceptional position. The experiments do not put in evidence the magnet properties of iron and nickel.

In any case indirect evidence in favour of Rayleigh's theory is not wanting. Liebenow gives the results of his calculation in a form involving two constants, which he calls the inner molecular heat conductivity of the metals. By assigning proper values to these the formula expresses well, in some cases such as gold and silver excellently, the relation

* *Annal. d. Physik*, vol. ii. p. 873 (1903).

between resistance and composition when applied to the measurements of Matthiessen, Feussner, Haas, and others. They are not, however, altogether convincing. The heat conductivities to be ascribed to the different molecules seem to bear no relation to the conductivities as usually measured in the mass. Thus, taking the heat-conduction of copper as unity, silver requires to be taken as 2·3 and gold as 2, while copper-zinc has the value 3 ascribed to it.

I might also point out that a frequent constituent of the high resistance alloys in common use is nickel, a metal for which the Peltier effect and other thermo-electric properties are strongly pronounced.

In conclusion I have to thank the Government Grant Committee of the Royal Society for help in defraying the cost of this research.

Cass Institute,
London, E.C.

Discussion.

Prof. J. A. Fleming expressed his interest in the paper and said Lord Rayleigh's theory was ingenious, but it seemed to him that it presupposed a coarseness of structure in the alloy. An alloy was probably a substance of the nature of a solution with a fine structure, and it was difficult to see how thermoelectric E.M.F.s could be brought into play without discontinuity of material. Considering an electric current as a passage of electrons, Prof. Fleming pointed out that some of the electrons would pass through the spaces between the molecules in the material and others would impinge against the molecules themselves or the molecular aggregates. The latter would set up perturbations which would produce heat and corresponding electric resistance. If an alloy were supposed to consist of molecular aggregates while a pure metal consisted of molecules, more electrons would pass through material in the latter case than in the former. This would produce a greater quantity of heat, and correspond to a higher resistance in the case of the alloy.

Mr. A. Campbell said that shortly after Lord Rayleigh advanced his theory he made some experiments to try to detect if alloys showed changes of resistance when tested

with alternating and direct currents. The alloy selected was ferro-nickel, as it was expected that this would show the effect strongly, as the constituents give a strong thermo-electric E.M.F. A current was sent through a long ferro-nickel wire doubled back on itself, and this current was measured by a Kelvin balance, while the voltage on the ends of the wire was measured by a Kelvin electrostatic voltmeter (reading to 1 in 1000). No difference could be detected between the apparent resistance with direct current and with alternating current of frequencies up to 80 per second (the usual correction for contact difference being found and applied).

Mr. W. DUDDELL suggested that the Author should proceed with his experiments using very much higher frequencies.

XXX. *Auroral and Sun-spot Frequencies Contrasted.* *By* C. CHREE, *Sc.D., LL.D., F.R.S.**

§ 1. DURING several recent investigations I have had occasion to contrast the annual variation in years of many and in years of few sun-spots of elements such as the diurnal range of the magnetic declination, or the frequency of occurrence of magnetic storms. The formula first advanced by Wolf

$$R = a + bS \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as connecting R, the range in the mean diurnal inequality of declination throughout the year, with S the corresponding sun-spot frequency—a and b being constants—can be applied with considerable accuracy to the range in individual months of the year, and to magnetic inclination, horizontal force and vertical force, as well as declination. But taking any one element, a and b are different for the different months of the year, and b/a is in general decidedly larger for winter than for summer.

Suppose, now, that dashed letters refer to a winter, undashed to a summer month, and that suffixes 1 and 2 relate

* Read November 23, 1906.

to two years in which sun-spots are respectively many and few. Then for the ratio of the ranges in the summer and the winter month concerned, we have in the year of many sun-spots,

$$R_1/R_1' = (a/a')\left(1+\frac{b}{a}S_1\right) \div \left(1+\frac{b'}{a'}S_1\right),$$

supposing for simplicity the sun-spot frequency the same for the two months. On the same hypothesis, we have for the corresponding ratio in the year when sun-spots are few,

$$R_2/R_2' = (a/a')\left(1+\frac{b}{a}S_2\right) \div \left(1+\frac{b'}{a'}S_2\right).$$

From these two equations we at once deduce

$$\frac{(R_2/R_2') - (R_1/R_1')}{R_1/R_1'} = \left(\frac{b'}{a'} - \frac{b}{a}\right)(S_1 - S_2) \div \left\{\left(1+\frac{b}{a}S_1\right)\left(1+\frac{b'}{a'}S_2\right)\right\}.$$

As already mentioned, observation shows that b'/a' exceeds b/a, and by hypothesis $S_1 - S_2$ is positive, thus

$$R_2/R_2' > R_1/R_1'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In temperate latitudes, whether sun-spots be many or few, the diurnal range of any magnetic element is larger in summer than in winter, *i.e.* R_1 exceeds R_1' and R_2 exceeds R_2'. Thus (2) shows that *relatively* considered the diurnal range is more variable throughout the year when sun-spots are few than when they are many. In other words, if the annual change of the diurnal range be illustrated by a curve whose ordinates represent the ratios borne by the ranges in individual months to their arithmetic mean for the twelve months, the maximum and minimum ordinates differ more when the year selected is one of few than when it is one of many sun-spots.

§ 2. Again, taking a list of the more considerable magnetic disturbances recorded at Greenwich from 1848 to 1903, as given by Mr. W. Maunder, I obtained the following figures * for the relative frequency at different seasons of the year, treating separately the fourteen years of largest (S max.)

* Phil. Mag. Sept. 1905, p. 314.

and the fifteen years of smallest (S min.) sun-spot frequency:

Winter. November to February.		Equinox. March, April, Sept. Oct.		Summer. May to August.	
S max.	S min.	S max.	S min.	S max.	S min.
35	28	38	48	27	24

The figures denote percentages of the totals for the whole year. The average *absolute* numbers of storms per annum were

Whole period.	S max. years.	S min. years.
13·0	18·4	8·5

With increased sun-spot frequency, the absolute number of storms increased in all three seasons of the year; but relatively considered the increase was least in the equinoctial months – the season when magnetic storms are most numerous at Greenwich—and the tendency obviously was towards a more uniform distribution throughout the year. The phenomenon is thus analogous to that described above in the case of the regular diurnal range.

§ 3. In temperate latitudes, as is well known, magnetic storms of any considerable intensity are usually associated with auroras. It was thus of interest to determine whether auroral frequency showed phenomena corresponding to those just described in Terrestrial Magnetism. It has long been known that auroral frequency, as observed in temperate latitudes, varies in a general way with sun-spot frequency. The results obtained, however, have not shown a very exact correspondence between the years of maximum and minimum in the two classes of phenomena. In the case of sun-spot frequency—except as regards data for recent years—practically the only source available has been the data published by Wolf and Wolfer, which extend back to 1749. It is improbable that the unit in Wolfer's latest table * represents an absolutely unchanging value throughout the whole period, but great care has been taken to make the table as homogeneous as possible, and the epochs it gives for the occurrence of sun-spot maximum and minimum are presumably, in at least the great majority of cases, very approximately correct.

* *Met. Zeit.* 1902, vol. xix. p. 195.

Auroral data are exposed to many more uncertainties. The observed frequency varies enormously at different parts of the Earth, and the number of auroras recorded in any specified area is largely dependent on the provision made for observing and recording them. Conspicuous auroras are unlikely to escape notice in populous countries where they are rare occurrences, but in Arctic latitudes where auroras are common many doubtless fail to be recorded. With the increase of population and the development of means of communication characteristic of the last 100 years, there has no doubt been a tendency to an increase in the proportion of auroras which come to be recorded.

Thus auroral frequency is a quantity which is certainly not expressed in terms of an invariable unit; and the various tables which have been published show irregularities due to temporary and local causes, whose disturbing influence it is practically impossible to assess. In the following investigations the methods adopted aim at reducing to a minimum the effect of the various uncertainties.

§ 4. The sun-spot frequencies made use of are derived exclusively from Wolfer's table, which gives data for each individual month during the 153 years 1749 to 1901. The auroral frequencies are from two sources, viz.: "Catalog der in Norwegen bis Juni 1878 beobachteten Nordlichter zusammengestellt von Sophus Tromholt, herausgegeben von J. Fr. Schroeter" (Kristiania, 1902), and Joseph Lovering's "On the Periodicity of the Aurora Borealis" (Mem. American Academy, New Series, vol. x. 1868).

Of the several tables in the former work, that employed is Table E, pp. 414–417, which gives auroral frequencies derived from the whole of Scandinavia from July 1761 to June 1878. In this table Schroeter has combined Tromholt's results for Norway with those of Rubenson for Sweden. In the original the yearly totals are for years commencing in July. In order, however, to obtain results more strictly comparable with Wolfer's mean annual sun-spot frequencies, I have calculated from the Scandinavian monthly totals data for years commencing in January. In addition to yearly totals—from July to June—for the whole of Scandinavia, Tromholt's Table E gives yearly totals for five subdivisions

of the country, numbered I. to V. according to latitude. I. includes all districts north of 68°·5, II. extends from 68°·5 to 65°, III. from 65° to 61°·5, IV. from 61°·5 to 58°·5, while V. includes the extreme south of Scandinavia from 58°·5 to 55°.

For a considerable time subsequent to 1761, observations from district I. were very few, a fact due probably more to lack of observers than anything else. This possesses some importance for the following reason. The annual variation in auroral frequency is largely dependent on the fact that aurora is seldom vivid enough to be visible until the sun is several degrees below the horizon. In high latitudes there is daylight throughout the whole 24 hours near midsummer, and no daylight near midwinter, and the auroral frequency in these regions, as was pointed out many years ago by Lovering, has a single maximum near midwinter, and a single minimum answering to a total absence of aurora near midsummer *. This state of matters is at least approached in district I., and to a lesser extent in district II. (*cf.* loc. cit. Table G, p. 420). Further south in Scandinavia the annual variation is similar to that in England, showing two maxima near the equinoxes, a principal minimum at midsummer, and a secondary minimum at midwinter. The mean annual variation deduced for the whole of Scandinavia will clearly depend to some extent on how far the several districts contribute to the general result. Assuming an increasing relative contribution from district I., if annual variations be calculated from two different periods, one may not unreasonably expect the later period to show the equinoctial maxima less prominently and the midsummer minimum more prominently than the earlier period. This is one of my reasons for contrasting one period of a special type with two of an opposite type, the one earlier the other later.

Lovering gives annual variations for a number of separate stations. Most of these, however, are based on too few years' observations to suit the present enquiry.

Of the data for separate stations or districts, I propose to use only those for New York State (l.c. p. 181). These extend over 26 years, including 1205 separate observations,

* *Cf.* A. Angot's 'Les Aurores Polaires,' Paris, 1895, p. 125.

and represent a latitude much lower than that of Scandinavia. The other data employed are from Lovering's General Catalogue, pp. 195–200. This comes down to 1864, and extends to earlier than the 14th century. Though enumerating nearly 10,000 auroras it is probably, judging by Tromholt's figures, very far from complete. The data are doubtless of a very heterogeneous character, and the same precautions appeared necessary as in the case of the Scandinavian data.

Recently Prof. Schuster * has gone very fully into the existence of periodic variations in Wolfer's sun-spot frequencies, and has concluded that there is evidence of the existence not merely of the ordinarily recognized period of 11·125 years, but of others which like it are submultiples of a period whose most probable value is 33·375 years. This is one of the reasons why I have dealt with three successive 33-year periods, viz. 1761–1793, 1794–1826, and 1827–1859.

§ 5. Before comparing sun-spots and auroras, I would draw attention to some features of Table I., which gives mean sun-spot frequencies for the several months of the year, as calculated by me from Wolfer's table for a number of specified combinations of years. It has been remarked by Mr. Ellis † —who based his remarks on Wolfer's data *for groups of 3 months* (February to April, &c.)—that sun-spot frequency shows no real annual period. The results in the first line of Table I. cannot be said to be decisive against the existence of a small annual term, though inconsistent with the existence of a large term having this period. The extent of the difference between the mean results for the several months is more easily realized in Table II., which expresses the monthly values in Table I. as percentages of the arithmetic mean for the 12 months. One would, I think, hardly have anticipated, in means based on 153 years' data, the difference of $6\frac{1}{2}$ per cent. shown between the values for January and May. On the other hand, if there were a true annual period, one would expect the percentage figures in Table II. for the three

* Proc. Roy. Soc. A. vol. lxxvii. p. 145.

† Monthly Notices R. A. S. vol. lx. p. 142.

TABLE I.—Wolf and Wolfer's Sun-spot Frequencies.

		Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Year.
153 years	1749 to 1901......	44·41	46·67	45·40	46·13	47·41	46·02	45·46	45·83	45·91	46 52	46·36	46·23	46·04
33 ,, ...	1761 to 1793......	63·94	64·58	57·93	63·80	70·92	65·70	63·79	62·56	68·21	66·97	71·11	67·46	65·59
,, ,, ...	1794 to 1826......	20·32	21·67	20·76	19·57	18·67	20·21	21·19	19·68	17·73	20·99	19·43	23·13	20·28
,, ,, ...	1827 to 1859......	55 53	56·95	56·95	56·28	54·76	53·09	50·94	55·67	57·38	60·96	55·43	58·86	56·07
39 years S max.	1760 to 1894......	81·76	85·78	84·13	89·24	89·66	89·21	88·35	89 03	88·04	90·93	90·23	90·08	88·05
,, ,, S min.	1754 to 1890......	13·63	14·36	14·44	12·09	13·06	11·34	9·87	10·26	9·63	12·45	12·95	12·12	12·19
33 years S max.	1760 to 1872......	83 76	88·16	88·36	90·52	92·98	92·21	90·75	91·79	92·57	94·84	94·44	94·42	91·25
,, ,, S min.	1754 to 1867......	14·67	16·01	15·87	13·29	14·11	12·21	10·52	10·65	9·55	13·64	13·72	13·38	13·14

TABLE II.—Sun-spot Frequencies. Percentages of mean year.

		Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Winter.	Summer.
153 years	1749 to 1901......	96·5	101·4	98·6	100·2	103·0	100·0	98·8	99·6	99·7	101·1	100·7	100·4	99·8	100·2
33 ,, ...	1761 to 1793......	97·5	98·5	88·3	97·3	108·1	100·2	97·3	95·4	104·0	102·1	108·4	102·9	99·6	100·4
,, ,, ...	1794 to 1826......	100·2	106·8	102·4	96·5	92·1	99·7	104·5	97·0	87·4	103·5	95·8	114·0	103·8	96·2
,, ,, ...	1827 to 1859......	99·0	101·6	101·6	100·4	97·7	94·7	90·9	99·3	102·3	108·7	98·9	105·0	102·5	97·5
39 years S max.	1760 to 1894......	92·9	97·4	95·6	101·4	101·8	101·3	100·3	101·1	100·0	103·3	102·5	102·3	99·0	101·0
,, ,, S min.	1754 to 1890......	111·9	117·9	118·5	99·2	107·2	93·1	81·0	84·2	79·0	102·2	106·3	99·5	109·4	90·6
33 years S max.	1760 to 1872......	91·8	96·6	96·8	99·2	101·9	101·1	99·4	100·6	101·5	103·9	103·5	103·5	99·4	100·6
,, ,, S min.	1754 to 1867......	111·7	121·9	120·8	101·2	107·4	93·0	80·1	81·1	72·7	103 8	104·5	101·9	110·7	89·3

successive 33-years periods to resemble one another more closely than they actually do.

The monthly means in these three periods show very considerable differences ; in the earliest period, for instance, the difference between the March and November means amounts to 20 per cent. of the mean value for the epoch.

The central period 1794–1826 comprises that period of 33 consecutive years which gives a minimum mean sun-spot frequency. That mean is only a third of the mean appropriate to the combined periods 1761–93 and 1827–59.

As a rule, three successive years of conspicuously high frequency occur at each sun-spot maximum, and three of conspicuously low frequency at each minimum. Wolfer's table includes 13 groups of these 3 extreme years of sun-spot maximum, and 13 of sun-spot minimum. The respective means from these 39 years of maximum and 39 years of minimum form the fifth and sixth rows of Tables I. and II. The last two lines in these Tables refer to shorter groups of 33 years of sun-spot maximum and 33 of minimum which correspond more closely to the auroral data presently to be described.

§ 6. Probably the simplest way of investigating the relationship between sun-spot frequency and the magnitude of any element is to form two mean values for the element, the one corresponding to years of many, the other to years of few sun-spots, and then to assume that the difference between these means depends on the corresponding difference in sun-spot frequency. In applying this method to data from the whole period covered by Wolfer's table, one would employ the sun-spot frequencies given in the 5th and 6th rows of Table I. Doing so, we should get differences of sun-spot frequency varying from 81·76–13·63, or 68·13, in January, to 78·77 in August. The existence of so large a difference between the sun-spot data for January and August is immaterial, provided there is a direct connexion, which possesses no lag, between sun-spot frequency and the element concerned. If, however, the connexion is of a less simple character, for instance if the element depends on the sun-spot frequency for some months previously, the application of the above method would lead to an overestimate of the influence of

TABLE III.—Auroral Frequency. Annual Variation.

		Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
Scandinavian,	1761–1877	12·40	11·78	13·18	7·82	0·93	0·04	0·26	4·17	11·87	13·52	12·23	11·80	9243	79·0
	1761–1793	10·59	10·55	12·62	9·71	1·99	0·08	0·60	6·13	14·37	13·02	10·51	9·83	2512	76·1
,,	1827–1859	13·29	11·42	12·07	7·54	0·43	0·00	0·18	3·66	11·67	14·26	12·68	12·78	2784	84·4
,,	Two periods combined. ...	11·94	10·99	12·34	8·63	1·21	0·04	0·39	4·90	13·02	13·64	11·60	11·30	5296	80·2
,,	1794–1826	13·96	14·49	14·65	4·52	0·31	0·00	0·08	1·92	9·12	13·42	14·49	13·03	1304	39·5
Lovering's General Catalogue.	Up to 1864	9·54	9·73	11·96	9·37	5·09	3·22	3·72	6·45	10·39	11·37	9·87	9·29	9766	...
,,	1761–1793	7·81	8·14	11·44	11·03	8·05	4·67	5·49	7·19	10·24	10·33	8·47	7·15	2421	73·4
,,	1827–1859	10·61	9·31	10·44	8·46	4·95	3·72	4·62	6·13	9·79	10·54	10·18	11·26	3900	118·2
,,	Two periods combined. ...	9·21	8·73	10·92	9 75	6·50	4·20	5·05	6·66	10·01	10·44	9·33	9·21	6321	95·8
,,	1794–1826	10·73	12·44	14·48	11·93	4·26	1·36	1·02	3·58	7·84	9·03	10·90	12·44	587	17·8

sun-spot frequency in January and an underestimate in August. If we use more exact methods, *e. g.* the method of least squares,—still assuming purely synchronous variation—the above source of uncertainty is less easily recognized, but it exists all the same. Considered absolutely, the difference between the mean monthly values in Table I. is greater for the 39 or the 33 years of S max. than for the corresponding group of S min. years; but relatively considered—*cf.* Table II.—the variability from month to month is greater for the S min. group. Thus in the 39 years of S min. the means for March and September differ by 39 per cent. of the mean from the 12 months. Moreover in the S min. groups of years there is a decided difference—some 20 per cent.—between the means derived from the 6 winter and the 6 summer months. There is a smaller difference, but in the same direction, between the winter and summer means from the 33-year period 1794–1826 remarkable for its low average sun-spot frequency. This is unquestionably somewhat suggestive of an appreciable real annual period in sun-spot frequency in years when sun-spots are few; but whereas in the 39 or 33 years of S min. the mean frequency is large for May and November, and small for July, in the period 1794–1826 the exact opposite is seen. Again, an appreciable excess in the winter over the summer mean also appears in the period 1827–59, when the average sun-spot frequency considerably exceeded the average from the whole 153 years.

§ 7. Proceeding to Table III., we have in the first line the annual variation of auroral frequency in Scandinavia as derived from 117 years. The monthly values represent percentages of the value for the whole year. The largest values occur in October and March, and a secondary minimum is recognizable in December. The dip in the February value arises really from the smaller number of days in that month. When referred to an equal number of days, the February frequency exceeds that in January in the ratio of 104 : 100.

In the second and third lines we have similarly annual variations from the two 33-year periods of high average sun-spot frequency, their mean appearing in the fourth line.

For the ratio, however, between the mean frequencies from the earlier and from the later of these two periods we have

From Sun-spots.	From Auroras.
100 : 85	100 : 111.

If we contrasted these two periods with one another we should thus associate an increase in auroras with a diminution in sun-spots.

Again, the annual variations from the two periods differ markedly. In the later period, as compared to the earlier, the summer frequencies fall and the winter frequencies rise.

The differences apparent between the two periods may represent a real change, but in all probability they are largely due to an increase in auroral observers, especially in the northern districts of Scandinavia. Taking the yearly data for the five districts mentioned above, we obtain for the total number of auroras observed the results given in Table IV.

TABLE IV.—Auroral Observations in Scandinavia.

	District.					All Scandinavia.
July to June.	I.	II.	III.	IV.	V.	
1761 to 1794	22	293	970	1453	555	2481
1794 to 1827	52	694	408	518	56	1316
1827 to 1860	827	873	352	1531	473	2811
1st & 3rd periods combined.	849	1166	1322	2984	1028	5292

An aurora is often seen in more than one district, but comparatively seldom in all. Obviously the increase in the auroral frequency in the latest as compared to the earliest of the three periods arises from the large increase of observations in the two most northern districts; and this being so, the tendency naturally is to bring the annual frequency nearer to the Arctic type with a single maximum at midwinter.

Coming now to the intermediate period in Table IV., we see that the development of district I. was mainly subsequent to 1827. Thus what we should *a priori* expect to observe in

the annual variation from this period in Table III. would be a form intermediate between those from the other 33-year periods, but approaching most closely that from the earlier period. What we actually do find is a variation differing from that of 1761–93 in the expected direction, but to an even *greater* extent than does the variation from the period 1827–59. Comparing the results from 1794–1826 in Table III. with the means from the preceding and succeeding 33-year periods we largely eliminate the influence of the change in observational conditions. The figures show that in the 33-year period characterized by few sun-spots, summer occurrences of aurora were relatively much fewer and winter occurrences more numerous than in the adjacent 33-year periods of high average sun-spot frequency. We thus have low sun-spot frequency associated with an exaggeration in the annual variation of auroras, the precise phenomenon already described in connexion with Terrestrial Magnetism.

§ 8. The mean sun-spot frequency for the two 33-year periods 1761–93 and 1827–59 combined was 60·83, whilst the corresponding auroral frequency for the year was 80·2.

Comparing these with the corresponding figures for 1794–1826, we have for a trebling of sun-spot frequency only a doubling of auroral frequency. This would suggest that if auroral frequency is connected with that of sun-spots by a formula of type (1), then the constant a does not vanish; *i. e.*, a total absence of sun-spots would not be accompanied by a total absence of auroras. A difficulty, however, arises here, which the figures in Table IV. for the several districts of Scandinavia will serve to explain.

The ratio borne to the frequency of auroras in the average year of the period 1794–1827 by the corresponding frequency for the combined periods 1761–94 and 1827–60 is roughly 2 : 1 for the whole of Scandinavia; but is 9 : 1 for district V., 3 : 1 for district IV., 3 : 2 for district III., and less than 1 : 1 for district II. Any comparison for district I. would be misleading.

These figures suggest that with decrease in sun-spot frequency the diminution in auroral frequency is enormously greater in the south than in the north of Scandinavia.

Several authorities have called attention to analogous phenomena, and the theory has even been advanced that the difference in auroral frequency in years of many and few sun-spots really arises from the alternate expansion and contraction of Fritz's isochasms * (curves of equal auroral frequency). The theory does not seem to be strongly supported, but there seems little if any doubt that a substantial difference really exists between the long period changes of auroral frequency in different regions. Results from several stations in Greenland—whose substantial accuracy seems accepted by Prof. A. Paulsen, one of the leading authorities on the subject—appear to indicate that auroral frequency is there very considerably *less* when sun-spots are many than when they are few.

§ 9. We now pass to the consideration of the figures from Lovering's general catalogue in the last five lines of Table III. As before, the monthly figures are expressed as percentages of the yearly total. The figures corresponding to the whole period covered by the table are given by Lovering † himself. Though derived from very heterogeneous data, they represent very fairly the type of annual variation which is characteristic of lower temperate latitudes. The midsummer minimum and the equinoctial maxima are less pronounced than they are even in the South of England.

Comparing the totals from the two periods 1761–93 and 1827–59, we see that the excess from the later period is even greater than it was in the case of Scandinavia. The annual variations from the two periods also differ, and in the same direction as before. From May to September the relative frequency is decidedly less, and at midwinter considerably greater, for the later period than for the earlier, and the equinoctial maxima are but indistinctly shown in the later period. As in the case of Scandinavia, northern stations may have contributed more to the means for 1827–59 than to those for 1761–93. It is clear, however, from the substantial frequencies in the summer months, that even in the period

* *Cf.* Angot, 'Les Aurores Polaires,' p. 138.

† *L. c.* pp. 200 & 216. The total for December on p. 200 should apparently be 907, and not 1007 as printed.

1827–59 data from temperate latitudes must largely have prevailed. Thus the closer approach to the Arctic or single maximum type is difficult to wholly explain, unless we admit that it is partly a real phenomenon, representing a real difference in the distribution of auroras throughout the average year of the two periods 1761–93 and 1827–59. As regards the intermediate period 1794–1826, we see that its mean annual variation differed from that of the neighbouring periods in the same direction as it did in the case of Scandinavia. The fall in the summer frequencies is very pronounced, even as compared to the period 1827–59 alone. Of the equinoctial maxima, that in March is much enhanced, but that in October seems to have vanished, unless it is represented by the maximum now shown in December.

The average year from the two periods 1761–93 and 1827–59 combined shows fully five times the auroral frequency of the average year of the period 1794–1826. This is intermediate between the ratios deduced for the Scandinavian districts IV. and V.

§ 10. Table V. shows the annual variation of auroral frequency in several pairs of groups of years characterized respectively by many and by few sun-spots. The first line gives results for the whole of Scandinavia from 30 years, made up of the 3 years of largest sun-spot frequency from each of the ten 11-year cycles covered by Tromholt's table; the second line gives the results for the corresponding 30 years of few sun-spots. Any *gradual* change in the nature of the observations should affect the two sets of results nearly equally. The third and fourth lines give corresponding data calculated from Lovering's general catalogue, also for 30 years of many and 30 years of few sun-spots.

Of the ten 11-year cycles employed in the two cases nine are identical; the Lovering data cover one cycle prior to the common nine, the Scandinavian one later. This was unfortunately necessary owing to the dates when the Lovering table ended and the complete Scandinavian table commenced. The two sets of data combined represent eleven consecutive 11-year cycles, and the sun-spot data for the 33 years of many and the 33 years of few sun-spots from these eleven cycles

TABLE V.—Auroral Annual Variation in years of many and few Sun-spots.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
Scandinavian.														
30 years S max. 1769–1872 ...	11·81	11·08	12·58	7·98	0·67	0·10	0·27	4·57	12·01	13·44	12·91	12·58	2997	99·9
,, S min. 1764–1867 ...	12·47	12·26	14·05	7·59	1·03	0·00	0·16	3·85	12·04	13·72	12·69	10·14	1844	61·5
Lovering's General Catalogue.														
30 years S max. 1760–1861 ...	8·44	8·56	9·80	9·84	5·68	4·48	4·84	7·48	10·08	11·12	9·92	9·76	2500	83·3
,, S min. 1754–1856 ...	10·86	10·13	12·92	10·92	5·33	3·34	3·82	4·97	9·76	8·79	9·95	9·22	1649	55·0
New York State.														
9 years S max. 1828–1849 ...	6·60	7·45	7·66	11·28	5·53	7·87	8·51	10·21	11·49	9·79	7·66	5·96	470	52·2
6 ,, S min. 1832–1844 ...	5·99	6·45	10·14	12·45	7·37	7·37	8·76	6·45	12·45	11·98	5·53	5·07	217	36·2

TABLE VI.—Auroral Annual Variation in years of many and few Auroras.

	Jan.	Feb.	Mar.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.	Total Auroras.	Average per year.
Scandinavian.														
30 years Auroral max. 1768–1872	12·98	10·72	12·78	8·07	0·71	0·09	0·30	3·99	11·52	13·97	12·66	12·21	3358	111·9
,, ,, min. 1765–1866	11·00	10·17	12·67	8·19	1·09	0·00	0·19	4·80	12·99	14·97	13·63	10·30	1563	52·1
New York State.														
9 years Auroral max. 1829–1850	6·35	9·20	10·87	9·87	7·69	7·02	8·36	11·04	9·03	10·20	5·02	5·35	598	66·4
,, ,, min. 1827–1851	5·26	3·86	7·02	11·23	10·18	3·16	6·67	10·88	16·49	12·98	4·91	7·37	285	31·7

are those given in the last two lines of Tables I. and II. For the purposes of the present enquiry, these sun-spot data answer sufficiently closely to either the Scandinavian or the Lovering auroral data.

The last two lines of Table V. give auroral data for New York State, derived from 3×3 years of largest sun-spot frequency and 2×3 years of least sun-spot frequency included between 1828 and 1849.

Each of the three pairs of comparative figures in Table V. shows a decided rise in the spring maximum in the years of few as compared to the years of many sun-spots.

The average monthly frequency from May to August is also less in the years of few sun-spots, very decidedly so in the case of Lovering's data; in the New York data the depression of the midwinter minimum is the more decided. On the whole, the phenomena resemble those already described, *i. e.* the annual variation is accentuated in the years of few sun-spots. The differences, however, between the selected years of many and the selected years of few sun-spots are less conspicuous than those between the 33-year periods; and such differences as exist may not unreasonably be partly ascribed to the differences in the annual variations of the sun-spot figures in the last two lines of Tables I. & II. This explanation cannot, at the same time, go very far, in view of the fact that there is no marked depression in the auroral frequency in September in the groups of 30 years of few sun-spots.

The differences between the average number of auroras in the year from the first two groups of contrasted years in Table V. are much less than the corresponding differences in Table III., notwithstanding that the sun-spot differences are larger in the case of Table V. The ratio of the mean frequency from the period or group of years of many sun-spots to the corresponding frequency from the period or group of years of few sun-spots takes the following approximate values in the several cases (p. 450).

Obviously the conclusions one would draw as to the extent of the influence of sun-spot on auroral frequency would vary immensely according to one's method of attacking the problem.

	From the 33-year periods. Tables I. and III.		From 10×3 years of S max. & S min. Tables I. and V.	
	Sun-spot ratio.	Auroral ratio.	Sun-spot ratio.	Auroral ratio.
Scandinavian data ..	3 : 1	2 : 1	7 : 1	1·6 : 1
Lovering's data......	3 : 1	5·5 : 1	7 : 1	1·5 : 1

§ 11. Table VI. deals with the same Scandinavian and New York data as the previous tables, but instead of selecting groups of 3 years of S max. and S min., it selected groups of 3 successive years of largest and 3 successive years of least auroral frequency. One of the groups for New York State was made up of the last and the two earliest years of the period of observation. The differences between the mean sun-spot frequencies for the contrasted groups of years in Table VI. are really much less than in the corresponding cases in Table V., but the opposite is true of the mean auroral frequencies. For the ratio between the auroral frequencies in the contrasted groups of years we have 21 : 10 in Table VI., both for Scandinavia and New York State, whereas in Table V. the corresponding ratios are respectively only 16 : 10 and 15 : 10.

As to annual variation in Table VI., the group of years of few auroras shows, as compared to that of many auroras, an enhanced maximum at one or both equinoxes, and a lower minimum both in winter and summer.

Lovering's general catalogue is not considered in Table VI. owing to the erratic way in which the auroral frequencies in it vary from year to year. In some of the 11-year cycles years of many and few auroras seem to occur almost promiscuously, and the selection of 3 successive years as representative of either high or low frequency presented difficulties. Even in the comparatively homogeneous Scandinavian data, the same phenomenon occurred to a certain extent.

§ 12. The difference between the results in Tables V. and VI., and the great irregularity in the variations from year

to year of auroral as compared to sun-spot frequency, point to one of two conclusions,—either

(1) Auroral data are so heterogeneous, or so intrinsically defective, that consecutive years' results are affected by large differential errors when treated as measures of the same quantity; or

(2) Auroral frequency depends, and to no small extent, on something more than the contemporaneous value of sun-spot frequency.

As to the question of a possible lag: In the case of Scandinavia the groups of 3 years selected from consideration of sun-spot frequency for use in Table V., represented an earlier epoch than those selected from consideration of auroral frequency for use in Table VI. in 14 cases, the same epoch in 2 cases, and a later epoch in only 4 cases. This is, to say the least, not unfavourable to the view that auroral frequency tends to lag behind sun-spot frequency.

In every case we have found the annual variation in auroral frequency, monthly values denoting percentages of the total number for the year, to be more uniform when sun-spots are numerous than when they are few. The differences, however, between the frequencies in the contrasted years are in some instances not very conspicuous, and may be partly due to chance. Also, supposing it to be a fact that the annual variation in temperate latitudes becomes more accentuated as sun-spots diminish, this may mean one of two things. There may be, as sun-spots decrease, a greater relative diminution in summer than in winter of the physical phenomena which appeal to our eyes as aurora, or there may only be a general diminution in the brightness of auroras throughout the whole year. From what happens during magnetic storms, it can hardly be questioned that the cause —presumably electric discharges in the upper atmosphere— to which auroral phenomena are due is often active when aurora is invisible. It may even conceivably be in continuous operation, though incapable of appealing to the eye, however

favourable the visual conditions, until a certain minimum intensity is reached.

Only exceptionally brilliant auroras have much chance of being seen until the sun is far below the horizon so that a general reduction of intensity might well be more prejudicial to visibility at midsummer than at other seasons.

In conclusion, I should like to draw attention to the utility for investigations such as the present of trustworthy auroral observations taken on a uniform plan, desirably throughout more than one sun-spot cycle, at a considerable number of stations suitably distributed over the earth.

Discussion.

Dr. J. A. Harker asked if in making calculations upon sun-spot frequencies any account was taken of the different sizes of spots.

Dr. Chree said the Astronomer Royal measured the areas of the spots, but Wolf used a "relative number" depending upon the total number of spots observed and upon the number of groups and isolated spots. The two methods of estimating the sun-spot frequency agreed well.

XXXI. *The Strength and Behaviour of Brittle Materials under Combined Stress.* *By* WALTER A. SCOBLE, *A.R.C.Sc., B.Sc., Whitworth Scholar.**

[Plate IX.]

1. *Previous Results.*—In a former paper† the writer has given the results of tests on a ductile material under combined stress, the material being steel in the form of turned bars, subjected to bending and twisting. It was found that at the yield-point, selected as the criterion of strength, both the maximum strain and maximum principal stress varied considerably, but the maximum shear was approximately constant. If we consider the great flow which takes place before fracture with ductile materials, the specimen behaving like a viscous fluid, the above result will be understood. It was pointed out that the results of the tests on ductile materials should not be applied to cases in which brittle metals are used. The small yield before fracture, and the positions of the planes of fracture, provide sufficient evidence to indicate that the behaviour is very different from that of ductile metals. Unfortunately there seems to be little information available relating to this subject.

2. *Separation of Metals into Ductile and Brittle.*—To clearly distinguish between ductile and brittle metals, the former may be defined as those which draw out considerably before fracture, fracture taking place across planes approximating to those of greatest shear. A brittle material shows little or no signs of drawing out before fracture, and usually breaks along the plane of principal stress.

3. *The Theory of Elastic Strength.*—Three views are held as to the condition which determines the limit of strength: (1) That the principal stress has a certain value; (2) The maximum strain must reach a certain amount; (3) The maximum shear is constant. Hitherto no distinction appears to have been made between brittle and ductile materials when considering the theory of elastic strength. Engineers have

* Read January 25, 1907.

† Proc. Phys. Soc. Lond. vol. xx.; also Phil. Mag. Dec. 1906.

accepted one theory to apply to all metals. For ductile specimens the maximum shear has been found to be almost constant at yield, but it is not justifiable to assume that the same holds for brittle materials at the point of fracture.

4. *The Criterion of Strength.*—With ductile materials the point selected as the criterion of strength was the yield-point, but with brittle materials fracture is evidently the best and most convenient. The only other condition available is that at the elastic limit, a point difficult to determine accurately and depending on the history of the specimen. As there is little drawing out there seems to be no important objection to using the point of fracture.

5. *The Method of Loading.*—The most important practical example of combined stresses in a ductile material is that of combined bending and twisting. On this account the tests on steel were made with this kind of loading, the resources of the writer allowing this to be done. Brittle materials would not be subjected in practice to this loading, but it is suitable for determining the conditions which accompany fracture, and it is better for this purpose than that which occurs more frequently. By using the same apparatus the results form a continuation of those obtained by the previous tests, and may be more easily compared with them.

6. *Apparatus and Specimens.*—The apparatus was described in detail in the previous paper, it being sufficient to state here that the bar was constrained at one end against twisting, but was free to take its natural slope under the bending load. At the other end it rested on rollers which formed the support and offered a minimum resistance to twisting. The bending load was applied directly at the centre, and the deflexion measured one inch from this point. The twist was measured at four points along the length of the bar, at three of these by means of mirrors attached to the bar and an arrangement similar to that used with galvanometers, and at the other by a pointer attached to the bar moving over a fixed graduated circle. A wooden pulley fitted over the squared end of the bar, and weights attached to wire ropes supplied the torque. The bars were of cast iron, as uniform as possible, machined all over, being turned to $\frac{3}{4}$ inch diameter and having squared ends. The distance between

Sept. 190

the bending supports was 30 inches, and the diameter of the wooden pulley was 29⅝ inches.

7. *The Tests Grouped.*—The tests may be divided into three groups: the first including those with one kind of loading only; the second those in which a definite fraction of the breaking torque was applied and bending load added to fracture; and in the third a given bending load was first put on and the torque increased until the bar was broken. By plotting the results they checked each other, and tests which did not appear to fit well were repeated. In this way variations due to differences in the material, which are sure to occur even with the most carefully selected specimens, were eliminated so far as possible.

8. *Making the Test.*—During the initial stages the weights added were large, but when the critical load was approached the increments were decreased to one or two pounds. In each case weights were directly applied. After each weight was added the wooden pulley was gently tapped to minimise the constraint due to friction. It may be mentioned that the forces due to friction were measured but found to be negligible, and were therefore neglected when working out the results.

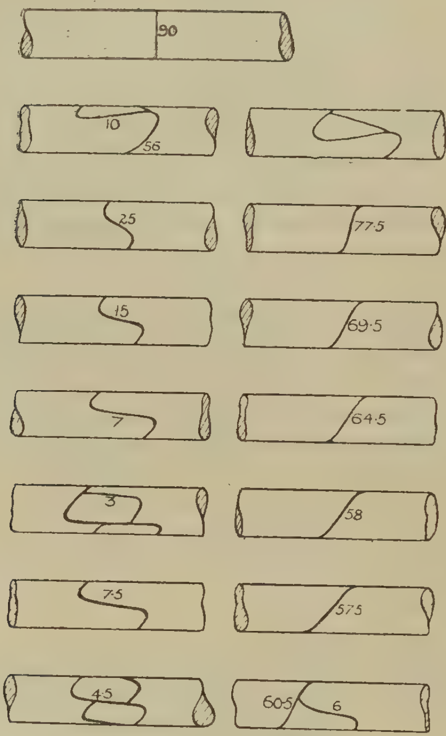
9. *Results of the Tests.*—

Group of Test.	Number of Test.	Bending Moment.	Torque.	Tensile Stress due to Bending.	Shear Stress due to Twisting.	Maximum principal Stress.	Minimum principal Stress.	Maximum Shear.	Remarks.
1.	1	2080	0	50550	0	50550	0	25275	
	2	0	2700	0	32800	32800	−32800	32800	
	10	0	3140	0	38200	38200	−38200	38200	Not broken.
	11	2152	0	52300	0	52300	0	26150	
2.	3	2154	681	52350	8270	53595	−1245	27420	
	4	1957	1363	47550	16550	52695	−5145	28920	
	5	1545	2045	37600	24830	49960	−12360	31160	
3.	6	525	3110	...	...	...	...	...	Not broken.
	6	1125	2760	27330	33570	49860	−22540	36200	
	7	1042	2550	25360	31000	46180	−20820	33500	
	8	1568	2136	38300	25950	51250	−12950	32100	
	9	105	2960	2560	36000	37300	−34700	36000	

The stresses are calculated on the assumption that there was no yield.

10. *The Angles of Fracture.*—The fractures were so interesting that a diagram is given, showing the angles (fig. 1)

Fig. 1.

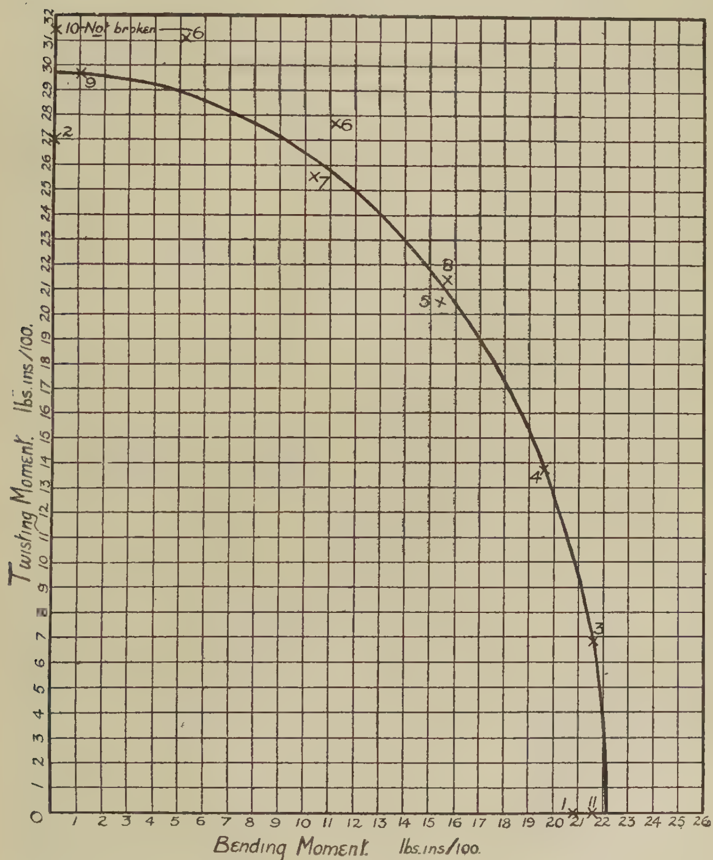


and photographs of the original bars (Pl. IX. fig. 2). The couples applied were left-handed. The part making a small angle with the axis invariably came under the knife-edge. When measuring the angles an attempt was made to determine if in any given bar the slope varied with the position referred to the bending stresses, it being thought possible that it might be different at the bottom of the bar, which was under tension, to that at the neutral axis. No difference could be detected. In certain cases the angle at the bottom was the greater, but in others the smaller. The slight irregularities in the fracture prevent a very accurate measurement over a small length.

11. *Examination of the Results.*—It will be noticed that

the maximum principal stress and maximum shear stress both vary considerably, even after allowing that certain bars were probably exceptionally strong. In fig. 3 the bending and twisting moments are plotted together, showing the relation between these and the differences due to experimental errors and inequalities in the bars. The points lie

Fig. 3.

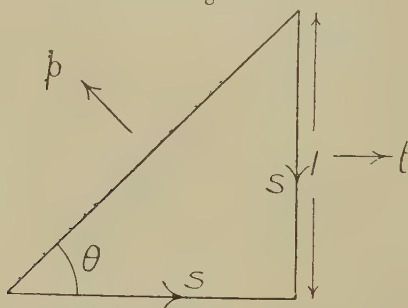


fairly well on an ellipse. A similar result was obtained with steel bars, the conclusion being that the maximum shear stress was approximately constant, but varied slightly on account of differences in the shear strength in different directions. In that case the bending moment was greater

than the twisting moment, 2660 and 2400 lbs. ins. Here the twisting moment is the greater, and the difference is so large that it could not be explained as above without strong supporting evidence. It was therefore necessary to examine the angles of fracture to find if they afforded any clue to the conditions determining fracture.

12. *The Angles of Fracture.*—The main part of the fracture appears at first sight to be along the plane of principal stress. Considering the bottom of the bar, which is in tension, then facing the axis (fig. 4) gives the distribution of stress

Fig. 4.



t being the tensile stress due to bending, s the shear stress due to twisting, and p the maximum principal stress, then

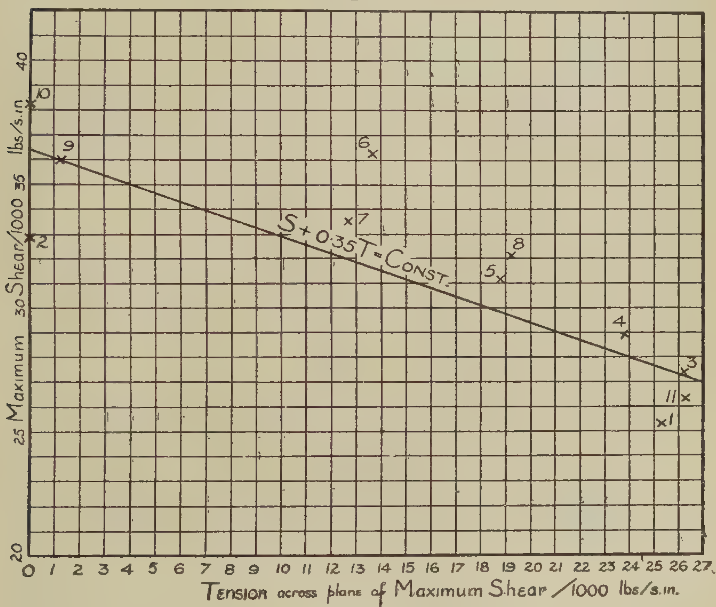
$$t + s \cot\theta = p.$$
$$s = p \cot\theta.$$
$$t/s \cot\theta + \cot^2\theta = 1.$$
$$\cot\theta = -t/2s \pm \sqrt{1 + t^2/4s^2}.$$

Using this to give the angles of the planes of greatest principal stress, the following values were obtained. The measured angles are also given for comparison.

Number of Test.	Measured Angle to Axis.	Calculated Angle to Axis.
	Degrees.	Degrees.
1.............	90	90
2.............	56	45
3.............	77·5	81
4.............	69·5	72·5
5.............	64·5	63·5
6.............	58	56
7.............	57·5	56
8.............	60·5	63

The measured angles of fracture and those calculated for the planes of principal stress agree so well, considering the difficulty of accurate measurement, that it is justifiable to state that the bars fractured along the planes of greatest principal stress. But this quantity is far from constant, Sec. 9. The angles of the part of the fracture roughly parallel to the axis are not so regular. The writer has not been able to identify these inclinations with those of any

Fig. 5.



other important plane. It seems that this part has no special significance, and merely serves to join the ends of the spiral; nevertheless the steady decrease of the angles with increase of torque, seen in tests 3, 4, and 5, leaves the impression that it is not accidental. When cast iron is tested in tension it fractures normal to the axis, and in compression across planes making about 35 degrees with the axis. So in tension and torsion fracture takes place across the plane of greatest stress, and in compression apparently by shearing modified by friction. This seems to afford no help towards arriving at any conclusion as to the conditions holding here. The

value of the coefficient of friction corresponding to the angle of fracture in compression is about 0·35. In fig. 5 (p. 459) the maximum shear and tension across the plane have been plotted, and there is a rough indication that the line $S+0{\cdot}35\,T=\text{constant}$ approximates to the points. There is a decided indication that the maximum shear decreases with an increase of tension across the plane. This could be explained if the straight part of the fracture at the top of the bar corresponded to the plane of maximum shear modified by friction, as then it would appear that this was the seat of the break. But there is no such coincidence of the planes.

13. *Further Examination of the Results.*—Either the maximum stress or greatest elongation being constant would account for the fracture coinciding with the principal plane. The maximum principal stress varied considerably, far too much to say it was even approximately constant. The only quantity which remains is the greatest elongation.

One principal stress is a tension and the other a compression. Representing Poisson's ratio by m, t being the tensile stress due to bending and s the shear stress due to twisting, if the greatest elongation is constant,

$$T/2+\sqrt{S^2+T^2/4}-m(T/2-\sqrt{S^2+T^2/4})=\text{constant},$$

or

$$(1-m)T/2+(1+m)\sqrt{S^2+T^2/4}=\text{constant}.$$

$T/2$ corresponds to T, and $\sqrt{S^2+T^2/4}$ to S of fig. 5. We may therefore write this as

$$(1-m)T+(1+m)S=\text{constant},$$

or

$$S+\frac{1-m}{1+m}T=\text{constant}.$$

The line which fits the points best is

$$S+0{\cdot}45T=\text{constant}.$$

Thus

$$\frac{1-m}{1+m}={\cdot}45; \qquad m=0{\cdot}38.$$

This would be nearly correct for compression, but it is too high for tension. In addition to this it will be seen that such a line is a very poor approximation to the results. If the

maximum and minimum principal stresses are plotted it is found that no such relation holds. The specimens appeared to twist abnormally, the modulus of rigidity being 530,000 lbs./sq. in. There was no relation between the measured twist and deflexion.

14. *The Material assumed to yield.*—So far no explanation has been found to fit the results. The stresses have been calculated on the assumption that Hooke's law was obeyed. No doubt the material yields, but the extent of the yield is doubtful. It has been thought well to tabulate the stresses obtained when it is assumed that the material yields sufficiently to cause the stresses directly due to bending and twisting to be uniform. To convert the tensile stress it is necessary to multiply by $3\pi/16$, and the shear stress by $3/4$. The results are tabulated :—

Test.	Tensile Stress due to Bending.	Shear Stress due to Twisting.	Maximum principal Stress.	Minimum principal Stress.	Maximum Shear.
1 L	29800	0	29800	0	14900
2 L	0	24600	24600	−24600	24600
10 H	0	28650	28650	−28650	28650
11 L	31000	0	31000	0	15500
3 C	30870	6200	32365	−1495	16930
4 C	28040	12400	32730	−4690	18710
5 L	22180	18630	32780	−10600	21690
6 H	16110	25170	34055	−17945	26000
7 L	14950	23230	31900	−16900	24400
8 H	22600	19460	33800	−11200	22500
9 C	1510	27000	27750	−26250	27000

From this it will be seen that the shear stress varies enormously, but the maximum principal stress is nearly constant. The letter against the number of the test indicates whether the load was low, high, or correct, according to fig. 5. As before the units are pounds and inches. The results which are sensibly out of agreement are those in which there was little or no bending. If the principal stresses are plotted against each other no relation can be traced, so there is no explanation of this fact; nor does it appear that the maximum strain was constant. However, keeping in view the low value found for the modulus of

rigidity, this method of checking the maximum strain is open to suspicion. Having new values for the stresses, it is necessary to find if the angles calculated for the principal planes agree with those measured from the fractures. These are given in the table.

Number of Test.	Measured Angle to Axis.	Calculated Angle to Axis.
	Degrees.	Degrees.
1	90	90
2	56	45
3	77·5	79
4	69·5	69·2
5	64·5	60·3
6	58	54·1
7	57·5	54·1
8	60·5	60

These agree quite as well as those given earlier, and a uniform distribution of stress would account for the constant angle of the fracture.

15. *Conclusions.*—There seems to be no explanation of the results unless it is assumed that the material yields sufficiently to allow a uniform distribution of the stresses directly due to bending and twisting. Allowing this, the principal stress is remarkably constant, especially if differences in the material are allowed for, but the values are slightly low in cases in which there was little or no bending load. The bars fractured across the planes of principal stress, the angles being constant around the bar, a result which supports the view as to yield. Cast iron undoubtedly fractures along the plane of maximum principal stress under tension and torsion, but in compression the plane is that across which $S+0{\cdot}35T=\text{constant}$. Apparently the maximum principal stress is constant when it is a tension, but in compression the only possible method of breaking is by shearing. It does not seem possible to condense these two results into one statement.

In conclusion, the writer wishes to thank Lord Blythswood for placing the resources of the laboratory at his disposal for the purpose of the tests.

Discussion.

The Secretary read a letter from Dr. Morrow stating that the author had attempted to find a criterion of rupture by examining what he has referred to as the maximum shear and maximum principal stresses in his specimens. To evaluate these stresses, he uses results of the theory of the bending and torsion of perfectly elastic bodies. The values which he thus arrives at for the stresses are far higher than can exist in the material, and are purely fictitious. If a theory of rupture were formulated by this means it would be valueless. The author meets with more success when he treats the stresses due to bending and the shear stress due to twisting as uniform over the section when fracture occurs. This is a convenient guess at what actually takes place ; but it may be nearer the truth than the assumption that the rules for perfectly elastic material may be applied to ordinary metal specimens when stressed to the breaking point. With this estimate of the stresses, the author finds that for brittle materials rupture is characterized by a limiting value of the principal stress. The writer had shown, in connexion with Mr. Scoble's previous paper, that for ductile materials also, if the stresses due to bending are estimated in a certain way, rupture may be determined by the value of the maximum principal stress. These papers show the great need of a more reasonable method for finding the stresses set up in a metal bar by bending action. More experiments of the kind described in the paper are required.

Mr. L. Bairstow said that the values of the "maximum principal stress" given in the first table in the paper were very high as compared with the ordinary breaking stress in tension, whilst the corresponding values in the later table were much nearer those usually obtained. It was therefore likely that the plastic deformation of the cast iron had a considerable effect on the distribution of the stresses. It would have been useful if the actual tensile strength had been given, for comparison with the principal stress, for the angles can be approximated to by calculation and this suggests that failure occurs in tension. That the planes of

fracture could be calculated approximately was not surprising, as plastic deformation has less effect on this quantity than on the actual values of the stresses. The discrepancies in the values of the maximum stress had been explained by the author on the assumption that the stresses throughout the material had become uniform, but he did not think this assumption could be made, as the outside must always have the greatest stress quite up to the point of fracture. A possible source of explanation depended upon the different distribution of the stresses due to torsion and those due to bending. In torsion the whole circumference of the bar is in the state of greatest stress and therefore takes the first permanent set, whilst in bending the maximum stresses occur at the top and bottom over a much more restricted area. The ratio of the stress due to torsion to the torque producing it is greater than the ratio of the stress due to bending and the bending moment. If, therefore, both stresses are overestimated, those due to bending will have a greater error than those due to torsion. Applying this to the actual experiments, it is seen that the principal stresses, calculated on the assumption of Hooke's law, increase as the bending moment increases or as the torque decreases. It seems probable that the cast-iron bars failed in tension, and there is no reason to suppose that the maximum principal stress is different from the ultimate tensile stress of the material.

Mr. Rollo Appleyard expressed his interest in the paper, and said he appreciated the difficulties of the subject. Some years ago he had made experiments on the hardness of steel, and he exhibited at the meeting several pieces of the fractured material.

The Chairman stated that Mr. Scoble did not favour the idea of internal friction, but the results which he had given in the paper did not contradict that theory.

Dr. Chree, being unable to attend the meeting, wrote to say that what the author described, § 3, as the third theory of elastic strength was apparently that ordinarily known to mathematicians, following Sir G. H. Darwin, as the *maximum stress difference* theory; and in his opinion the stress difference was a more convenient quantity than the shearing stress or

strain on which to concentrate attention. The simplest way of describing the stresses at a point was by means of the stress quadric, and the natural thing was to refer the quadric to its principal axes. In the case of simple traction, the three theories were really identical. Supposing T the greatest permissible simple traction, and E Young's modulus, then in the general case if P_1, P_2, P_3 were the three principal stresses at a point in descending (algebraical) order, T was a superior limit on the greatest stress theory to P_1, and in the stress difference theory to $P_1 - P_3$; while on the remaining theory T/E was a superior limit to the greatest strain. The interrelationships of the three theories were thus very readily seen.

A shearing stress S in two perpendicular planes, both containing say the x axis, gave for the stress quadric $2\,Syz=1$; which when referred to its principal axes became $S(y^2-z^2)=1$, showing at once that $2\,S$ was the maximum stress difference. If the case treated in the author's § 12 were approached from the standpoint of the stress quadric, one obtained at once in the author's notation (in that paragraph) $\tan 2\theta = -2s/t$. This result was equivalent to that given by the author, but much simpler for computation.

He, Dr. Chree, had pointed out many years ago * that none of the three theories mentioned could be universally true. To show this for the stress difference and greatest strain theories, one had only to consider a solid sphere under *uniform* normal forces. Whether these forces were pressures or tensions, the three principal stresses at every point were equal, and hence the stress difference (or shear) zero; if the forces were pressures the strains were all negative. The greatest strain theory thus set no limit to the intensity of the forces when pressures; while the maximum stress difference set no limit whether they were pressures or tensions. It was obvious, however, that the smallness of the strains and the linearity of the relationship between strains and stresses postulated by the ordinary mathematical theory were incompatible with indefinite reduction or increase in the radius of the sphere.

* Phil. Mag. Sept. 1891, pp. 242 &c.

In view of Kelvin & Tait's adoption of m to denote one of the two elastic constants of isotropy, an example followed in many mathematical papers, the use of that letter to denote Poisson's ratio was to be deprecated. The letters in most common use for the purpose were η and σ. The author did not seem to give Poisson's ratio for his material; but his remarks in § 13 seemed to suggest that the values were different for tension and compression. If this were the case, the material did not possess the properties usually assigned to elastic solids. As a matter of fact, several authorities * had described experiments which seemed to show that cast iron did not follow Hooke's law sufficiently closely to be treated mathematically as an elastic solid.

It was difficult to see how any inference could be drawn as to the theories of strength mentioned from the shape of the fractured surface. Even supposing the mathematical equations to hold right up to rupture, it should be remembered that the stresses before rupture varied over the cross section, and that the critical stress, or strain, was first reached at a single point or along a single line. The fracture might be sudden, but would not be strictly instantaneous. On its commencement, at say a definite point on the surface of the bar—where the stresses and strains were greatest,—the instant the material parted company there, a redistribution of stress and strain took place through the still intact material. The commencement of a rift altered the shape of the cross section, and the distribution of stress and strain existing before rupture commenced threw no direct light on what the stresses and strains were at each point on the surface of fracture at the instant when the material in its immediate vicinity parted company. The subject seemed to call for more consideration.

* Todhunter & Pearson's 'History of Elasticity,' vol. i. art. [1411] and elsewhere.

XXXII. *Improvements in the Hüfner Type of Spectrophotometer.* By F. Twyman *.

The following is a description of a Spectrophotometer of the Hüfner type to which an addition has been made which converts it into a Polarimeter, by means of which optical rotations can be measured for light of any wavelength in the visible part of the spectrum. Incidentally an improvement in design is described whereby greater accuracy is attained.

Fig. 1.

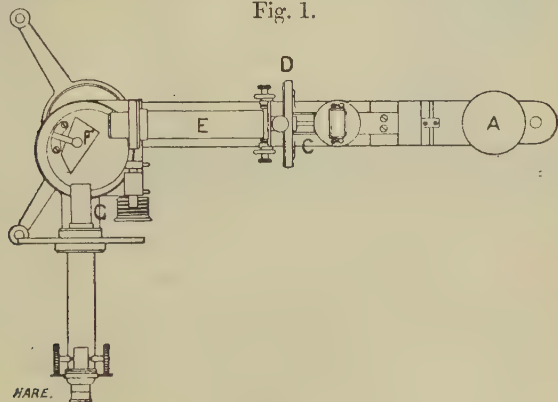
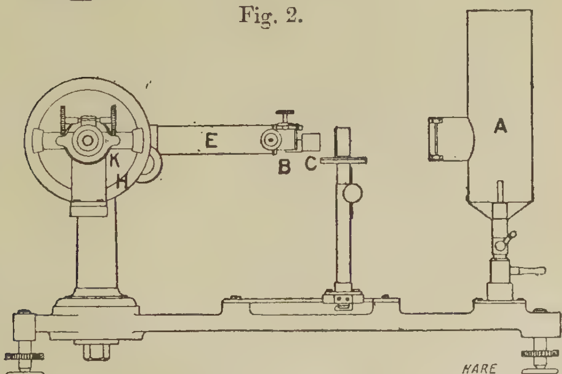


Fig. 2.

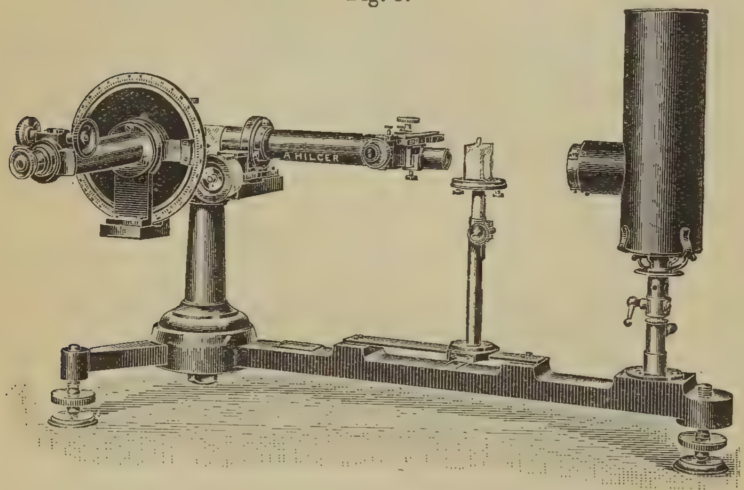


The Spectrophotometer part of the apparatus (shown in plan and elevation in figs. 1 and 2 and in fig 3) was designed and constructed in the workshop of Messrs. Adam Hilger Ltd.,

* Read January 25, 1907.

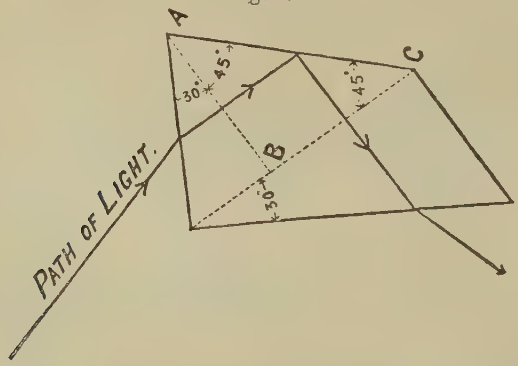
or Dr. C. E. Kenneth Mees in 1904, and embodies many suggestions of his and of Mr. S. E. Sheppard's. The Optical Train is the same as that of the Hüfner Spectrophotometer

Fig. 3.



as made by Albrecht; with the exception that instead of the dispersion prism being the usual prism of 60°, the now fairly well-known constant-deviation prism is employed.

Fig. 4.

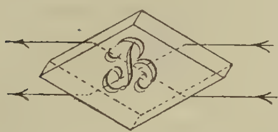


The prism is shown in fig. 4. It may be considered as built up of two 30° prisms and one right-angled prism from the hypotenuse of which light is internally reflected as shown in the figure. The prism is, however, made in one piece. This

construction of prism has the very important mechanical advantage that the telescope and collimator are both fixed, being rigidly attached to a strong cast-iron base. The passing from one part of the spectrum to another is effected by rotation of the prism by a micrometer screw.

The light source, A, is fixed to the cast-iron bed of the instrument. The light passes partly above and partly through a polarizing prism C. The upper beam passes first through the substance whose absorption is to be measured and then through a thin wedge of neutral-tint glass, by the lateral translation of which the two beams may be made equal in intensity. Both beams then pass through the rhomb of glass B (fig. 5) by means of which the two beams are brought into close juxtaposition. Immediately behind this rhomb is the slit of the spectroscope, the edge of the rhomb and the jaws of the slit both being in focus simultaneously when one observes them in the telescope.

Fig. 5.



The light after collimation in the usual way then undergoes dispersion by the constant-deviation prism, and passes on to the second polarizing prism G, and so on to the telescope. The second polarizing prism is mounted on a circle H divided into degrees and reading by a vernier to minutes, and by rotation of this the polarized beam of light can be reduced in intensity in the usual way till the intensity of the top and bottom beams is equal; and the density of the absorbing substance under test is then determined by means of the $\cos^2$ law (the intensity of the light passing through crossed nicols being proportional to the square of the cosine of the angle between their planes of polarization).

The instrument as used as a spectrophotometer will be found very completely described (together with an admirable bibliography on the subject of Spectrophotometry) in a paper by Messrs. C. E. Kenneth Mees and S. E. Sheppard printed in the 'Photographic Journal' for July 1904.

I might, however, again mention here the fact that the wave-length of the light under observation is read off direct on the helical drum of the micrometer-screw, see fig. 1. This is an arrangement which is found of great convenience

in practice. The wave-lengths can be relied on to an average accuracy of well within 10 Angström units.

The present paper is principally concerned

(A) with the mode of correction of a serious error to which polarization spectrophotometers are liable;

(B) with the use of the instrument as a polarimeter.

(A) The error referred to is that due to the polarization of the light on transmission through the dispersing prisms whether it be an ordinary prism of 60° or other angle, or a constant-deviation prism as in the present instrument.

Considering the case where a beam of light, collimated but initially unpolarized, is passed through the prism. Such a beam is the upper beam in the present instrument, which passes over the first nicol. The beam when it emerges is partially polarized, owing to the unequal reflexion of vibrations in and at right angles to the plane of incidence.

(In the case of a beam initially polarized in a plane neither in nor at right angles to the plane of incidence, matters are further complicated by the reflexion within the prism, which causes elliptical polarization; a state of things which has to be avoided by very carefully setting up the first nicol, until it is found that satisfactory extinction of the polarized beam can be obtained by rotation of the second nicol.)

Considering the appearance of the field of view as seen in the eyepiece, in the zero position of the second nicol: the position, namely, in which its plane of polarization is parallel to that of the first nicol. One sees a slit of light wider or narrower according to the width to which one may think it advisable to open the shutters in the eyepiece which limit the field of view. This slit of light is bisected by a fine horizontal line, the edge of the rhomb in front of the slit, which is almost invisible when the intensities above and below are equalized by the adjustment of the wedge of neutral-tint glass. The light-absorbing medium, whose density it is required to measure, is introduced into the top beam, and the second nicol is then rotated to obtain equality of intensity. The lower half varies in intensity according to the $\cos^2$ law; but owing to the polarization due to the

dispersion prism the top beam also varies in intensity, and the absorption if determined simply by the application of the $\cos^2$ law will consequently be incorrect. The amount of the error can be determined by applying Fresnel's equations to any particular case. We will consider the actual state of things in the original instruments made, *i. e.*, a prism of refractive index 1·658 for the light under observation.

Let θ and θ_1 represent respectively the angles of incidence and refraction at the surface of the dispersion prism.

According to Fresnel's equations, if unpolarized light of intensity a^2 fall on a reflecting surface at an angle of incidence θ, and if θ_1 represent the angle of refraction, the intensity of that part of the reflected light which is polarized in the plane of incidence is (see Schuster's 'Theory of Optics,' first edition, p. 49)

$$\tfrac{1}{2}a^2\frac{\sin^2(\theta_1-\theta)}{\sin^2(\theta_1+\theta)}.$$

The intensity of the transmitted light if that light be polarized in the plane of incidence will then be

$$\tfrac{1}{2}a^2\left\{1-\frac{\sin^2(\theta_1-\theta)}{\sin^2(\theta_1+\theta)}\right\}.$$

As in a prism at the angle of minimum deviation the angle of incidence on entering the prism is equal to that of refraction on leaving it, the intensity of the transmitted light if that light be polarized in the plane of incidence, is, after passing right through the prism,

$$\tfrac{1}{2}a^2\left\{1-\frac{\sin^2(\theta_1-\theta)}{\sin^2(\theta_1+\theta)}\right\}^2.$$

For the case under consideration this works out to $\frac{a^2}{2}\times \cdot 651$.

The polarization due to the prism in the plane at right angles to this will be found, when evaluated, to be negligible.

When, then, a spectrum produced by such a prism is observed through a polarizing prism, and the latter is rotated, the beam varies in intensity from 1 to ·651 (for the particular wave-length for which the μ of the prism $= 1\cdot658$).

It will be seen how extremely serious is this polarizing effect of the dispersion prism. Want of leisure has prevented me from reading up fully the bibliography of the subject, but although this error has been previously mentioned and seems fairly well known, there certainly seems to have been insufficient attention brought to bear on the subject, as there are still polarization photometers made which suffer from the full amount of this error.

The Hüfner instrument even in its original form is to a very great extent protected from this failing by the presence of the so-called Hüfner rhomb in front of the slit. It will be seen at once that this produces a partial polarization of the upper beam of light in a plane perpendicular to that of the partial polarization produced by the dispersion prism. By applying Fresnel's equations as in the case of the dispersion prism above, we can find the amount of this partial polarization. The rhombs in the first instrument made were of Jena borosilicate crown of $\mu_D = 1{\cdot}517$, the angle of the front and back edges of the rhomb being 70° (which makes the angle of incidence of the beam upon the face of the rhomb 55°). With these values we find that the intensity of the light transmitted by the rhomb, if that light be polarized after transmission in the plane of incidence, would vary from 1 to 0·732: the partial polarization due to the rhomb in a plane at right angles to this being negligible.

Thus if the beam after passage through both rhomb and dispersion prism be polarized, it will vary from 0·732 to 0·651 as the analyzing nicol is turned.

This variation in the intensity of the top beam produces errors of the following amounts:—

Density of the absorbing medium under test (*i. e.* $\log_{10} \frac{I_1}{I_0}$, I_1 being the intensity of the incident beam and I_0 that of the transmitted beam).	Percentage error in the density, to be added or subtracted according as the first nicol is set with its plane of polarization vertical or horizontal.
0·054	11·5
0·231	9·4
0·602	6·4
1·521	3·3

The errors of reading are considerably less than the errors in the above table, so that it is very important to get rid of the latter. This is done in the more recent instruments by the simple and obvious plan of making the rhomb of the same glass as the prism, and of such angles that the angles of incidence of the beam on its surfaces equal those on the surfaces of the dispersion prism.

The following table will show the effect of this in several actual instances. A piece of neutral-tint glass was retained as a standard, and its density measured on various instruments, some with the old form of rhomb and some with the new. The plane of polarization of the front nicol was sometimes vertical, sometimes horizontal.

Spectrophotometer number.	Density of standard glass at W.L. 5550 as given by the spectrophotometer.	Error due to polarization by the Hüfner rhomb and dispersion prisms. (calculated).	Corrected reading.
1	1·81	+ ·05	1·86
2	1·92	− ·05	1·87
3	1·92	− ·05	1·87
3 (nicol rotated through 90°)	1·84	+ ·05	1·89
3 (new rhomb)	1·88	0	1·88
4	1·83	+ ·05	1·88
5	1·86	0	1·86
6	1·86	0	1·86

It was in the testing of Nos. 2 and 3 that we first became fully aware of the magnitude of the error.

Having calculated the amount on the hypothesis that the error was caused in the manner I have described, the front nicol was rotated through 90°, and an altered reading of the expected amount was obtained. On replacing the Hüfner rhomb by a fresh one of the same glass as the dispersion-prism

and of corresponding angles, a reading was obtained exactly the mean between the two former ones. (See No. 3 of table.)

Several instruments were made before readings on a test density were taken, and instrument No. 4 was one of these sent back for repair. The opportunity was taken of replacing the rhomb, after which it gave the correct reading, although I have no actual record of the final test reading.

One of the great sources of error in polarization photometers is the difficulty of obtaining pure plane-polarized light, and to keep it plane-polarized so that when it arrives at the second nicol it can be entirely extinguished. The extinction of the last two instruments was so good, that by rotation of the second nicol from the zero position to the 90° position, the intensity varied from about 1 to 1/10,000. Imperfect annealing of the optical train and strain by clamping or cementing, or flaws in the first nicol are the chief difficulties met with—they give considerable but not insurmountable difficulties, and are readily detected in the instrument. They all tend to make densities appear too high.

(B) The use of the instrument as a polarimeter was first suggested by me, I believe, to Dr. K. C. Browning in conversation. Some while after that he commissioned Messrs. Hilger to undertake the construction of one of these instruments with the suggested polarimeter addition, which consists merely in the extension of the telescope support sufficiently far to enable one to interpose between the dispersion prism and the second nicol the medium of which one desires to measure the optical rotation.

The readings are taken by putting into the top beam a suitable density. This density is matched by rotation of the second nicol before and after the interposition of the substance under test. The difference of readings gives, of course, the optical rotation for the part of the spectrum under test.

In the instrument exhibited room is provided for the interposition of a 100 mm. tube for liquids, but there is of course no difficulty in extending this to any desired amount. It will be seen that one measures by reducing one half of the field to the intensity of the other half which is already fairly

dark. Thus one would expect the sensitiveness to be somewhat less than that obtained by the use of polarizers, in which one half of the field decreases while the other half increases in intensity. But it is an exceedingly useful addition to the spectrophotometer and produces a polarimeter free from some serious defects of most instruments excepting those with the Lippisch and similar polarizers. Moreover it is, of course, suitable either for use by monochromatic light or for the measurement of optical rotations for different wave-lengths.

I may add that a photographic negative is the best neutral-tint medium with which I am acquainted, and is what is employed when using the instrument as a spectro-polarimeter.

Discussion.

Mr. Milne congratulated the author upon the ingenious method of getting rid of the errors due to the polarization produced by the dispersion-prism. A special piece of apparatus was usually employed for this purpose, but the author had achieved his object without this.

Mr. Selby expressed his interest in the paper, and said that when the light went through the prism before being separated the difficulty did not arise.

Mr. Twyman said he did not think the error was altogether absent when the light traversed the prism first.

Tf 12

XXXIII. *The Magnetic Field and Inductance Coefficients of Circular, Cylindrical, and Helical Currents.* By

ALEXANDER RUSSELL, *M.A., M.I.E.E.**

TABLE OF CONTENTS.

1. *Introduction.*

SERIES formulæ † are usually given for the components of the magnetic force at a point in the neighbourhood of a current flowing in a circular filament. They are generally deduced from the expression for the magnetic potential at the point expressed in a series of spherical harmonics. In many cases the series converge very slowly. It seemed desirable, therefore, to attempt to find formulæ which can be evaluated more readily.

By means of Laplace's formula ‡ for the magnetic force at any point due to an element of current, formulæ for the component forces can be found very easily in terms of

* Read February 8, 1907.

† E. Mascart and J. Joubert, *Leçons sur l'Electricité et le Magnetisme*, vol. ii. § 566 (1897).

‡ A. Russell, 'Alternating Currents,' vol. i. p. 29. This formula is sometimes attributed to Ampère in English text-books.

elliptic integrals, the values of which are given in Mathematical Tables *. The results found are also useful in hydrodynamics in connexion with the theory of the circular vortex filament. The formulæ give directly the mutual inductance between two coaxial circular currents and a close approximation to the self-inductance of a thin circular current. The author has not found it necessary to assume Neumann's theorem in proving any formula for self or mutual inductance. An expression is found for the axial magnetic force at any point due to a cylindrical current sheet, and particular cases are noticed. It is proved that the mutual inductance between two cylindrical current sheets is the same as that between one of them and a certain helical current of the same diameter and axial length as the other sheet. This theorem can also be immediately deduced from a formula given by Viriamu Jones †.

The exact formula for the mutual inductance between a cylindrical current sheet and a coaxial helical filament is expressed both in terms of elliptic integrals and in a series which in general converges rapidly. Viriamu Jones left the solution in the form of a definite integral. By the formulæ given in his paper ‡ this can be expressed without difficulty in terms of complete and incomplete elliptic integrals. By utilising Jones's results, Professor Coffin in the 'Bulletin of the Bureau of Standards' (p. 118, June 1906) has done this when the axial length of the two helices is the same. The author, however, starting from Laplace's formula, directly deduces the complete solution in forms adapted for easy computation. Lorenz's formula § for the self-inductance of a helical filament is a particular case of the general formula for mutual inductance given in this paper. It is shown also that a well-known formula due to Lord Rayleigh is a particular case of Lorenz's formula.

2. *Mathematical Formulæ.*

It is convenient to collect together for reference the mathematical definitions and theorems in connexion with

* For instance, Dale's 'Mathematical Tables,' p. 76.

† Proc. Roy. Soc. p. 203 (1898). ‡ *L. c. atne.*

§ Wiedemann's *Annalen*, vii. p. 170 (1879).

Elliptic Functions which we shall require in proving the theorems which follow. Most of them are well known, and they are nearly all to be found in Legendre's *Traité des Fonctions Elliptiques* (1825). For proofs the reader is referred to A. Cayley's 'Elementary Treatise on Elliptic Functions.'

The definitions of the complete elliptic integrals E and F are

$$E=\int_0^{\pi/2}\Delta d\phi, \qquad \text{and } F=\int_0^{\pi/2}\frac{d\phi}{\Delta},$$

where $\Delta=(1-k^2\sin^2\phi)^{1/2}$, and k is the modulus.

It follows at once from the Integral Calculus that

$$E=\frac{\pi}{2}\left\{1-\left(\frac{1}{2}\right)^2k^2-\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{k^4}{3}-\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\frac{k^6}{5}-\dots\right\}$$

$$F=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2k^6+\dots\right\}.$$

Hence when k is small E and F can be readily computed. When k is nearly equal to unity (which is its maximum value) Legendre * has shown that we can use the formulæ

$$E=1+\frac{1}{2}k_1^2\left(\log_\epsilon\frac{4}{k_1}-\frac{1}{1\,.\,2}\right)$$
$$+\frac{1^2\,.\,3}{2^2\,.\,4}k_1^4\left(\log_\epsilon\frac{4}{k_1}-\frac{2}{1\,.\,2}-\frac{1}{3\,.\,4}\right)+\dots, \quad (1)$$

$$F=\log_\epsilon\frac{4}{k_1}+\left(\frac{1}{2}\right)^2k_1^2\left(\log_\epsilon\frac{4}{k_1}-\frac{2}{1\,.\,2}\right)$$
$$+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k_1^4\left(\log_\epsilon\frac{4}{k_1}-\frac{2}{1\,.\,2}-\frac{2}{3\,.\,4}\right)+\dots, (2)$$

where $k_1^2=1-k^2$, and so k_1 is a small quantity. For instance, when $k=0{\cdot}99$, $k_1=0{\cdot}14$ approximately. For a proof of (1) and (2) see A. Cayley, 'Elliptic Functions,' 2nd edition, p. 54.

* A. le Gendre (A. M. Legendre), "*Mémoire sur les Intégrations par arcs d'ellipse,*" Histoire de l'Académie Royale des Sciences, 1786.

The following theorems can be easily proved by elementary Integral Calculus :—

$$\int_0^{\pi/2}\frac{\sin^2\phi}{\Delta}d\phi=\frac{1}{k^2}(F-E), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\int_0^{\pi/2}\frac{\cos 2\phi}{\Delta}d\phi=\frac{2}{k^2}(E-F)+F, \quad . \quad . \quad . \quad (4)$$

$$\int_0^{\pi/2}\frac{\sin^2\phi\cos^2\phi}{\Delta}d\phi=\frac{2-k^2}{3k^4}E-\frac{2-2k^2}{3k^4}F, \quad . \quad . \quad (5)$$

$$\int_0^{\pi/2}\frac{d\phi}{\Delta^3}=\frac{E}{1-k^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\int_0^{\pi/2}\frac{\sin^2\phi d\phi}{\Delta^3}=\frac{E}{k^2(1-k^2)}-\frac{F}{k^2}, \quad . \quad . \quad . \quad (7)$$

$$\int_0^{\pi/2}\sin^2\phi\Delta d\phi=\frac{2k^2-1}{3k^2}E+\frac{1-k^2}{3k^2}F. \quad . \quad (8)$$

The following theorems for transforming the modulus of elliptic functions are of fundamental importance. By their means simple formulæ can be found from which, as Legendre has shown, the elliptic functions can be calculated to any required degree of accuracy with extreme ease. The first formula* was given by Landen.

$$\left.\begin{aligned} F&=(1+k')F' \quad . \quad . \quad . \quad . \\ E&=\frac{2}{1+k'}E'-(1-k')F' \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (9)$$

where the modulus of E′ and F′ is k', and

$$k'=(1-\sqrt{1-k^2})/(1+\sqrt{1-k^2})$$

and $$k=2\sqrt{k'}/(1+k').$$

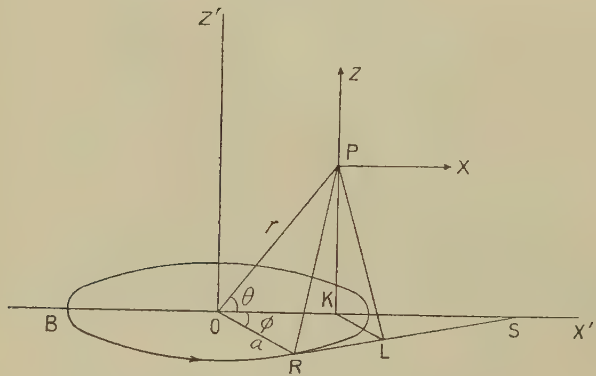
3. *Formulæ for the magnetic force at any point due to the current in a circular filament in rectangular, polar, and bipolar coordinates.*

In fig. 1, BR is the conducting circular filament and OZ′ is its axis. It is obvious from symmetry that the lines of force

* A proof is given in Todhunter's 'Integral Calculus,' Chapter X.

will all be in planes passing through OZ'. We shall first find the component forces Z and X, parallel to OZ' and OX' respectively, at the point P in the plane $Z'OX'$. Let the radius of the circular filament be a, let PO be r, and let the angles POX' and $X'OR$ be θ and ϕ respectively. We shall consider the force at P due to the current in an element $ad\phi$ of the filament at R. Let RS be the tangent to the circle at R, and draw PK and KL perpendicular to OX' and RS respectively. Then, noticing that $KL = a - r\cos\theta\cos\phi$, and

Fig. 1.



X and Z are the components of the magnetic force at P due to the current in the circular filament.

that $\cos POR = \cos\theta\cos\phi$, so that $PR^2 = a^2 + r^2 - 2ar\cos\theta\cos\phi$, we get by Laplace's formula,

$$dZ = \frac{iad\phi}{PR^2}\sin PRL \,.\, \sin KPL$$

$$= \frac{iad\phi}{PR^2}\cdot\frac{PL}{PR}\cdot\frac{KL}{PL}$$

$$= \frac{ia(a - r\cos\theta\cos\phi)d\phi}{(a^2 + r^2 - 2ar\cos\theta\cos\phi)^{3/2}};$$

and therefore,

$$Z = 2ia\int_0^\pi \frac{(a - r\cos\theta\cos\phi)d\phi}{(a^2 + r^2 - 2ar\cos\theta\cos\phi)^{3/2}} \quad . \quad . \quad (10)$$

Similarly we find that

$$X=2iar\int_0^\pi \frac{\sin\theta\cos\phi d\phi}{(a^2+r^2-2ar\cos\theta\cos\phi)^{3/2}} \quad . \quad . \quad (11)$$

Now (10) can be written

$$Z=i\int_0^\pi \frac{d\phi}{(a^2+r^2-2ar\cos\theta\cos\phi)^{1/2}}$$
$$+i(a^2-r^2)\int_0^\pi \frac{d\phi}{(a^2+r^2-2ar\cos\theta\cos\phi)^{3/2}}.$$

Hence, writing $\phi=\pi-2\phi'$, we get by (6) after a little reduction

$$\left.\begin{aligned} Z&=\frac{2i}{r_1}\left(\frac{a^2-r^2}{r_2^2}\mathrm{E}+\mathrm{F}\right) \quad . \\ \text{Similarly}\quad X&=\frac{2i\tan\theta}{r_1}\left(\frac{a^2+r^2}{r_2^2}\mathrm{E}-\mathrm{F}\right) \end{aligned}\right\}, \quad . \quad . \quad . \quad (12)$$

where
$$r_1^2=a^2+r^2+2ar\cos\theta,$$
$$r_2^2=a^2+r^2-2ar\cos\theta,$$

and the modulus k of the elliptic functions is given by $k^2=1-r_2^2/r_1^2$.

If R and T be the component magnetic forces along and perpendicular to OP (fig. 1), we have

$$\left.\begin{aligned} \mathrm{R}&=\mathrm{Z}\sin\theta+\mathrm{X}\cos\theta=\frac{4ia^2\sin\theta}{r_1r_2^2}\mathrm{E}, \quad . \quad . \quad . \quad . \quad . \quad . \\ &\text{and} \\ \mathrm{T}&=\mathrm{Z}\cos\theta-\mathrm{X}\sin\theta=\frac{2i}{r_1r_2^2}\left[\frac{a^2\cos 2\theta-r^2}{\cos\theta}\mathrm{E}+\frac{r_2^2}{\cos\theta}\mathrm{F}\right] \end{aligned}\right\}. \quad (13)$$

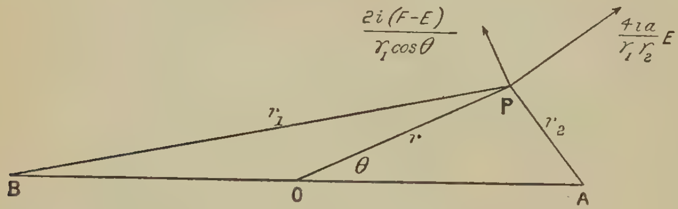
In rectangular coordinates, we have

$$\left.\begin{aligned} X&=\frac{2i}{r_1}\left[\frac{2az}{r_2^2}\mathrm{E}-\frac{z}{x}(\mathrm{F}-\mathrm{E})\right], \quad . \\ Z&=\frac{2i}{r_1}\left[\frac{2a(a-x)}{r_2^2}\mathrm{E}+(\mathrm{F}-\mathrm{E})\right], \end{aligned}\right\} \quad . \quad . \quad (14)$$

where
$$r_1^2=(a+x)^2+z^2,$$
$$r_2^2=(a-x)^2+z^2,$$
and
$$k^2=1-r_2^2/r_1^2.$$

If ϕ be the angle APB (fig. 2), OP$=r$, the angle POA$=\theta$, and OA$=$OB$=a$, we have AP$=r_2$ and PB$=r_1$.

Fig. 2.



The magnetic force at P due to the circular current is the resultant of the forces $2i(\mathrm{F}-\mathrm{E})/(r_1 \cos\theta)$ and $4ia\mathrm{E}/r_1r_2$ acting at right angles to OP and AP respectively.

Hence in bipolar coordinates

$$\left.\begin{aligned} \mathrm{Z} &= -\frac{2i}{r_2}\mathrm{E}\cos\phi + \frac{2i}{r_1}\mathrm{F}, \\ \mathrm{X} &= \frac{2i}{r_2}\cdot\frac{r_1^2+r_2^2}{r_1^2-r_2^2}\cdot \mathrm{E}\sin\phi - \frac{2i}{r_1}\cdot\frac{2r_1r_2}{r_1^2-r_2^2}\cdot \mathrm{F}\sin\phi, \end{aligned}\right\} \quad (15)$$

where $\cos\phi=(r_1^2+r_2^2-4a^2)/(2r_1r_2)$, and $k^2=1-r_2^2/r_1^2$.

A simple and useful way of considering the force at any point P due to a circular current is to consider that it is the resultant (fig. 2) of two forces at right angles to AP and OP respectively. The component at right angles to AP equals $(4ia/r_1r_2)\mathrm{E}$, and the component at right angles to OP equals $2i(\mathrm{F}-\mathrm{E})/(r_1\cos\theta)$, the modulus of E and F being $(1-r_2^2/r_1^2)^{1/2}$.

4. *Applications of the Formulæ.*

i. *Force in the plane of a circular current.*

From fig. 2 we see at once that the force at any point in the plane of the circular current and inside the circle is given by

$$\begin{aligned} \mathrm{Z} &= \frac{2i}{r_2}\mathrm{E} + \frac{2i}{r_1}\mathrm{F} \\ &= \frac{2i}{a-r}\mathrm{E} + \frac{2i}{a+r}\mathrm{F}, \end{aligned} \quad (16)$$

the modulus being $2(ar)^{1/2}/(a+r)$.

Again, at points in the plane of the circular current, but

outside the circle, we have

$$Z = -\frac{2i}{r-a}E + \frac{2i}{r+a}F$$
$$= \frac{2i}{a-r}E + \frac{2i}{a+r}F.$$

Hence, whether the point be inside or outside the circle, formula (16) gives the magnetic force. When the point lies within the circle the author at first used the formula *

$$Z = \frac{4ai}{a^2-r^2}E'$$
$$= \frac{2i}{a-r}E' + \frac{2i}{a+r}E',$$

where the modulus is r/a. By means of (9) we see that the formulæ give identical results. For points outside the circle, however, we find by (16) and (9), or otherwise,

$$Z = -\frac{4ri}{r^2-a^2}E'' + \frac{4i}{r}F'',$$

the modulus now being a/r. It will be seen that it is best to consider (16) as the standard formula, as it gives the force at points in the plane of the filament, both outside and inside the circle.

It is worth noticing that if r be not greater than $0{\cdot}7a$, the maximum inaccuracy of the formula

$$Z = \frac{\pi i}{2(a^2-r^2)}\left\{a + \sqrt{a^2-r^2} + \sqrt{4a^2-2r^2}\right\}$$

is less than 1 in 10,000.

Let us now find the value of the magnetic force very close to the filament. In this case, as the modulus of E and F in (16) is nearly unity, we can use Legendre's formulæ (1) and (2) to find E and F. Hence

$$Z = \frac{2i}{a-r}\left[1 + \frac{1}{2}\left(\frac{a-r}{a+r}\right)^2\left\{\log\frac{4(a+r)}{a-r} - \frac{1}{2}\right\} + \dots\right]$$
$$+ \frac{2i}{a+r}\left[\log\frac{4(a+r)}{a-r} + \frac{1}{4}\left(\frac{a-r}{a+r}\right)^2\left\{\log\frac{4(a+r)}{a-r} - 1\right\} + \dots\right]$$
$$= \frac{2i}{a-r} + \frac{2i}{a+r}\log\frac{4(a+r)}{a-r}, \quad \dots \dots \dots \quad (17)$$

* 'The Electrician,' vol. xxxi. p. 212 (1893) or 'Alternating Currents,' vol. i. p. 30.

when $(a-r)^2/(a+r)^2$ is negligibly small compared with unity. The first term on the right-hand side of this equation is very great compared with the second when r is nearly equal to a, and, in this case, we may write

$$Z=\frac{2i}{a-r}, \text{ very approximately.}$$

Similarly, when r is greater than a, we have

$$Z=-\frac{2i}{r-a}+\frac{2i}{r+a}\log\frac{4(r+a)}{r-a}, \quad . \quad . \quad . \quad (18)$$

approximately.

ii. *The mutual inductance of two coaxial circular currents.*

Let the radii of the circles be a and b respectively, and let c be the distance between their planes. We shall calculate the flux M through the circle whose radius is b due to unit current in the circle whose radius is a. Let Z be the magnetic force at an element $r'd\theta'dr'$ of the area of the circle whose radius is b, then

$$M=\int_0^b\int_0^{2\pi} Zr'd\theta'dr',$$

and noticing that $r^2=r'^2+c^2$, and $r\cos\theta=r'$, we get

$$\frac{M}{4\pi a}=\int_0^b\int_0^{\pi}\frac{(a-r'\cos\phi)r'd\theta'dr'}{(a^2+r'^2+c^2-2ar'\cos\phi)^{3/2}}.$$

Changing to rectangular coordinates, we have

$$\frac{M}{4\pi a}=\int_0^b\int_{-\sqrt{b^2-y^2}}^{+\sqrt{b^2-y^2}}\frac{(a-x)dxdy}{\{(a-x)^2+y^2+c^2\}^{3/2}}$$

$$=\int_0^b\Big[1/\{(a-x)^2+y^2+c^2\}^{1/2}\Big]_{-\sqrt{b^2-y^2}}^{+\sqrt{b^2-y^2}}dy.$$

Put $y=b\sin\phi$ and simplify. In the second integral also, write $\phi'=\pi-\phi$. We get

$$\frac{M}{4\pi ab}=\int_0^{\pi}\frac{\cos\phi d\phi}{(a^2+b^2+c^2-2ab\cos\phi)^{1/2}}.$$

Now putting $\phi=\pi-2\theta$, we have

$$\frac{M}{4\pi ab}=-\frac{2}{r_1}\int_0^{\pi/2}\frac{\cos 2\theta}{\Delta}d\theta, \quad . \quad . \quad . \quad (19)$$

and therefore, by (4)

$$M = 4\pi\sqrt{ab}\{(2/k-k)F-(2/k)E\}, \quad . \quad . \quad (20)$$

where $\quad r_1^2 = (a+b)^2 + c^2, \quad$ and $k^2 = 4ab/r_1^2$.

Formula (20) agrees with that given by Helmholtz * for the analogous problem of the vortex ring.

If we transform (20) by the Landen-Legendre formulæ (9), we get

$$M = \frac{8\pi\sqrt{ab}}{\sqrt{k'}}(F'-E'), \quad . \quad . \quad . \quad . \quad (21)$$

where

$k' = (r_1 - r_2)/(r_1 + r_2)$, $r_1^2 = (a+b)^2 + c^2$, and $r_2^2 = (a-b)^2 + c^2$.

This is Maxwell's formula.

iii. *Approximate formula for the self-inductance of a ring.*

We shall suppose that the radius of the ring is a, that the radius of the cross section is r, and that the current density at all points of the cross section is the same. We suppose also that r/a is small compared with unity. From the construction given in fig. 2 we see that the lines of force near a circular filament are very approximately circles with their centres on the circular axis of the filament. The field is therefore similar to that round a straight cylindrical conducting tube, in which the current flow is parallel to the axis. Now it is well known † that the magnetic force outside a straight cylindrical conducting tube can be calculated as if the current were concentrated along the axis and that the magnetic force inside the tube is zero. Let us suppose that the circular conductor is built up of an infinite number of infinitely thin ring-tubes, every point on any tube having the same minimum distance from the circular axis, and make the assumptions that the current in each ring-tube produces no magnetic field inside it, and that the field outside can be calculated as if the current were concentrated along the axis. From the corresponding electrostatic problem of a charged conducting ring we see, by the method of duality, that the first assumption could be made rigorously true by assuming that

* Crelle's *Journal*, vol. lv. p. 25 (1858), or *Ges. Abh.* t. 1, p. 101.

† A. Russell, 'Alternating Currents,' vol. i. p. 32.

the density of the current distribution over the cross section of the infinitely thin ring-tube was proportional to the surface density of the electrostatic charge. In this case, however, the magnetic force at points outside the tube is not the same as if the current were concentrated along the circular axis, and therefore the gain in rigour by making the first assumption accurate is problematical. When r is very small compared with a the surface density is approximately constant over the ring *. In this case, therefore, when the current density is uniform over the circular tube, the magnetic force produced inside is negligible, and therefore our assumptions are legitimate in obtaining an approximate solution.

Let Φ_1 be the flux linked with the whole current when unit current is flowing in the ring, and Φ_2 the flux linked with part of the current only. By (21) we have

$$\Phi_1 = 8\pi a(\mathrm{F}-\mathrm{E}),$$

where the modulus is $(a-r)/a$.

By (17), the force Z at a point in the plane of the circular axis and at a distance $a-\xi$ from its centre is given by

$$\mathrm{Z} = \frac{2i'}{\xi} + \frac{i'}{a}\log\frac{8a}{\xi},$$

approximately, where $i' = \xi^2/r^2$, since the whole current in the ring is unity.

Hence, on our assumptions

$$\Phi_2 = 2\pi\int_0^r (a-\xi)\mathrm{Z}(\xi^2/r^2)d\xi.$$

This integral can be easily found. Since, however, we are supposing that r and ξ are very small, we may write $\mathrm{Z}=2i'/\xi$ simply, and thus $\Phi_2 = \pi a$, approximately.

If L, therefore, denote the self-inductance, we have

$$\mathrm{L} = \Phi_1 + \Phi_2,$$
$$= 8\pi a(\mathrm{F}-\mathrm{E}) + \pi a.$$

Substituting for F and E their approximate values (1/2) log $(8a/r)$ and 1, we get

$$\mathrm{L} = 4\pi a\{\log(8a/r) - 1\cdot75\}, \quad . \quad . \quad . \quad (22)$$

approximately.

* F. W. Dyson, Phil. Trans. vol. clxxxiv. A, p. 67 (1893).

The formula given by Lord Rayleigh * and Sir W. D. Niven is

$$L=4\pi a\{\log (8a/r)-1\cdot75\}+(\pi r^2/2a)\{\log (8a/r)+1/3\},$$

and by Max Wien †

$$L=4\pi a\{\log (8a/r)-1\cdot75\}+(\pi r^2/2a)\{\log (8a/r)-0\cdot0664\}.$$

When $r/a=1/100$ we should expect (22) to give L with considerable accuracy. We see that in this case, the value of L found by it differs from that given by either Rayleigh's or Wien's formula by less than the five-hundredth part of one per cent.

If we suppose that the conductivity of the ring is infinite so that the current flows entirely on the surface, then our method of proof shows that

$$L=\Phi_1=8\pi a(F-E)=4\pi a\{\log (8a/r)-2\},$$

approximately. When $r/a=1/100$, the value of L given by this formula is more than five per cent. lower than that given by any of the preceding formulæ. It will be seen, therefore, that even with a thin wire the manner in which the current is distributed over the cross section has an appreciable effect on the inductance. It must also be noticed that when r/a is small, $dL/dr(=-4\pi a/r)$ is large, and hence L varies rapidly with r and a small error made in measuring r may lead to a large error in the calculated value of L.

It is interesting to compare the formula for the self-inductance of a ring with the formula for its electrostatic capacity. Formulæ for the capacity K have been found by W. M. Hicks ‡ and F. W. Dyson §.

When the radius of the ring is large compared with the radius of the circular section, they both agree in giving the formula as

$$K=\pi a/\{\log (8a/r)\}.$$

Hence, if the resistivity of the ring be zero, we have

$$KL=(2\pi a)^2\{1-2/\log (8a/r)\}. \quad . \quad . \quad . \quad (25)$$

* 'Scientific Papers,' vol. ii. p. 15.

† Wiedemann's *Annalen*, liii. p. 934 (1894).

‡ Phil. Trans. vol. clxxii. p. 643 (1881).

§ F. W. Dyson, p. 68, *l. c. ante*.

When $100r=a$, this equation becomes

$$KL=0{\cdot}7(2\pi a)^2, \text{ approximately.}$$

It will be seen, therefore, that we can only use the equation $KL=(2\pi a)^2$ when r/a is an exceedingly small fraction. The corresponding equation $KL=l^2$, for two long parallel cylinders of infinite conductivity is strictly true (see A. Russell, 'Alternating Currents,' vol. i. p. 141).

5. *Applications to Hydrodynamics.*

Lord Kelvin has shown ('Papers on Electrostatics and Magnetism,' p. 444) that the problem of finding the velocity at any point of an incompressible fluid near a vortex filament is, from the point of view of mathematical analysis, the same as the problem of finding the magnetic force near a current filament. The strength m of the vortex corresponds to the strength i of the current and the fluid velocity to the magnetic force. If r be the radius of the cross section (supposed circular) of a vortex ring and ω the angular velocity of points on the ring round the circular axis, the strength * m of the vortex is generally defined to be $\pi r^2\omega$. In the electrical problem the magnetic force tangential to the ring is $2i/r$, and in the hydrodynamical problem the corresponding velocity is ωr. Hence to convert the formulæ for the magnetic force at a point into the corresponding formulæ for the fluid velocity at that point we must multiply by $\omega r^2/2i$, that is, by $m/2\pi i$. For instance, we see from the construction given in fig. 2 that the velocity at a point P in the fluid near a vortex ring can be found as follows:—The component velocity at right angles to AP (fig. 2) equals $(2ma/\pi r_1 r_2)E$, and the component at right angles to OP equals

$$m(F-E)/(\pi r_1 \cos\theta),$$

the modulus of E and F being $(1-r_2^2/r_1^2)^{1/2}$. At points on the surface of the vortex ring we have $r_2=r$, and we may write $r_1=2a$, $\cos\theta=1$, $E=1$, and $F=\log(8a/r)$. Hence points on the surface of the ring have a velocity $m/\pi r$, that is, ωr about the circular axis and a linear velocity w upwards,

* It is sometimes defined as $2\pi r^2\omega$.

where

$$w = \frac{m}{2\pi a}\left(\log\frac{8a}{r} - 1\right). \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We see, therefore, that the vortex ring moves bodily upwards with the velocity w. We see also at once that the velocity of the fluid at the centre of the vortex ring is $2\pi i/a \times m/2\pi i$, that is, m/a.

Again, if the components of the velocity of an element of the volume dv of the fluid be u and w, and if ρ be the density of the fluid, we have

$$\mathrm{T}_1 = \tfrac{1}{2}\rho\Sigma(u^2 + w^2)dv = \frac{\rho m^2}{8\pi^2 i^2}\Sigma(\mathrm{X}^2 + \mathrm{Z}^2)dv$$

$$= \frac{\rho m^2}{\pi i^2}\Sigma\left(\frac{\mathrm{R}^2}{8\pi}\right)dv,$$

where T_1 is the kinetic energy of the moving fluid and R=the resultant magnetic force at dv in the electrical problem. But by Kelvin's formula

$$\Sigma(\mathrm{R}^2/8\pi)dv = \mathrm{L}i^2/2,$$

and thus

$$\mathrm{T}_1 = \frac{\rho m^2}{2\pi}\mathrm{L},$$

where $\mathrm{L}i^2/2$ is the energy due to the linkages of the lines of force with the whole of the electric current.

Hence, by the preceding section

$$\mathrm{T}_1 = \frac{\rho m^2}{2\pi}4\pi a\left(\log\frac{8a}{r} - 2\right),$$

$$= 2\rho m^2 a\left(\log\frac{8a}{r} - 2\right).$$

Let us now suppose that the core of the vortex-ring is solid so that the angular velocity of every point in it round the circular axis is constant. This corresponds to assuming that the current density over the cross section of the ring in the electrical problem is constant. Let us also suppose that the density of the core is the same as that of the fluid. The kinetic energy T_2 of the core is given by

$$\mathrm{T}_2 = \tfrac{1}{2}\rho\pi r^2 2\pi a\{(r^2/2)\omega^2 + w^2\},$$

$$= \tfrac{1}{2}\rho m^2 a + \text{small terms.}$$

Hence the total kinetic energy of the system T_1+T_2 equals

$$2\rho m^2 a\left(\log\frac{8a}{r}-1{\cdot}75\right).$$

This is the formula given by Lamb *. By means of (22) we could have written it down at once. The formula, however, Lamb gives for the velocity of translation of a vortex ring is

$$w=\frac{m}{2\pi a}\left(\log\frac{8a}{r}-\frac{1}{4}\right). \quad . \quad . \quad . \quad . \quad (b) \dagger$$

This formula is due to Lord Kelvin ‡. In Basset's 'Hydrodynamics' § both formulæ are given. The author thinks that when r/a is small formula (a) must be the more accurate. If this be not the case, then the electrical analogy must break down at points contiguous to the filament. Chree ‖, in his paper 'Vortex Rings in a Compressible Fluid,' gives formula (a). He also points out reasons for the slight divergences in the formulæ for the motion of vortex rings given by various physicists. In another important paper ¶ by the same author, the similarity of the equations of vorticity to those of electrodynamics is shown very clearly. When the compressibility of the fluid is considered, the hydrodynamical problem is much the more difficult.

6. *The magnetic force due to a cylindrical current sheet.*

We shall suppose that there are n bands of current per unit length. We shall first find the magnetic force at a point P in a plane through the base of the cylinder and at a distance r from the axis. Let the length of the axis of the cylinder be l and its radius a. Then, by (10),

$$Z=2ia\int_0^\pi\int_0^l\frac{a-r\cos\phi}{(z^2+a^2+r^2-2ar\cos\phi)^{3/2}}\cdot\frac{ndz}{l}\cdot d\phi,$$

where i is the current in a filament.

* 'Hydrodynamics,' p. 227, 3rd edition.

† 'Hydrodynamics,' p. 227. [Note that $\kappa=2m$.]

‡ Phil. Mag. [4] vol. xxxiii. Supp. p. 511 (1867).

§ Vol. ii., (5) p. 62, and (86) p. 87.

‖ Proc. Edin. Math. Soc. vol. vi. p. 59 (1887).

¶ Proc. Edin. Math. Soc. vol. viii. p. 43 (1889).

Hence
$$Z=\frac{2ni}{l}\int_0^\pi \frac{a(a-r\cos\phi)}{\Delta^2}\left[\frac{z}{(z^2+\Delta^2)^{1/2}}\right]_0^l d\phi$$
$$=2ni\int_0^\pi \frac{a(a-r\cos\phi)}{\Delta^2(l^2+\Delta^2)^{1/2}}\,d\phi,$$
where $\Delta^2=a^2+r^2-2ar\cos\phi$.

The magnetic force, therefore, at a point distant $h+z$ from one end and $h-z$ from the other end of a cylindrical current sheet is given by
$$Z=\frac{N_1 i}{h_1}\int_0^\pi \frac{a(a-r\cos\phi)}{\Delta^2}\left[\frac{h_1+z}{\{(h_1+z)^2+\Delta^2\}^{1/2}}\right.$$
$$\left.+\frac{h_1-z}{\{(h_1-z)^2+\Delta^2\}^{1/2}}\right]d\phi, \quad . \quad (26)$$
where $2h_1$ = the axial length of the cylinder, and $N_1=2h_1 n$.

Now noticing that
$$\frac{h_1+z}{\{(h_1+z)^2+\Delta^2\}^{1/2}}=\frac{h_1+z}{d}\left[1+\tfrac{1}{2}\cdot\frac{2ar\cos\phi}{d^2}\right.$$
$$\left.+\frac{1}{2}\frac{3}{.4}\cdot\left(\frac{2ar\cos\phi}{d^2}\right)^2+\ldots\right],$$
where $d^2=(h_1+z)^2+a^2+r^2$, and that
$$\int_0^\pi \frac{a(a-r\cos\phi)}{\Delta^2}\,d\phi=\pi,$$
$$\int_0^\pi (2ar\cos\phi)\frac{a(a-r\cos\phi)}{\Delta^2}\,d\phi=\pi r^2,$$
$$\int_0^\pi (2ar\cos\phi)^2\frac{a(a-r\cos\phi)}{\Delta^2}\,d\phi=\pi r^2(r^2+2a^2),$$
$$\cdots\cdots\cdots\cdots,$$
we see that,
$$Z=\frac{\pi N_1 i}{h_1}\left[\frac{h_1+z}{d}\left\{1+\frac{1}{2}\cdot\frac{r^2}{d^2}+\frac{1.3}{2.4}\frac{r^2(r^2+2a^2)}{d^4}\right.\right.$$
$$+\frac{1.3.5}{2.4.6}\cdot\frac{r^4(r^2+3a^2)}{d^6}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{r^4(r^4+4r^2a^2+6a^4)}{d^8}$$
$$\left.+\frac{1\ldots9}{2\ldots10}\cdot\frac{r^6(r^4+5r^2a^2+10a^4)}{d^{10}}+\ldots\right\}$$
$$\left.+\frac{h_1-z}{d'}\left\{1+\frac{1}{2}\cdot\frac{r^2}{d'^2}+\ldots\right\}\right], \quad . \quad . \quad . \quad (27)$$
where $d'^2=(h_1-z)^2+a^2+r^2$.

The terms in the bracket multiplying $(h_1-z)/d'$ can be found at once from the corresponding multiplier for $(h_1+z)/d$, by writing d' for d in the terms in the latter bracket. As a rule the series converges rapidly.

Particular cases of the formula are interesting. For instance, when h_1 is infinite we get

$$Z=4\pi\left(\frac{N_1}{2h_1}\right)i,$$

and when r is zero,

$$Z=\frac{\pi N_1 i}{h_1}\left[\frac{h_1+z}{d}+\frac{h_1-z}{d'}\right].$$

Both of these results are well known. It is not difficult to show that the latter result is true for a helical current i, the radius of the helix being a, the axial length $2h_1$, and the number of turns N_1.

When z is zero, that is at points on the plane bisecting the axis of the cylindrical current sheet at right angles, we have

$$Z=\frac{2\pi N_1 i}{\{h_1^2+a^2+r^2\}^{1/2}}\left[1+\frac{1}{2}\cdot\frac{r^2}{h_1^2+a^2+r^2}+\frac{3}{8}\cdot\frac{r^2(r^2+2a^2)}{(h_1^2+a^2+r^2)^2}\right.$$
$$\left.+\frac{5}{16}\cdot\frac{r^4(r^2+3a^2)}{(h_1^2+a^2+r^2)^3}+\dots\right]. \quad (28)$$

When r^2 is small compared with $h_1^2+a^2+r^2$, the series converges with great rapidity. This result, however, is not true for a helical current.

7. *Formulæ for the mutual inductance between two coaxial cylindrical-current sheets.*

Let us suppose that there is a cylinder of radius b and axial length $2h_2$, coaxial and concentric with the cylinder considered above. We shall suppose that b is less than a and that we have N_2 filaments equally spaced on the inner cylinder so that $N_2/(2h_2)$ gives the turns per unit length. We have now to calculate the linkages M of the flux, due to unit current in the outer cylinder, with the filaments of the inner cylinder.

If dM denote the linkages due to the flux through the element of area $rd\theta\,dr$ from $z=h_2$ to $z=-h_2$, we have

$$dM=\int_{-h_2}^{+h_2} Zr\,d\theta\,dr\,N_2(dz/2h_2),$$

and hence, by (26),

$$dM=\frac{N_1N_2}{h_1h_2}\int_0^b\int_0^\pi \frac{ar\,d\theta(a-r\cos\phi)}{\Delta^2}\left[\{(h_1+h_2)^2+\Delta^2\}\right.$$
$$\left.-\{(h_1-h_2)^2+\Delta^2\}^{1/2}\right]d\phi\,dr.$$

Therefore

$$M=\frac{2\pi N_1N_2}{h_1h_2}\int_0^b\int_0^\pi \frac{ar(a-r\cos\phi)}{\Delta^2}\left[\{(h_1+h_2)^2+\Delta^2\}^{1/2}\right.$$
$$\left.-\{(h_1-h_2)^2+\Delta^2\}^{1/2}\right]d\phi dr. \quad . \quad . \quad (29)$$

It can be proved mathematically that this formula is equivalent to the following:

$$M=4\pi abc^2\frac{N_1N_2}{h_1h_2}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{1-c^2\sin^2\theta}\left[R_1(1-k_1^2\sin^2\theta)^{1/2}\right.$$
$$\left.-R_2(1-k_2^2\sin^2\theta)^{1/2}\right]d\theta, \quad . \quad . \quad (30)$$

where $c^2=4ab/(a+b)^2$; $R_1^2=(a+b)^2+(h_1+h_2)^2$; $R_2^2=(a+b)^2+(h_1-h_2)^2$; $k_1^2=4ab/R_1^2$ and $k_2^2=4ab/R_2^2$.

As we give a proof of the identity of (29) and (30) in the next section, it is unnecessary to give a mathematical proof here.

It is not difficult to find a series for M by expanding the radicals in (29) by the binomial theorem and then integrating the terms. The author has found several series formulæ for M. The following two approximate formulæ deduced from them are simple, and when $h_1^2+h_2^2+a^2$ is greater than b^2+2ab they give accurate results. The first formula given (31) is much the more accurate, especially when h_1 and h_2 are not very different.

$$M=\frac{2\pi N_1N_2(\pi b^2)}{d}\left\{\frac{2d}{(d^2+2h_1h_2)^{1/2}+(d^2-2h_1h_2)^{1/2}}\right.$$
$$\left.+\frac{3a^2b^2}{8d^4}-\frac{5a^2b^4}{16d^6}\right\}, \quad . \quad . \quad (31)$$

and

$$M = \frac{2\pi N_1 N_2 (\pi b^2)}{d}\left\{1 + \frac{3a^2b^2 + 4h_1^2h_2^2}{8d^4}\right\}, \quad . \quad . \quad . \quad . \quad (32)$$

where $d^2 = h_1^2 + h_2^2 + a^2$.

Let us suppose, for instance, that $N_1 = 300$, $N_2 = 200$, $a = 5$, $b = 4$, $h_1 = 15$, and $h_2 = 2 \cdot 5$, the lengths being measured in centimetres. Formula (31) makes $M = 0 \cdot 0011995$ henry, and (32) makes $M = 0 \cdot 0011992$. By the complete formula given in the next section $M = 0 \cdot 0011999$. It will be seen that the accuracy of the approximate formulæ in this case is quite satisfactory.

From considerations in connexion with the magnetic potential of a cylindrical current sheet, G. F. C. Searle* has deduced the formula

$$M = \frac{2\pi^2 N_1 N_2 b^2}{d_1}\left(1 - \frac{a^2}{2d_1^4} \cdot \frac{4h_2^2 - 3b^2}{4} - \frac{a^2(4h_1^2 - 3a^2)}{8d_1^8} \cdot \frac{8h_2^4 - 20h_2^2b^2 + 5b^4}{8} - \ldots\right),$$

where $d_1^2 = h_1^2 + a^2$. The value of M found by this formula in the case considered is $0 \cdot 0011999$ henry.

We shall find a formula in the next section by means of which M can always be evaluated to any specified degree of accuracy.

8. *Formulæ for the mutual inductance between a helix and a coaxial cylindrical current sheet.*

We shall first find Jones's formula † for the mutual inductance between a circle and a coaxial helix.

From (19) we see that the mutual inductance M between two coaxial circular currents is given by

$$M = -8\pi ab \int_0^{\pi/2} \frac{\cos 2\theta \, d\theta}{\{(a+b)^2 + z^2 - 4ab \sin^2\theta\}^{1/2}},$$

where z is the distance between their planes, and a and b are their radii. Hence the mutual inductance dM between a

* G. F. C. Searle and J. R. Airey, 'The Electrician,' Dec. 8th, 1905.

† Proc. Roy. Soc. vol. lxii. p. 247 (1897).

sector of the circle whose radius is b and the circle whose radius is a is given by

$$dM = -4ab\,d\psi \int_0^{\pi/2} \frac{\cos 2\theta\, d\theta}{\{(a+b)^2 + z^2 - 4ab\sin^2\theta\}^{1/2}},$$

where $d\psi$ is the angle of the sector.

Let $2\pi p$ be the pitch of the helix, and let the coordinates of points on it be given by $x = b\cos\psi$, $y = b\sin\psi$, and $z = p\psi$.

We shall first find the mutual inductance between this helix and a coaxial circle in the plane XY (fig. 3). Let us find the linkages of the helix with the flux due to unit current in the circular filament. Consider two contiguous points P_1 and P_2 on the helix and draw P_1N_1 and P_2N_2 perpendicular to OZ. Also draw P_2P parallel to OZ to meet the plane passing through P_1N_1 and perpendicular to OZ at P. Join PN_1 and PP_1. Now every line of force due to the circular current must lie in a plane which passes through OZ. Hence, since P_1P_2 are infinitely close together all the lines of force linked with the circuit $P_1P_2N_2N_1$ pass through the sector N_1P_1P. Hence if the helix have an integral number of turns and its ends be joined by a conductor lying in a plane containing the axis of the helix, we get

$$M = -4ab\int_0^{\pi/2}\int_{\psi_1}^{\psi_2} \frac{\cos 2\theta\, d\psi\, d\theta}{\{(a+b)^2 - 4ab\sin^2\theta + p^2\psi^2\}^{1/2}},$$

where ψ_1 and ψ_2 are the values of ψ at the ends of the helix.

Hence, integrating, we get

$$\frac{pM}{4ab} = \int_0^{\pi/2} \cos 2\theta\Big[\log\big(p\psi_1 + \sqrt{p^2\psi_1^2 + \Delta^2}\big) - \log\big(p\psi_2 + \sqrt{p^2\psi_2^2 + \Delta^2}\big)\Big]d\theta,$$

where $\Delta^2 = (a+b)^2 - 4ab\sin^2\theta$.

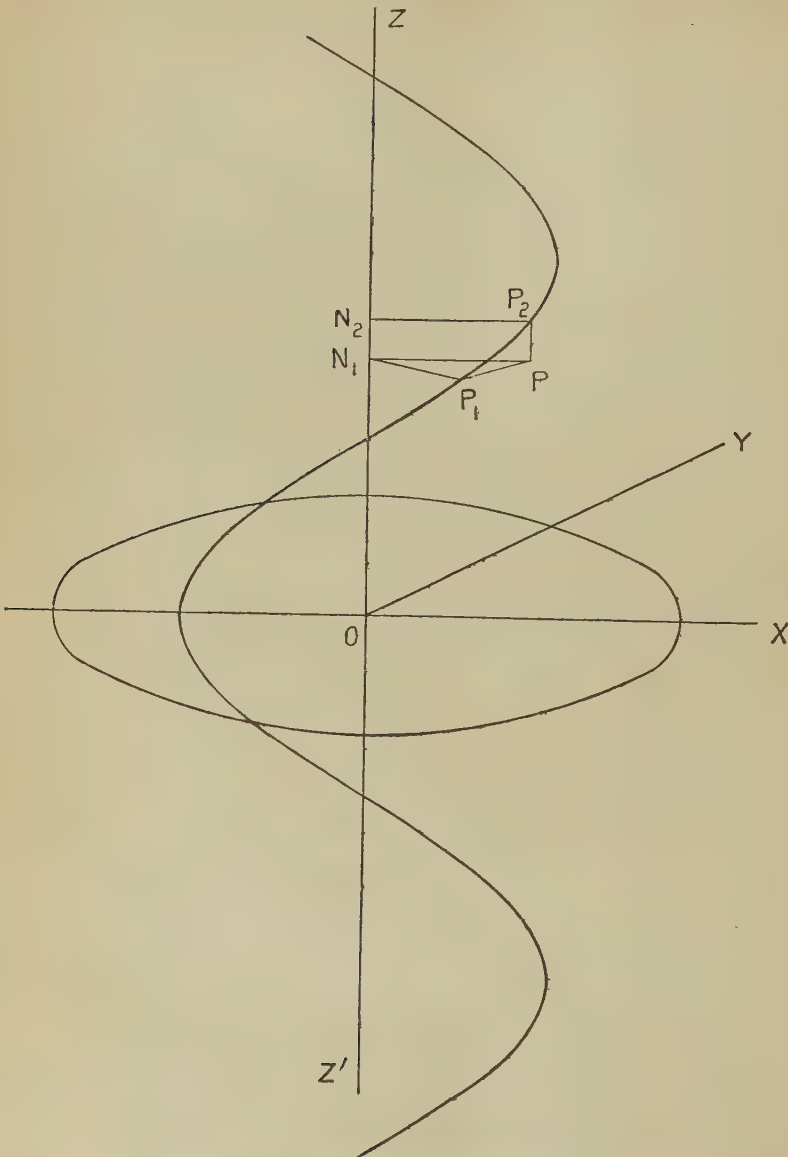
Integrating by parts and noticing that $\sin 2\theta$ vanishes at both limits, we find that

$$\frac{pM}{4ab} = 4ab\int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta\, d\theta}{(p^2\psi_1^2 + \Delta^2)^{1/2}(p\psi_1 + \sqrt{p^2\psi_1^2 + \Delta^2})} - \dots$$

$$= 4ab\int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta}{\Delta^2(p^2\psi_1^2 + \Delta^2)^{1/2}}\Big(\sqrt{p^2\psi_1^2 + \Delta^2} - p\psi_1\Big)d\theta - \dots$$

$$= 4ab\int_0^{\pi/2} \frac{\sin^2\theta\cos^2\theta}{\Delta^2}\left\{\frac{p\psi_2}{\sqrt{p^2\psi_2^2 + \Delta^2}} - \frac{p\psi_1}{\sqrt{p^2\psi_1^2 + \Delta^2}}\right\}d\theta.$$

Fig. 3.



Helix and Circle. ZOZ′ is the axis of the helix and XOY is the plane of the circle. The axial length of the helix is $2h_1$, its radius b and the number of turns N_1. The radius of the circle is a.

Therefore

$$\frac{\mathrm{M}}{4ab}=c^2\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{1-c^2\sin^2\theta}\left\{\frac{\psi_2}{\mathrm{R}_2\Delta_2}-\frac{\psi_1}{\mathrm{R}_1\Delta_1}\right\}d\theta, \quad (33)$$

where $c^2=4ab/(a+b)^2$; $\mathrm{R}_1{}^2=(a+b)^2+p^2\psi_1{}^2$;

$\mathrm{R}_2{}^2=(a+b)^2+p^2\psi_2{}^2$; $k_1{}^2=4ab/\mathrm{R}_1{}^2$; $k_2{}^2=4ab/\mathrm{R}_2{}^2$;

$\Delta_1{}^2=1-k_1{}^2\sin^2\theta$, and $\Delta_2{}^2=1-k_2{}^2\sin^2\theta$.

If $-\psi_1=\psi_2=\psi$, $\mathrm{R}_1=\mathrm{R}_2=\mathrm{R}$, and $k^2=4ab/\mathrm{R}^2$, we have

$$\frac{\mathrm{M}}{4ab}=\frac{c^2\psi}{\mathrm{R}}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta\,d\theta}{(1-c^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \quad . \quad (34)$$

Noticing that

$$c^2\sin^2\theta\cos^2\theta=-\frac{1-c^2}{c^2}+(1-c^2\sin^2\theta)\left(\sin^2\theta+\frac{1-c^2}{c^2}\right),$$

and using (3), we get

$$\mathrm{M}=2\psi(a+b)ck\left[\frac{1}{k^2}(\mathrm{F}-\mathrm{E})+\frac{1-c^2}{c^2}(\mathrm{F}-\Pi)\right], \quad . \quad (35)$$

where

$$\Pi=\int_0^{\pi/2}\frac{d\theta}{(1-c^2\sin^2\theta)(1-k^2\sin^2\theta)^{1/2}},$$

and is consequently the complete elliptic integral of the third kind in which $n=-c^2$. This is the final form in which Jones gave his formula.

If we put $\psi=\pi$, and $p=0$, so that $c=k$, then, by (6), we get

$$\mathrm{M}=4\pi\sqrt{ab}\{(2/k-k)\mathrm{F}-(2/k)\mathrm{E}\},$$

which is formula (20).

M may be found from (35) by a formula, due to Legendre, given on p. 138 of Vol. I. of his *Traité des Fonctions Elliptiques*, and by the tables given in Vol. II.

Legendre's formula may be written in the form

$$\mathrm{F}-\Pi=\frac{c}{k'^2\sin\alpha\cos\alpha}[\mathrm{E}\,.\,\mathrm{F}(k',\alpha)+\mathrm{F}\,.\,\mathrm{E}(k',\alpha)$$
$$-\mathrm{F}\,.\,\mathrm{F}(k',\alpha)-\pi/2]. \quad . \quad (36)^*$$

* See also Cayley's 'Elliptic Functions,' Chapter V. p. 139, 2nd edition.

In this formula $k'^2=1-k^2$, and $F(k', \alpha)$, $E(k', \alpha)$ represent elliptic integrals of modulus k' and amplitude α, where $\sin\alpha=(1-c^2)^{1/2}/k'$.

We shall now find the mutual inductance between a helix and a coaxial concentric cylindrical current sheet. Let N_1 be the number of turns of the helix, $2h_1$ its axial length, and a its radius. If p be the pitch of the helix, $h_1=pN_1\pi$. Let also N_2, $2h_2$, and b be the number of turns, axial length and radius of the cylindrical sheet. Let us consider a filament of the cylinder at a height z above the median plane. Then its distances from the ends of the axis of the helix will be h_1+z and $_1-z$ respectively. Hence if dM be the linkages of the flux due to unit current in the helix embraced by the circular filament $N_2(dz/2h_2)$, we get by (33)

$$dM=\frac{4abc^2}{p}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{1-c^2\sin^2\theta}\left[\frac{h_1+z}{\{(h_1+z)^2+\Delta^2\}^{1/2}}\right.$$
$$\left.+\frac{h_1-z}{\{(h_1-z)^2+\Delta^2\}^{1/2}}\right]d\theta\frac{N_2dz}{2h_2},$$

and therefore, integrating from $-h_2$ to $+h_2$ we get

$$M=4\pi abc^2\frac{N_1N_2}{h_1h_2}\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{1-c^2\sin^2\theta}\left[R_1(1-k_1^2\sin^2\theta)^{1/2}\right.$$
$$\left.-R_2(1-k_2^2\sin^2\theta)^{1/2}\right]d\theta. \quad (37)$$

Now

$$c^2\sin^2\theta\cos^2\theta\sqrt{1-k_1^2\sin^2\theta}$$
$$=\frac{(c^2-k_1^2)\sin^2\theta\cos^2\theta+k_1^2\sin^2\theta\cos^2\theta(1-c^2\sin^2\theta)}{\sqrt{1-k_1^2\sin^2\theta}},$$

and thus, by (5)

$$M=4\pi ab\frac{N_1N_2}{h_1h_2}\left[R_1\left(1-\frac{k_1^2}{c^2}\right)\left\{\frac{1-c^2}{c^2}(F_1-\Pi_1)+\frac{1}{k_1^2}(F_1-E_1)\right\}\right.$$
$$+R_1k_1^2\left\{\frac{2-k_1^2}{3k_1^4}E_1-\frac{2-2k_1^2}{3k_1^4}F_1\right\}$$
$$-R_2\left(1-\frac{k_2^2}{c^2}\right)\left\{\frac{1-c^2}{c^2}(F_2-\Pi_2)+\frac{1}{k_2^2}(F_2-E_2)\right\}$$
$$\left.-R_2k_2^2\left\{\frac{2-k_2^2}{3k_2^4}E_2-\frac{2-2k_2^2}{3k_2^4}F_2\right\}\right]$$

$$
\begin{aligned}
=4\pi ab\frac{N_1N_2}{h_1h_2}\Big[&R_1\left(1-\frac{k_1^2}{c^2}\right)\left(\frac{1-c^2}{c^2}\right)(F_1-\Pi_1)\\
&+R_1\left(\frac{1}{3k_1^2}-\frac{1}{c^2}+\frac{1}{3}\right)(F_1-E_1)+\frac{1}{3}R_1F_1\\
&-R_2\left(1-\frac{k_2^2}{c^2}\right)\left(\frac{1-c^2}{c^2}\right)(F_2-\Pi_2)\\
&-R_2\left(\frac{1}{3k_2^2}-\frac{1}{c^2}+\frac{1}{3}\right)(F_2-E_2)-\frac{1}{3}R_2F_2\Big]. \quad (38)
\end{aligned}
$$

By the aid of (36), therefore, and the tables given in Legendre's treatise, M may be calculated by this formula to any desired accuracy. If a four-figure accuracy suffice, the tables given on pp. 68–75 of Dale's 'Mathematical Tables' will be found ample.

It is to be noticed that provided a, b, h_1 and h_2 remain the same, the formula for the mutual inductance between a helical current and a cylindrical current sheet may be written in the form mN_1N_2 where m is a constant. By considering the case, therefore, of N_1 infinitely great, that is, when the helix is practically identical with a cylindrical current sheet of the same axial length and diameter, we see that the formula for two cylindrical current sheets must also be mN_1N_2. We thus see that the formulæ (29) and (38) must be identical.

It has to be remembered that the above formula needs to be modified when both the coils are helical. If the pitch of *one of the helices* be a very small quantity the correcting factor will be small. It is advisable, however, when actually measuring the mutual inductance to investigate whether rotating * one of the helices round the axis through various angles alters the mutual inductance.

Even when Legendre's tables are accessible, the evaluation of (38) is laborious. We shall, therefore, give a series formula for M which is convenient in many cases. By the binomial theorem, we always have

$$
\begin{aligned}
&\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{1-c^2\sin^2\theta}(1-k_1^2\sin^2\theta)^{1/2}\,d\theta\\
&\qquad=P_1-\tfrac{1}{2}P_2k_1^2-\frac{1\,.\,1}{2\,.\,4}P_3k_1^4-\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}P_4k_1^6-\dots,
\end{aligned}
$$

where

$$P_n = \int_0^{\pi/2} \frac{\sin^{2n}\theta \cos^2\theta}{1-c^2\sin^2\theta}\, d\theta$$

$$= \frac{P_{n-1}}{c^2} - \frac{1}{c^2}\int_0^{\pi/2} \sin^{2n-2}\theta \cos^2\theta \, d\theta$$

$$= \frac{P_{n-1}}{c^2} - \frac{1}{2n} \cdot \frac{1.3.5\ldots.2n-3}{2.4.6\ldots.2n-2} \cdot \frac{\pi}{2} \cdot \frac{1}{c^2}. \quad . \quad . \quad (39)$$

Now it is easy to show that

$$P_1 = \frac{\pi}{4(1+\sqrt{1-c^2})^2} = \frac{\pi}{16} \cdot \frac{(a+b)^2}{a^2},$$

and if we denote P_n/P_1 by q_n we get

$$q_n = \frac{(a+b)^2}{4ab} q_{n-1} - \frac{1}{n} \cdot \frac{1.3.5\ldots.2n-3}{2.4.6\ldots.2n-2} \cdot \frac{a}{b}. \quad . \quad (40)$$

Thus the successive values of q_n can be readily computed.

Since P_n is always less than P_{n-1}, and k_1 is always less than unity, it follows that the series given above is always convergent. The formula for M can, therefore, be written in the form

$$M = \pi^2 b^2 \frac{N_1 N_2}{h_1 h_2}\Big[R_1\Big\{1 - \frac{1}{2} q_2 k_1^2 - \frac{1.1}{2.4} q_3 k_1^4 - \ldots\Big\}$$

$$- R_2\Big\{1 - \frac{1}{2} q_2 k_2^2 - \frac{1.1}{2.4} q_3 k_2^4 - \ldots\Big\}\Big], \quad (41)$$

where $R_1^2 = (a+b)^2 + (h_1+h_2)^2$; $k_1^2 = 4ab/R_1^2$,
and $R_2^2 = (a+b)^2 + (h_1-h_2)^2$; $k_2^2 = 4ab/R_2^2$.

This formula gives a complete practical solution of the problem.

Let us take the case of the two cylindrical coils for which $a=5$, $b=4$, $h_1=15$, $h_2=2{\cdot}5$, $N_1=300$, and $N_2=200$, mentioned in the preceding section. By (39) we easily obtain

$q_2=0{\cdot}7$, $q_3=0{\cdot}5525$, $q_4=0{\cdot}4618$, $q_5=0{\cdot}3992$,
$q_6=0{\cdot}3527$, $q_7=0{\cdot}3170$, &c.

* Lord Rayleigh, British Association Report (1899), p. 241.

Also $R_1^2=(a+b)^2+(h_1+h_2)^2=387{\cdot}25$; $R_1=19{\cdot}6787$;
$R_2^2=(a+b)^2+(h_1-h_2)^2=237{\cdot}25$; $R_2=15{\cdot}4029$;
$k_1^2=80/387{\cdot}25$; $k_2^2=80/237{\cdot}25$.

Thus, substituting in (41), we have

$$
\begin{aligned}
M = \frac{(4\pi)^2\, 6\,.\,10^4}{37{\cdot}5}\big[19{\cdot}6787(1&-0{\cdot}07230-0{\cdot}00295\\
&-0{\cdot}00026-0{\cdot}00003-\ldots)\\
-15{\cdot}4029(1&-0{\cdot}11802-0{\cdot}00785\\
&-0{\cdot}00111-0{\cdot}00020\\
&-0{\cdot}00004-0{\cdot}00001-\ldots)\big]\,10^{-9}
\end{aligned}
$$

$=0{\cdot}0011999$ henry,

which agrees with the answer found from Searle's formula.

When $h_1=h_2=h$,

$$k_1^2=4ab/\{(a+b)^2+4h^2\} \quad \text{and} \quad k_2^2=4ab/(a+b)^2=c^2.$$

In this case the series of terms with the suffix 2 in (41) may converge slowly. It is better, therefore, to express the second half of the formula on the right-hand side of (41) in terms of elliptic integrals. The formula * now becomes :

$$M=\pi^2b^2\frac{N_1N_2}{h^2}\left[R_1\left\{1-\frac{1}{2}q_2k_1^2-\frac{1\,.\,1}{2\,.\,4}q_3k_1^4-\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}q_4k_1^6-\ldots\right\}\right]$$

$$+\frac{8\pi ab}{3(a+b)}\cdot\frac{N_1N_2}{h^2}[(a^2+b^2)(F-E)-2abF], \quad (42)$$

where the modulus of the elliptic functions is $2(ab)^{1/2}/(a+b)$.

For the calculation of the mutual inductance standards required for calibrating the galvanometers used in iron-testing, this formula is suitable. In order that the coils be as compact as possible, it is necessary to make $h=a$ and b large, but even in this case the series can be easily summed.

Let us now consider the case when the cylinder has the same radius as the helix. We may suppose, for instance, that both coils are wound on the same insulating cylinder. The

* For another formula in terms of incomplete elliptic integrals, see J. G. Coffin, p. 124, *l. c. ante*.

formula (37) now becomes

$$M = 4\pi a^2 \frac{N_1 N_2}{h_1 h_2} \int_0^{\pi/2} \sin^2\theta \left[R_1(1-k_1^2\sin^2\theta)^{1/2} - R_2(1-k_2^2\sin^2\theta)^{1/2}\right] d\theta.$$

Hence, from (8),

$$M = 4\pi a^2 \frac{N_1 N_2}{h_1 h_2}\left[R_1 \frac{2k_1^2-1}{3k_1^2}E_1 + R_1\frac{1-k_1^2}{3k_1^2}F_1 - R_2\frac{2k_2^2-1}{3k_2^2}E_2 - R_2\frac{1-k_2^2}{3k_2^2}F_2\right], \quad (43)$$

where

$$R_1^2 = 4a^2+(h_1+h_2)^2, \quad R_2^2 = 4a^2+(h_1-h_2)^2,$$
$$k_1^2 = 4a^2/R_1^2, \quad \text{and} \quad k_2^2 = 4a^2/R_2^2.$$

The value of M can be readily found from this formula with the help of Dale's Mathematical Tables or by the series for E and F given in § 2.

If we have, in addition, $h_1 = h_2 = h$, then

$$R_1 = 2(a^2+h^2)^{1/2}, \; R_2 = 2a, \; k_1^2 = a^2/(a^2+h^2), \; \text{and} \; k_2^2 = 1.$$

Hence

$$M = \frac{8\pi a^2}{3h}N_1 N_2\left[\left(1+\frac{a^2}{h^2}\right)^{1/2}\left\{E_1 + \frac{h^2}{a^2}(F_1 - E_1)\right\} - \frac{a}{h}\right]. \quad (44)$$

If the height $2h$ equal the diameter $2a$, so that the coils are as compact as possible, we may write

$$M = 13{\cdot}589\, N_1 N_2 a. \quad \ldots \ldots \quad (45)$$

9. *Formulæ for the self-inductance of a helical current when the pitch of the helix is small.*

If we have a wire closely wound on a smooth insulating cylinder, the diameter of the wire being small compared with that of the cylinder, then the mutual inductance between it and a cylindrical current sheet of the same radius as the helix formed by the axis of the wire and having the same axial length, will be approximately equal to the self-

inductance L of the coil. Hence, by (44), we find that

$$L = \frac{8\pi a^2}{3h} N^2 \left[\left(1+\frac{a^2}{h^2}\right)^{1/2} \left\{E_1 + \frac{h^2}{a^2}(F_1 - E_1)\right\} - \frac{a}{h}\right], \quad (46)$$

where N is the number of turns of the helix and $a/(a^2+h^2)^{1/2}$ is the modulus of the elliptic functions.

From a paper by Rosa*, the author has recently discovered that this formula has been previously given by Lorenz †. Before finding this out, however, he had given the formula to his students, and several of them had constructed standards of self-inductance by simply turning cylinders of well-seasoned teak in a lathe and winding cotton-covered wire round them. The agreement between measurement and calculation was found quite satisfactory.

When h is less than a, we get by the Binomial Theorem and (1) and (2),

$$L = 4\pi a N^2 \left[\log\frac{4a}{h} - \frac{1}{2} + \frac{h^2}{8a^2}\left(\log\frac{4a}{h} + \frac{1}{4}\right)\right]. \quad . \quad (47)‡$$

This is in exact agreement with a formula previously given by Lord Rayleigh (‘Scientific Papers,’ vol. ii. p. 15).

When h equals $(3/4)a$, the number found by this formula is too small by about the quarter of one per cent. For smaller values of h the numbers found by (46) and (47) are in practical agreement. When h is small, a small percentage error in determining it introduces a large percentage error into the calculated value of L. Hence, in making standards, it is advisable to have h not less than $2a$. In this case we can use the following remarkably simple formula :

$$L = \frac{2\pi^2 a^2}{h} N^2 \left[1 - \frac{4a}{3\pi h} + \frac{a^2}{8h^2} - \frac{a^4}{64h^4}\right]. \quad . \quad . \quad (48)$$

When $h=2a$, the inaccuracy of the formula (48) as compared with the more accurate elliptic integral formula is less

* Bull. Bureau of Standards, p. 162, Aug. 1906.

† Wied. *Ann.* vii. p. 161 (1879).

‡ The much more elaborate formula given by Coffin (*l. c. ante*, p. 113) may be deduced in the same way from (46).

than 1 in 8000. If h be greater than $2a$, therefore, it is sufficiently accurate for all practical purposes.

It is convenient to write

$$L = \beta(a^2/h)N^2, \quad \ldots \ldots \quad (49)$$

and tabulate the values of β for various values of h/a :—

h/a	1/100.	1/10.	1/5.	1/4.	1/2.	3/4.	1.	2.
β	0·6901	4·013	6·313	7·214	10·38	12·34	13·59	16·15
h/a	3.	4.	5.	6.	7.	8.	9.	10.
β	17·22	17·80	18·16	18·41	18·59	18·73	18·84	18·91

When h is greater than $40a$ we may write

$$L = 19{\cdot}74(1-4a/3\pi h)(a^2/h)N^2; \quad \ldots \quad (50)$$

and when h is greater than $400a$, we get the well-known formula

$$L = 19{\cdot}74(a^2/h)N^2.$$

[*Note added April* 15, 1907.]—Kirchhoff * in 1864 found a series formula for the self-inductance of a helical current by supposing that it is exactly equivalent to a series of parallel circular currents †, and Strasser ‡ has elaborated the formula. Rosa § has worked out numerical examples, and it is interesting to find that in the cases considered the difference between the results computed by the series and the Lorenz formula is small. It must be noticed, however, that for very thin wires wound on large cylinders the two formulæ give very different results. The author has not yet worked out the exact formula taking the helicity into

* *Ges. Abh.* p. 176.

† The strength of the magnetic field at all points on the axis of the coil is the same in the two cases but it is not the same at all other points.

‡ *Ann. d. Phys.* xvii. p. 763, 1905.

§ *L. c. ante.*

account, but judging from his experimental results he considers the Lorenz formula more accurate than the series formula when the wires are thin and the diameter of the helix is large.

The following description of two cylindrical coils wound by the author's assistant, Mr. J. N. Alty, is of interest as it shows that a mutual inductance standard having a maximum inaccuracy of less than the half of one per cent. may be constructed in a few hours. Both coils are wound on ordinary glass cylinders similar to those supplied for instrument covers. The mean diameter of the outer cylinder is 20·50 cm., and of the inner 19·03. The outer cylinder is wound with 174 turns of No. 20 double silk-covered copper wire, and the inner with 289 turns of No. 24 double silk-covered copper wire. The axial length ($2h$) of each coil is 19·0 cm., and the diameters ($2a$ and $2b$) of the two helical axes of the wires are 20·60 and 19·09 cm. respectively. By (42) we find that

$$M = 0{\cdot}00624 \text{ henry,}$$

and this was found to agree within the half of one per cent. with the value found by comparing it with a standard condenser.

The self-inductance coefficients of the two cylindrical coils can also be readily calculated by (46). We find that

$$L_1 = 0{\cdot}00447 \quad \text{and} \quad L_2 = 0{\cdot}0106 \text{ henry.}$$

These numbers agree within one per cent. with the measured values.

The leakage factor σ of the coils is given by

$$\sigma = 1 - M^2/L_1L_2 = 0{\cdot}177.$$

The author finds it useful when experimenting with Duddell currents to make the coils of his air-core transformers cylindrical in shape, as the inductance coefficients can then be readily calculated.

Discussion.

Mr. A. Campbell said the accurate calculation of mutual inductances was of great practical importance because it formed the basis of several of the chief methods of determining the absolute electrical units: it was therefore highly

satisfactory that Mr. Russell had succeeded in obtaining the complete solution for the most general case required. The author's series solution (41) was a most interesting one. When the length of one of the coils was very small compared with that of the other the series became practically o/o. This was the case used by Prof. Jones. Mr. Campbell suggested that from the author's series formula a suitable series might be obtained by differentiating with regard to h_2 (omitting the h_2 in the denominator). In calculating mutual inductance by Mr. Russell's complete formula (38) or by that of Prof. Viriamu Jones, it was necessary to find the incomplete integrals $E(c, \phi)$ and $F(c, \phi)$. These could often be found to a high degree of accuracy by the following two formulæ due to Legendre, viz.:

$$E(c, \phi) = \frac{1}{c}\sin c\phi + \frac{l^2c^2}{30}\phi^5 - \frac{l^2c^2}{1260}\phi^7(4-11c^2),$$

and

$$F(c, \phi) = \frac{1}{c}\log_e\left(\frac{\pi}{4} + \frac{c\phi}{2}\right) - \frac{l^2c^2}{30}\phi^5 + \frac{l^2c^2}{1260}\phi^7(4-41c^2),$$

where $l=1-c^2$. The first formula is accurate to 7 decimal places if $\phi<30°$ and the second to a similar accuracy if $\phi<20°$ or if $\phi<30°$ when $c<\sin 35°$ or $>\sin 75°$.

Mr. A. Russell, in reply, after thanking Mr. Campbell for his remarks, called attention to the important relations between the equations of vorticity in hydrodynamics and the equations of electrodynamics. He had experienced difficulty in conceiving the nature of the stress in the medium surrounding an electric current. Was the energy stored in the medium when a current was flowing kinetic or potential? Some physicists spoke as if the state of stress in the medium was due to streams of ether whirling round the electric current; others, that it was due to a compression of the medium.

Fig. 1.

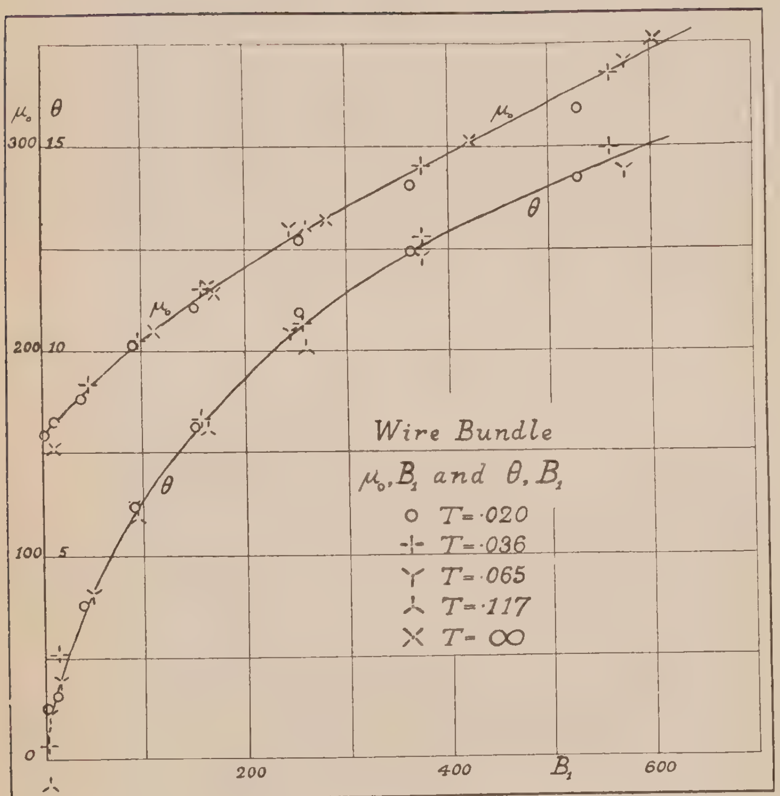


Fig. 2.

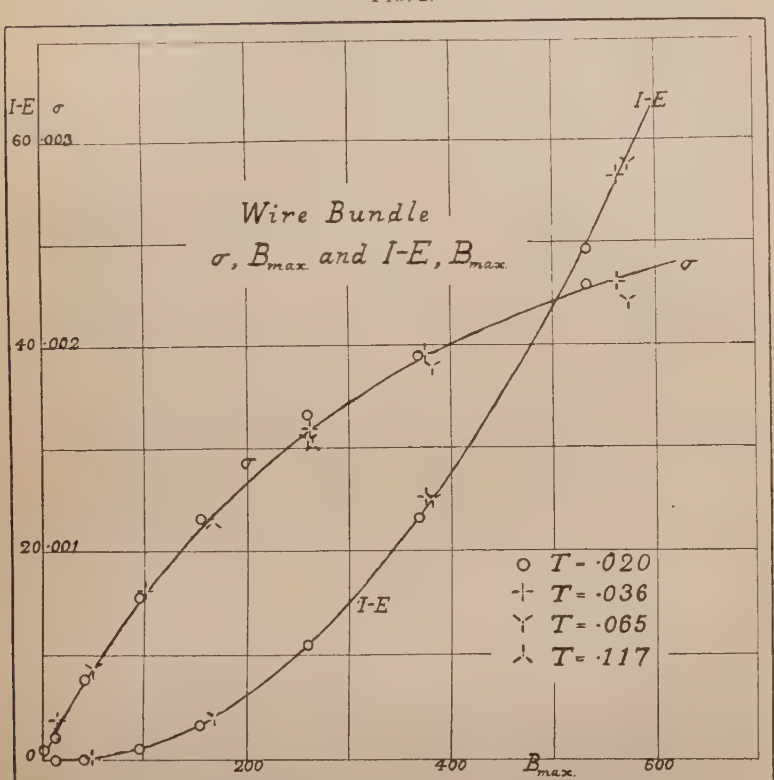


Fig. 3.

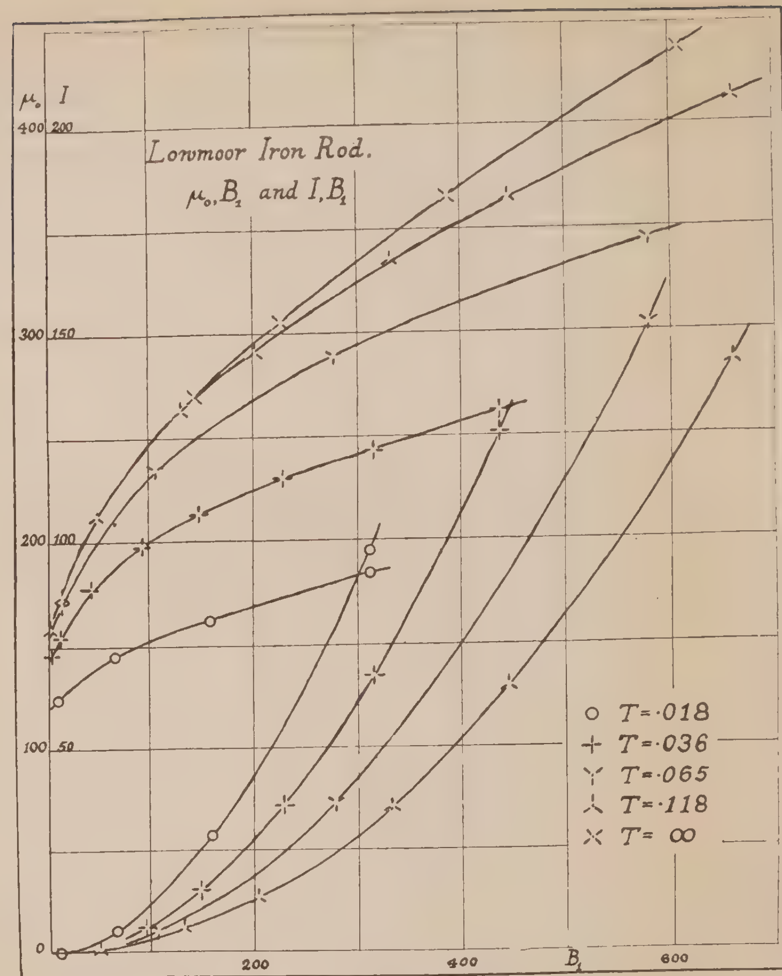


Fig. 4.

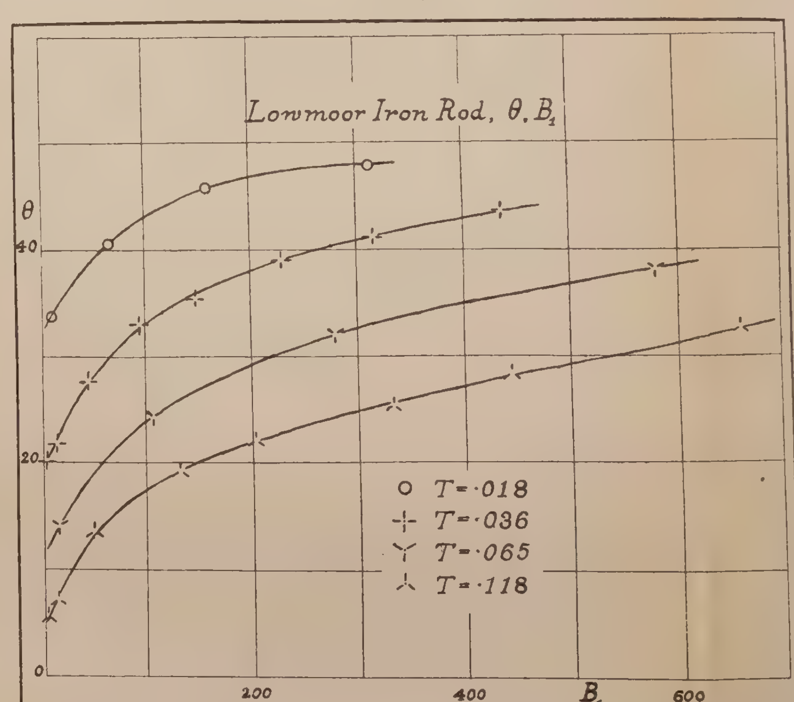


Fig. 5.

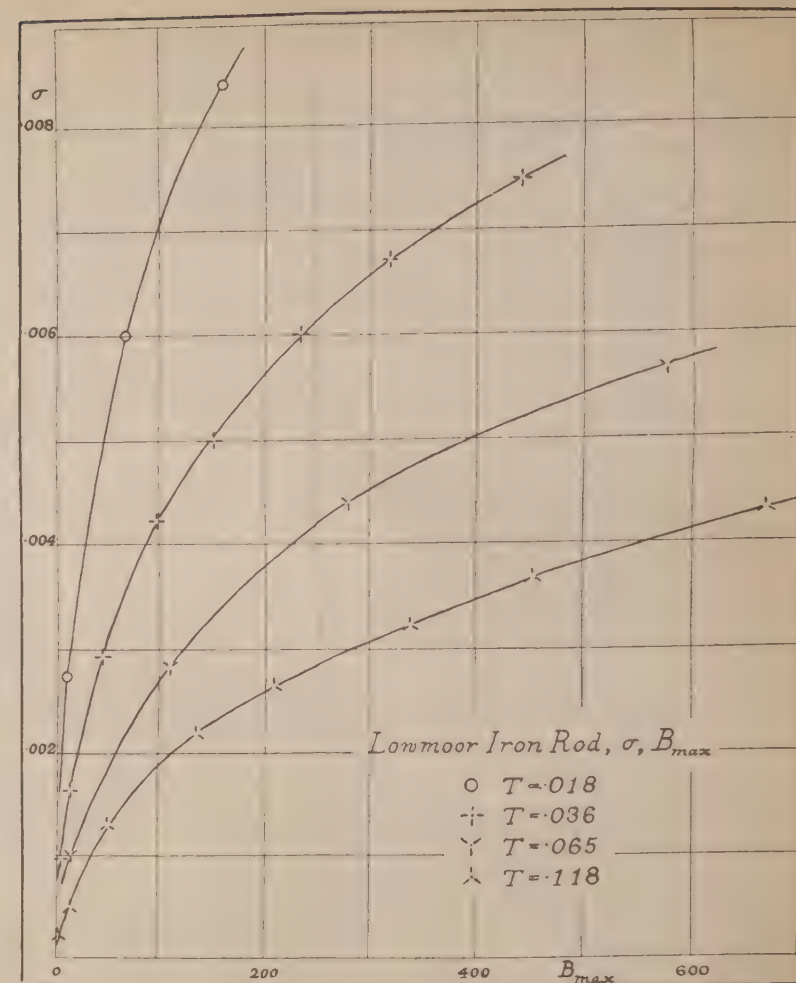


Fig. 6.

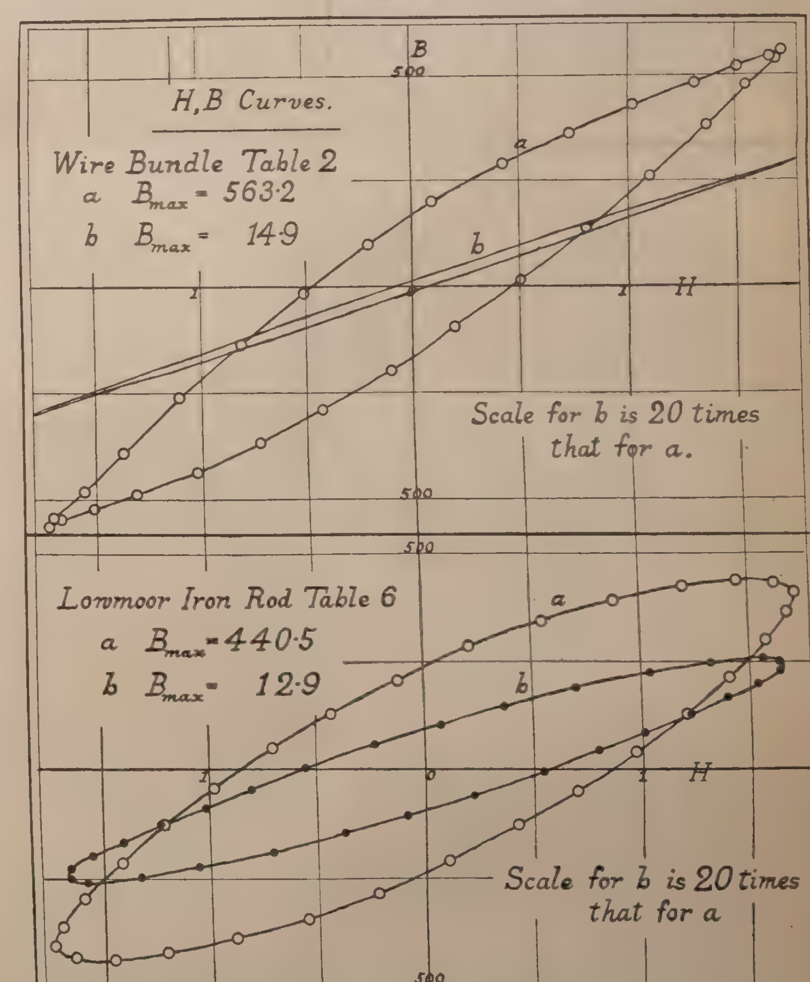
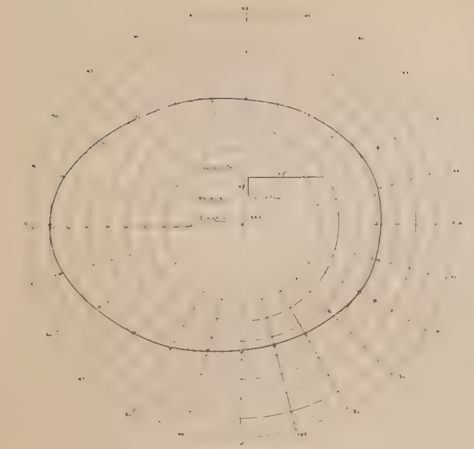


Fig. 1.

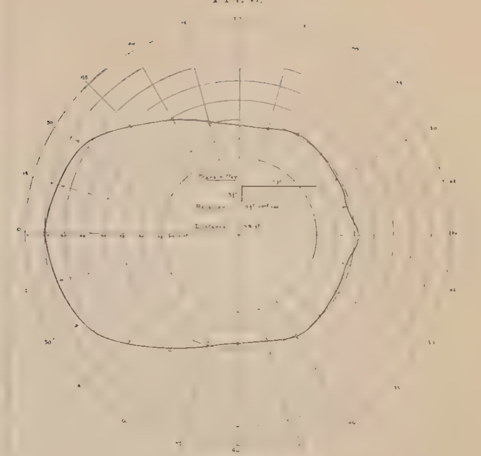
Transmitter 5 : 15.

Fig. 2.



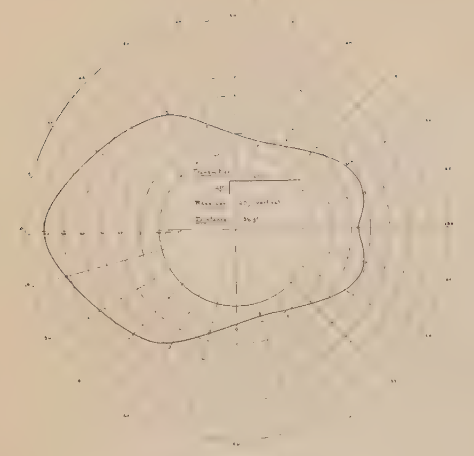
Transmitter 4 : 16.

Fig. 3.



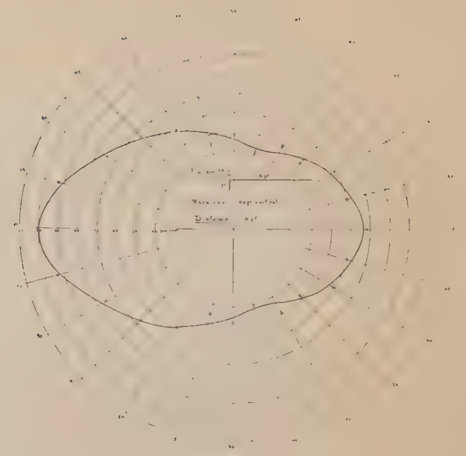
Transmitter 3 : 17.

Fig. 4.



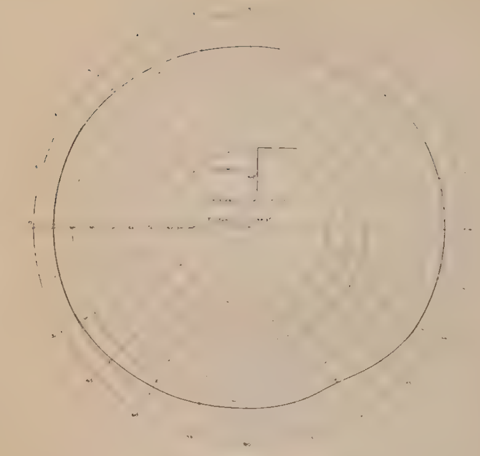
Transmitter 2 : 18.

Fig. 5.



Transmitter 1 : 19.

Fig. 6.

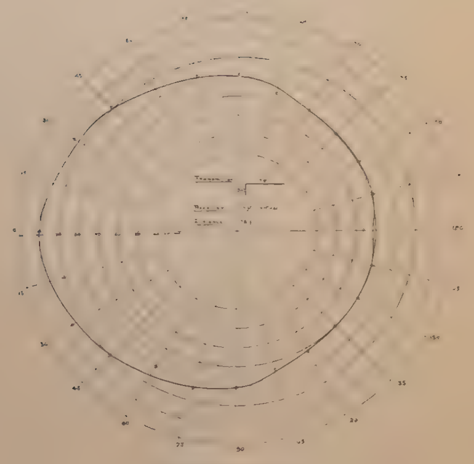


Transmitter 10 : 10

Fig. 7.

Transmitter 3 : 7.

Fig. 8.



Transmitter 3 : 7.

Fig. 9.

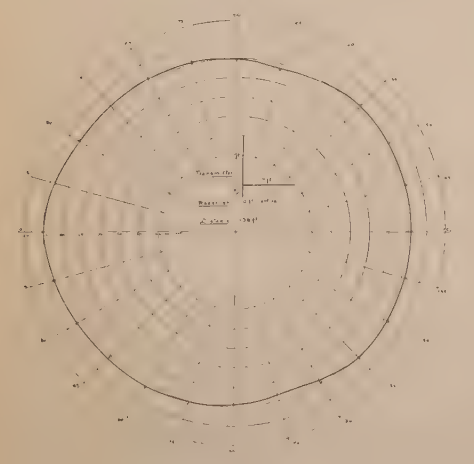


Fig. 10.

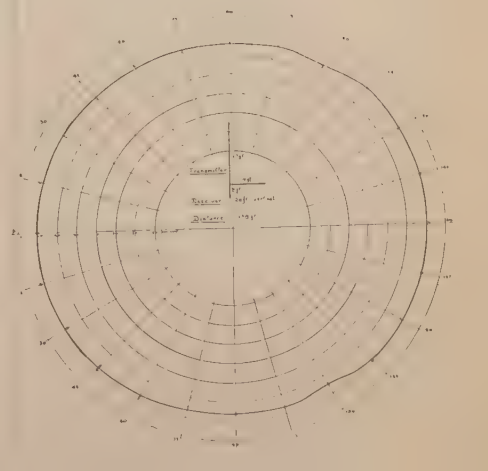


Fig. 11.

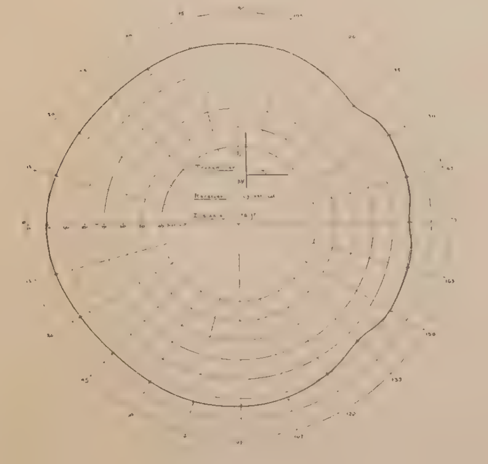


Fig. 12.

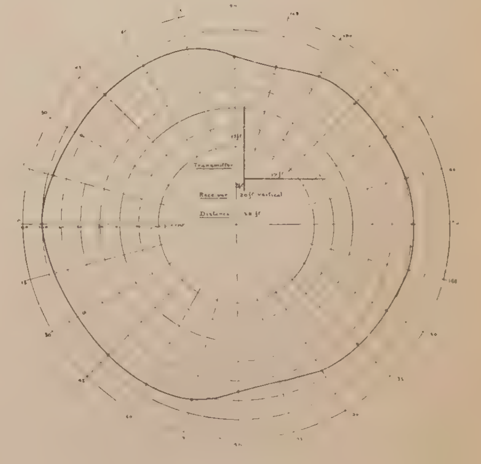
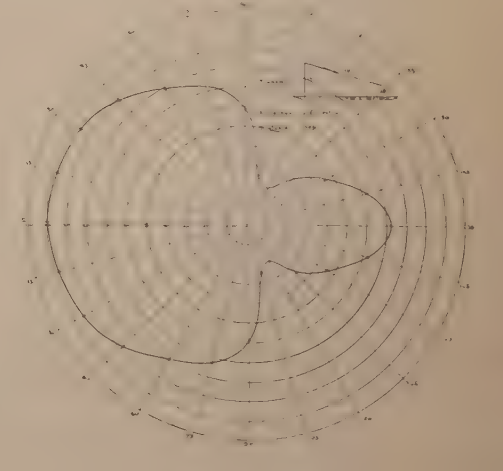


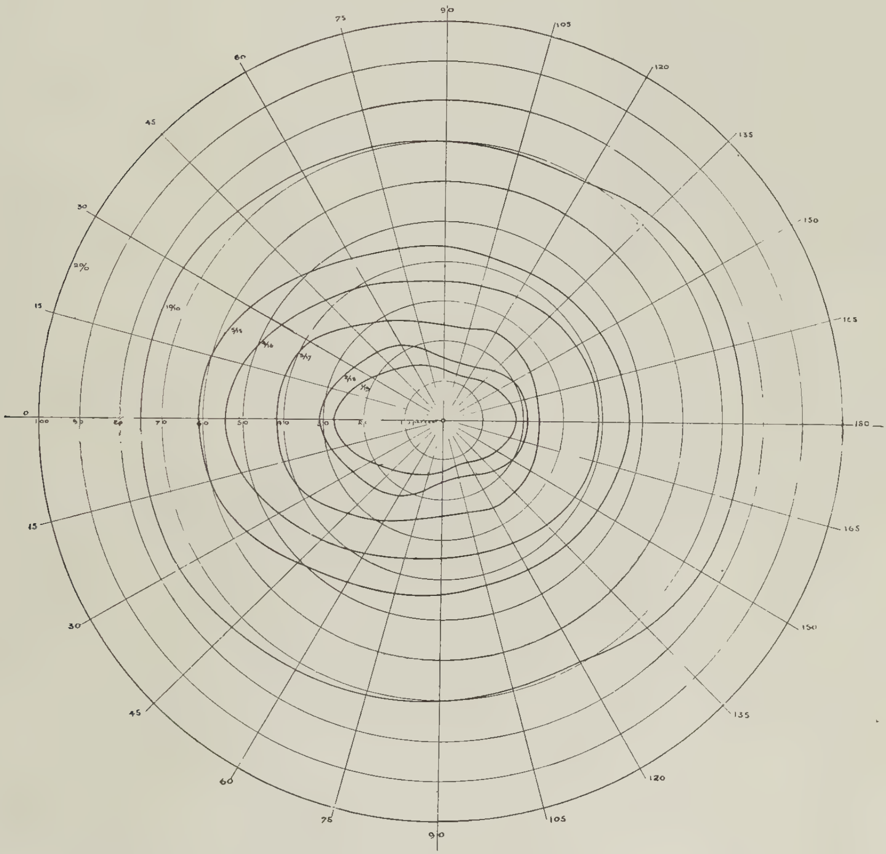
Fig. 13.

Fig. 14.



EXPERIMENTS WITH BENT ANTENNÆ.

TRANSMITTING ANTENNA. Total length 20 ft.
RECEIVING „ Height 21 ft.
DISTANCE. 138 ft.

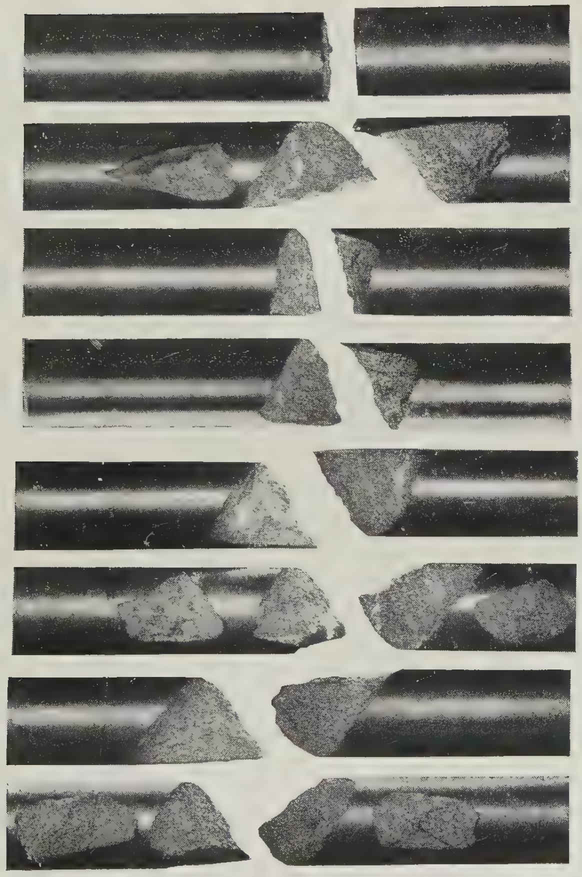


The radii of the curves are proportional to the currents in the receiving antenna when the bent transmitter wire has its vertical to horizontal parts in the ratio indicated by the fractions 10/10, 5/15, 4/16, 3/17, 2/18, 1/19 marked on the curves: the receiver current when the transmitter is vertical is taken as 100.

The zero position of the transmitter is that in which the horizontal part is directed away from the receiver.

SCOBLE.

FIG. 2.



Test

1

2

3

4

5

6

7

8

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1906-1907.

February 23rd, 1906.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

The following Candidate was elected a Fellow of the Society :—
E. M. Modi.

The following Papers were read :—

1. A Note on Talbot's Lines. By Mr J. Walker.

2. Secondary Röntgen Radiation. By Dr C. G. Barkla.

3. Records of the difference of potential between railway-lines when a train passes and at other times, and a suggested method for the observations of earth currents and magnetic variations. By Messrs C. W. S. Crawley and F. B. O. Hawes.

March 9th, 1906.

Meeting held at the Royal College of Science.

Dr Chree, Vice-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—

H. C. Silver.

The following Papers were read:—

1. The Velocities of the Ions of Alkali Salt Vapours at High Temperatures. By Prof. H. A. Wilson.

2. Some Experiments on Earth-Currents at Kew Observatory. By Dr J. A. Harker.

March 23rd, 1906.

Meeting held at University College.

Prof. Perry, President, in the Chair.

The following Papers were read:—

1. Unilateral Electric Conductivity over Damp Surfaces. By Prof. F. T. Trouton.

2. The Construction and Use of Oscillation Valves for Rectifying High-Frequency Electric Currents. By Prof. J. A. Fleming.

3. The Use of the Cymometer for the Determination of Resonance Curves. By Mr G. B. Dyke.

April 27th, 1906.

Meeting held at the Royal College of Science.

Dr Chree, Vice-President, in the Chair.

The following Papers were read:—

1. Some Simple Questions on the Images of Microscopes, Telescopes, and other problems. By Mr W. B. Croft.

2. The Lateral Vibration of Bars subjected to Forces in the direction of their Axes. By Mr J. Morrow.

Dr C. V. Boys gave a description of a Gas Calorimeter which he had designed, and which was shown in action.

May 11th, 1906.

Meeting held at the Royal College of Science.

Dr CHREE, Vice-President, in the Chair.

The following Candidate was elected a Fellow of the Society:—

J. MORROW.

The following Paper was read:—

The Dead Points of a Galvanometer Needle for Transient Currents. By Mr A. RUSSELL.

Prof. H. A. WILSON exhibited a Lippmann Capillary Electromotor and Dynamo.

Mr DUDDELL exhibited an apparatus for demonstrating the Movements of the Diphragms in telephone transmitters and receivers and the currents flowing.

May 25th, 1906.

Meeting held at the Royal College of Science.

Dr CHREE, Vice-President, in the Chair.

The following Papers were read:—

1. Colour Phenomena in Photometry. By Mr J. S. Dow.

2. Theory of Moving-Coil and other kinds of Ballistic Galvanometers. By Prof. H. A. WILSON.

Mr SKINNER exhibited an Automatic Arc Lamp on behalf of Mr H. Tomlinson and the Rev. G. T. Johnstone.

Mr A. CAMPBELL exhibited and described a Bifilar Galvanometer free from zero creep.

June 8th, 1906.

Meeting held at the Royal College of Science.

Prof. PERRY, President, in the Chair.

The following Candidate was elected a Fellow of the Society:—

W. H. WHITE.

The following Papers were read:—

1. On the Solution of Problems in Diffraction by the Aid of Contour Integration. By Mr H. DAVIES.

2. Fluid Resistance. By Col. de Villamil.

Mr Goold's Experiments with a Vibrating Steel Plate were exhibited by Messrs. Newton & Co.

June 22nd, 1906.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

The following Candidate was elected a Fellow of the Society:—
E. Beckwith.

The following Papers were read:—

1. The Effect of Radium in facilitating the visible electric discharge in vacuo. By Mr A. A. C. Swinton.

2. The Effect of the Electric Spark on the Actinity of Metals. By Mr T. A. Vaughton.

3. The Dielectric Strength of Thin Liquid Films. By Dr P. E. Shaw.

4. The Effect of Electrical Oscillations on Iron in a Magnetic Field. By Dr W. H. Eccles.

October 26th, 1906.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

The following Papers were read:—

1. The Strength and Behaviour of Ductile Materials under Combined Stress. By Mr W. A. Scoble.

2. The Behaviour of Iron under Weak Periodic Magnetizing Forces. By Mr J. M. Baldwin.

3. Fluorescence and Magnetic Rotation Spectra of Sodium Vapour and their Analysis. By Prof. R. W. Wood.

November 9th, 1906.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

Mr G. F. C. Searle exhibited and described a number of apparatus for Students' practical work in Physics.

November 23rd, 1906.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

The following Candidates were elected Fellows of the Society:—

E. I. Spiers and W. G. Swann.

The following Papers were read:—

1. The Electric Radiation from Bent Antennæ. By Dr J. A. Fleming.

2. Auroral and Sun-Spot Frequencies Contrasted. By Dr C. Chree.

3. The Electrical Resistance of Alloys. By Dr R. S. Willows.

December 14th, 1906.

Meeting (informal) held at the Royal College of Science.

An Exhibition of Apparatus was given by the following firms:— Messrs R. & J. Beck, Ltd.; The Cambridge Scientific Instrument Co.; Casella & Co.; Crompton & Co.; Elliott Bros.; Everett, Edgcumbe & Co.; Evershed & Vignoles; Gambrell Bros.; A. Hilger; Nalder Bros. & Thompson; Newton & Co.; R. W. Paul; James Pitkin & Co.; Reynolds & Branson, Ltd.; Ross, Ltd.; J. Swift & Son; Dr W. M. Watts; Alexander Wright & Co.; and Carl Zeiss.

January 25th, 1907.

Meeting held at the Royal College of Science.

Prof. Perry, President, in the Chair.

The following Candidates were elected Fellows of the Society :—
S. G. Burrow, C. Lean, and R. S. Willows.

The following Papers were read :—

1. The Strength and Behaviour of Brittle Materials under Combined Stress. By Mr W. A. Scoble.

2. Recent Improvements in Spectrophotometers. By Mr F. Twyman.

A number of photographs of Electric Discharge were shown by Mr K. J. Tarrant.

Annual General Meeting.

February 8th, 1907.

Meeting held at the Royal College of Science.

The President in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting twelve ordinary Science Meetings and one informal Meeting of the Society have been held. Of these, twelve were held at the Royal College of Science, and one, that on March 23rd, was held at University College. The average attendance at the Meetings has been 50.

The second Annual Exhibition of Apparatus by Manufacturers was held on December 14th, when the attendance of Fellows and Visitors amounted to 200.

The number of Fellows now on the roll is 426. Twelve new Fellows have been elected, and two Honorary Fellows, namely, Prof. F. Kohlrausch and Prof. A. A. Michelson. There have been 10 resignations, and the Society has to mourn the loss by death of two Honorary Fellows, Prof. S. P. Langley and Prof. Ludwig Boltzmann, and two Fellows, namely, A. G. Bessemer, Junr., and Prof. J. Purser.

The Council are pleased to report that arrangements have been completed whereby Fellows now have the privilege of obtaining Section A of the 'Proceedings' of the Royal Society at a reduced rate.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The following Candidates were elected Honorary Fellows of the Society:—

GABRIEL LIPPMANN and SIMON NEWCOMB.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. J. PERRY, F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; The Lord KELVIN, D.C.L., LL.D., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Prof. W. E. AYRTON, F.R.S.; Prin. Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.; SHELFORD BIDWELL, M.A., LL.B., F.R.S.; Prin. Sir OLIVER J. LODGE, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.; R. T. GLAZEBROOK, D.Sc., F.R.S.; Prof. J. H. POYNTING, M.A., D.Sc., F.R.S.

Vice-Presidents.—C. CHREE, Sc.D., LL.D., F.R.S.; H. M. ELDER, M.A.; Prof. J. A. FLEMING, M.A., D.Sc., F.R.S.; W. WATSON, D.Sc., F.R.S.

Secretaries.—W. R. COOPER, M.A.; Prof. W. CASSIE, M.A.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—Prof. H. L. CALLENDAR, M.A., LL.D., F.R.S.

Librarian.—W. WATSON, D.Sc., F.R.S.

Other Members of Council.—T. H. BLAKESLEY, M.A.; A. CAMPBELL, B.A.; W. DUDDELL, F.R.S.; W. H. ECCLES, D.Sc.; A. GRIFFITHS, D.Sc.; J. A. HARKER, D.Sc.; T. MATHER, F.R.S.; A. RUSSELL, M.A.; S. SKINNER, M.A.; S. W. J. SMITH, M.A.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Board of Education.

The President then delivered his Address.

Treasurer's Report.

The difficulty of collecting arrears of subscriptions, alluded to in the report for 1905, has been severely felt during the past year. The Arrears now stand at a higher figure than for some time past. Many correspondents attribute their inability to keep up to date to the long continued prevalence of a high income tax, which hits many members of our Society rather seriously. With this exception the position of the Society is fairly prosperous. The value of the invested funds has not appreciably declined, the Bank balance shows a substantial increase, and there are no outstanding liabilities.

HUGH L. CALLENDAR,
Hon. Treasurer.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY 1ST, 1906, TO DECEMBER 31ST, 1906.

Dr.	£ s. d.	£ s. d.
Jan. 1, 1906. Balance in Bank		119 17 2
Entrance-Fees	12 12 0	
Subscriptions	375 18 0	
Extra Subscriptions	31 8 0	
Arrears	39 18 0	
Abstracts etc.	8 6 7	
Students	1 11 6	
Donation (Exors. of M. Gray)	10 10 0	
		480 4 1
Dividends:—		
Furness Debenture Stock	15 3 10	
Metropolitan Board of Works	6 13 0	
Lancaster Corporation Stock	11 8 0	
New South Wales	8 9 0	
Midland Railway	38 0 0	
Great Eastern Railway	19 0 0	
London, Brighton, & South Coast Railway	26 14 5	
		125 8 3
Sales of Publications		42 3 0
		£767 12 6

Cr.	£ s. d.	£ s. d
Science Abstracts		276 12 0
Taylor and Francis:—		
Proceedings	102 13 9	
Abstracts	77 13 1	
Distribution	15 3 6	
Separate Copies	14 4 0	
		209 14 4
Periodicals		3 9 0
Attendance		6 10 0
Reporting (2 years)		61 14 2
Royal Asiatic Society (2 years)		4 4 0
Binding		1 0 0
Account Book		10 0
Petty Cash, Treasurer and Secretaries		19 7 9
Refreshments Dec. 14th		6 11 4
Subscription Overpaid and Returned		1 1 0
Cheque Book and Commissions		11 4
Balance in Bank		176 7 7
		£767 12 6

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

11*th January*, 1907.

ED. F. HERROUN,
CHARLES SELBY WHITEHEAD, } *Auditors.*

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, December 31, 1906.

Assets.	£	s.	d.	Liabilities.	£	s.	d.
Subscriptions due, Treasurer's estimate	70	0	0	Balance	4101	14	1
£533 Furness Ry. Co. 3 per cent. Debenture Stock	453	0	0	Cheque not presented:—			
£1600 Midland Railway $2\frac{1}{2}$ per cent. Preference Stock	1152	0	0	Treasurer, Petty Cash	1	13	6
£200 Metropolitan Board of Works $3\frac{1}{2}$ per cent. Consolidated Stock	203	0	0				
£400 Lancaster Corporation 3 per cent. Redeemable Stock	340	0	0				
£284 2*s.* 9*d.* New South Wales $3\frac{1}{2}$ per cent. Inscribed Stock	254	0	0				
£500 London, Brighton, and South Coast Railway Ordinary Stock	670	0	0				
£500 Great Eastern Railway 4 per cent. Debenture Stock	585	0	0				
Publications in Stock, estimated	200	0	0				
Balance in Bank	176	7	7				
	£4103	7	7		£4103	7	7

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

11*th January*, 1907.

ED. F. HERROUN,
CHARLES SELBY WHITEHEAD, } *Auditors.*

Presidential Address to THE PHYSICAL SOCIETY OF LONDON,

8th February, 1907.

PROFESSOR JOHN PERRY, D.Sc., F.R.S.

I STARTED a discussion at Johannesburg in August 1905 on the teaching of Elementary Mechanics. An account of the discussion has been published, supplemented by written remarks from men who were not present at Johannesburg *.

In my opening remarks I advocated the use of engineers' units and also of the term "Centrifugal Force." I may say that all the speakers and writers were in fair agreement with me except in regard to these two things; and as I thought that our differences were due to mere misunderstanding, I tried in a reply to put forward my ideas more clearly than I had done before. May I venture to say that these ideas never have had a fair hearing. Nobody who gives a fair consideration to the business can deny that engineers' units based on the weight of 1 lb. in London as the unit of force, are as absolute as any other units employed by scientific men; and every man who has done *advanced* work in Kinetics must recognize the usefulness of the term "Centrifugal Force." All writers of advanced treatises, Dr. Love, Dr. Routh, Dr. Chree, Mr. Whittaker and the rest, when they deal with a Kinetic problem convert it into a problem in Statics, and as surely as they do this they use the term Centrifugal Force †. They do it mostly under inverted commas, to show that they are acting against those principles which, when they were younger, they considered to be orthodox, but at all events they do it! I affirm that what they do is natural and right. Centrifugal force is natural and right to any school boy, and if his schoolmaster does not teach him through the ideas that he has, that he finds easy to understand, the schoolmaster is doing much worse than showing that he has learnt

* Discussion on the Teaching of Elementary Mechanics, edited by John Perry: Macmillan & Co., London.

† At this day it is not I hope necessary to say that D'Alembert's Theorem is simply Newton's great law of motion. See page 585 of my 'Applied Mechanics.'

c

his twopenny-halfpenny worth of Kinetics altogether from the astronomical side.

It is the same with the Pound as a unit of force. The two hugger-mugger ideas (as Fitzgerald called them), that quantity of matter is a simple thing to understand, and that quantity of matter and inertia are the same and must be called *mass*, these are metaphysical, grown up, mixtures of abstractions which are quite unnatural and occult to young persons whose experience has been with forces and weights. Now for all this I must refer you to the discussion: I ask you to read what I said and not to criticize me without reading. I feel sure that if you do read what I have said, I shall get from you what I have not yet had from men in general—an unprejudiced judgment. In forming a judgment I hope you will give weight to the fact that all English-speaking practical men habitually employ the pound (or units such as the ton, founded upon it) as the unit of force, in thought and when expressing their thoughts. Whether boys are or are not going to be engineers, they know that they would excite ridicule if they spoke of poundals or foot poundals to practical people; the results of all their calculations must be reduced to pounds and foot-pounds. Again, no good teacher has any difficulty in teaching through engineers' units; that there is such a difficulty is an academic delusion.

To come to a less important matter, almost all my critics seemed to think that it was impossible to give to the average young student the great fundamental idea that force is (vector) rate of change of momentum. I can only say that if this idea is not made familiar to a boy, a most powerful weapon is deliberately kept away from him. A young engineer or physicist familiar with the idea can at once and easily solve many problems that look hopelessly difficult to other people.

There are many pieces of apparatus to illustrate the fact that when no external forces act, the total momentum of systems of bodies in any direction remains constant. My own apparatus to illustrate centrifugal force is found to be easy to understand, and the conical pendulum afterwards becomes most useful. Attwood's machine may be found useful, but only in its very cheap forms. Like other pieces of apparatus designed for grown-up people the usual Attwood has so many contrivances for helping to obtain accuracy that it frightens a boy and becomes quite uneducational. To assist in giving the general idea that force is rate of change of

momentum, quite simply and accurately, I know of nothing better than the following piece of apparatus. It is a suspended vessel which can be kept in position by one or more spring-balances or the twist of a suspending wire. It is supplied with water in a vertical jet, the water leaving the vessel by one or more orifices. Or the water may leave the vessel vertically, the supply being one or more horizontal jets. The size of an orifice and the quantity of water flowing per second or per minute enable momentum per second to be calculated. It is, unfortunately, the only very simple thing that I know of, and it presents the idea in only one of its aspects. I wish that some of our ingenious Fellows who understand boys would give a little thought to the subject. Such illustrations are of great use in developing a boy's mathematical faculties.

It is important that Statics should be known from its graphical side, but it is a great mistake to neglect the analytical side of the subject. The trigonometric functions enable Graphics to be translated into Algebra, and such translation furnishes one of the best of simple mathematical exercises. After all it is on Algebra that we must mainly rely. By Graphics, including the use of squared paper, we are able to solve difficult concrete examples and illustrate general principles, but we use Algebra in getting general formulæ which we can at once apply to any particular case. The most elementary notions of Vector Algebra are of considerable value, but I am now rather speaking of ordinary Algebra leading up to the Calculus. Without the simple ideas of the Calculus it is impossible to get an elementary knowledge of Kinetics, and all attempts to teach Kinetics through evasions of the Calculus have in my opinion been utter failures. The engineering profession is filled with men who use formulæ, graphical and algebraical, which they are always desirous to understand and which they cannot possibly understand because they have not had a few hours' teaching in certain simple principles. They have been told that the principles of the Infinitesimal Calculus cannot be understood unless there is a preparatory study for years of difficult, uninteresting, preliminary mathematics. This causes me to refer to a matter in which I have been greatly misunderstood.

The newspaper writers in reviewing the discussion have ridiculed my standard average boy of 14 who is in the future to know something of the infinitesimal calculus, and they have classed him with Macaulay's fabulous schoolboy. But there are, I am glad to say, some schoolmasters who know that this standard boy of 14 is a boy

of the near future. There are some schools in which he is already being developed.

My critics do not seem to know that a great reform is now in progress. It is acknowledged by the best teachers that the whole system on which we used to be taught mathematics was quite wrong. The new system is really old enough: it was in use always by good teachers, but unfortunately the examination system compelled even the best teachers, like the people of Samaria, not only to worship God but to bow down before idols. There is now a healthy recognition of the fact that much of the old devotion to a crystallized system, an unchangeable ritual, was mere idolatry.

I, like you, had to learn all sorts of unnecessary things before I began the Calculus. I had an intricate course of Geometrical Conics which might have been easy for a boy of good memory, but I had a bad memory. I did Spherical Trigonometry. I developed all sorts of trigonometric series, basing my reasoning on De Moivre. It never seems to enter the head of the orthodox mathematician that much of the work he tries to give the schoolboy is work that ought not to be touched by a man till after he is a wrangler. In elementary Algebra alone, think what unnecessary work there is. Anomalous forms; Involution and Evolution; Surds; a hundred curious things about the roots of a quadratic equation, leading to numerous tricky exercises, when the straightforward information wanted might be put in four lines; the same is true of imaginary expressions, of ratio and proportion and scales of notation, of the progressions and continued fractions. Think of the worry over curious simultaneous equations involving quadratics and also of mathematical induction. Why is it that the subject of Permutations and Combinations must precede the Binomial Theorem? Simply to enable tricky questions to be asked;—"What is the middle or what is the rth term of the expansion of $(x+a)^n$?" "What is the greatest coefficient of $(1+x)^n$?" "Show that the coefficient of the rth term from the beginning is equal to that of the rth term from the end." "What is the sum of the coefficients of $(1+x)^n$?" There is the multinomial theorem and the rigid proof of the exponential theorem and learned disquisitions on the convergency of series; equation of payments; the reduction of curious surds to continued fractions; indeterminate equations of all degrees; recurring series; the theory of numbers; the theory of probability. Why, just consider the summation of series alone? What a time would be saved if the merest elementary notion of the

Calculus of Differences were given to a boy; but of course it never is given to him, it would make his work too interesting, there would be too little mathematical gymnastics! Surely if gymnastics is wanted it can be obtained in large enough quantity by the use of the Calculus in its interesting applications. I say, leave out all this unnecessary, tantalizing, stupefying, unprogressive and disheartening work and let a boy get on to the Calculus while he still retains some enthusiasm, before he feels jaded and stale.

We are finding in evening classes on Practical Mathematics that the Board School boy who has greatly forgotten whatever he knew, even his multiplication in Arithmetic, is able at the end of fifty lessons to use the idea of a *rate*, to differentiate and integrate practically all the functions which are of use in those kinds of applied science which we call engineering, and, furthermore, his sequence of study has been strictly logical. He can apply his knowledge to new problems with success, for he really does understand what he is doing.*

I know that it seems difficult of belief, but it is a fact all the same, that these evening students do rapidly acquire a working knowledge of the Calculus which is marvellous, and this knowledge does allow them to advance in their other studies with great rapidity. If an evening class in Pure Mathematics is started, it is found that the attendance drops off rapidly, but attendance at a Practical Mathematics class is quite steady throughout the winter. That is because the students are intensely interested all the time, because they understand what they are doing. How much of what you did when you were young was uninteresting because you did not understand it? It was occult! It was like meretricious ornament to a building, not only useless but harmful, because it hid from you the straightforward simple principles which are essential and must be clearly understood. I wonder if it is possible to bring to the thousand wranglers who are teaching in Great Britain a consideration of the awful waste not merely of time but of mental power that is going on in our common methods of teaching Mathematics. A boy who was bullied when a fag became as a consequence a bully himself. In the same way a man whose enthusiasm and energy and individuality have been destroyed by endless solutions of conundrums and puzzles in elementary mathematics, drives his

* See "Practical Mathematics": Summary of six lectures by John Perry. Wyman & Sons. Price 6*d*. See also Examination Papers on this subject of The Board of Education, 1899–1906.

flock of pupils along the same soul-destroying path that led to his own perdition. But happily the fag system and the old system of teaching Mathematics are disappearing, and presently they will be like the two giants whom Bunyan's hero passed by in safety as he came from the Valley of the Shadow. We often wonder at the conditions under which people lived in old times. We wonder why it was that they had so little knowledge and that they used the little knowledge they had with so little common sense, but we never seem to see that we ourselves show in all sorts of ways that we are lineally descended from those illogical people. Of course you may disagree with me, you may see some magical good in the study of the parasitic parts of Mathematics, but I hope you will give weight to the fact that I have myself had considerable experience in teaching Mathematics since 1870, when I became a master in Clifton College.

I hope you will not join the crowd in laughing at my standard average boy. He knows decimals at 8 because he cannot help it; the knowledge comes without effort in his weighing and measuring, and it is useless to tell him that he cannot understand such a difficult thing as a decimal until he is 13. He will use squared paper to record his own experimental results at the age of 9. He will solve very interesting problems using squared paper and logarithms and tables of sines and cosines at the ages of 10 and 11. He will get the notion of a rate long before he is 12, and he will have an elementary working knowledge of the Infinitesimal Calculus before he is 14. This will occur because he is also making experiments in the Mechanics and Physics laboratory. I do not ask for more time to be devoted to Mathematics than at present, but I certainly do wish more time to be given to Mechanics and Physics. Even if you cannot admire the attainments of my standard boy, I feel sure you must admit his possible existence, and this is what my critics deny. I assure you that he is coming along.

There are one or two details to be mentioned. If men are absurd enough to insist on strict proofs of everything in my sequence, taking nothing for granted, then there are several good easily understood proofs that the differential coefficient of x^n is nx^{n-1}. There is no proof of Taylor's theorem to which some exception may not be made: the earliest is good enough for a beginner and is very easily understood. Having proved Taylor, I would regard the Binomial as an example of Taylor. I say these things to meet the objections of the severely logical mathematician.

But in truth I do not see why we may not ask the beginner to take the Binomial on faith and leave Taylor to a much later date.

I think that Cambridge is greatly responsible for the fact that elementary mathematics is full of tricks, conundrums, and puzzles, and her responsibility is due to her method of examination. In preparing a list of wranglers in order of merit, if in the first four or five papers the questions were straightforward, every wrangler would get full marks for every question. To differentiate between them therefore it was necessary to make the questions tricky, and so we find all the exercises in all Algebras and books on Trigonometry to be tricky. Now that Cambridge reform has been voted I hope to see a great change in regard to this in all the text-books. I hope indeed in time to see Part I. at Cambridge deal also with many problems that require the use of the Calculus. I consider the elementary use of the Calculus and even the solution of easy differential equations, a school and not a university subject. Permutations and Combinations and the theory of probability are postgraduate subjects, like the study of the very beautiful 5th Book of Euclid. How many of you here, I wonder, have read again in your older days Euclid's reasoning on proportion: it is a poem! Euclid used to be studied by old Philosophers; it ought now to be a postgraduate study. I consider Spherical Harmonics and Bessel Functions to be undergraduate studies, but of course they ought to be reached with the shortest possible logical sequence and then used in all sorts of physical problems, with actual curve drawing as a finish, for the mere algebraic solution is no solution for practical purposes.

XXXIV. *On Transformer Indicator Diagrams.* *By* THOMAS R. LYLE, *M.A., Sc.D., Professor of Natural Philosophy in the University of Melbourne* *.

[Plate X.]

1. THE term "transformer indicator diagram" has been applied by Professor Fleming to any series of periodic curves which gives the forms, relative phase-positions, and magnitudes of the waves of current and E.M.F. on both the primary and secondary sides of a transformer when working. Such diagrams have been obtained by many investigators in different ways, but by none of the methods hitherto used has it been possible to determine directly and independently either the wave of magnetic flux F in the core, or the wave of magnetizing-current turns usually represented by the vector sum $n_1C_1+n_2C_2$.

Both these quantities are of fundamental importance in the theory of the transformer. When they are known for any given load, all the other quantities (currents and E.M.F.s) can be determined for the same load when the primary and secondary turns, resistances, and leakage coefficients are known †.

In addition, since, as will be shown later, the integral

$$\int(n_1C_1+n_2C_2)dF$$

for one cycle is equal to the total iron loss per cycle, the advantage of being able to determine both $n_1C_1+n_2C_2$ and F directly and accurately is apparent.

Theoretically, $n_1C_1+n_2C_2$ can be obtained by the vector addition of n_1C_1 and n_2C_2, but as the latter quantities are, when the transformer carries a load, approximately equal in magnitude and opposite in phase, their vector sum is a small quantity compared with either of them. Hence small errors in the magnitudes of n_1C_1 and n_2C_2 may cause a large percentage error in the magnitude of $n_1C_1+n_2C_2$, while very small errors in the magnitudes and phase-difference of n_1C_1

* Read February 22, 1907.

† See Lyle: "The Alternate Current Transformer," Proc. Roy. Soc. Victoria, vol. xviii. pt. 1.

and n_2C_2 may render the phase of $n_1C_1+n_2C_2$ calculated from them utterly unreliable.

By means of the wave-tracer * designed by the author not only can the E.M.F. and current waves be accurately determined, but also the wave of magnetic flux pulsating in the core of the transformer, and in addition, as will be shown in the sequel, we can obtain by its means the magnetizing-current wave, $n_1C_1+n_2C_2$, with the same accuracy as any of the other quantities.

Incidentally will be given new methods of comparing mutual inductances, and of measuring both mutual and self inductances in terms of a resistance and a time.

2. In the paper just quoted I have shown that if a periodic current C flows in the primary of a pair of coils whose mutual inductance is M, and if the secondary be joined through a suitably arranged commutator (running synchronously with the generator of the periodic current), which commutes twice per period, to a large resistance r and thence to a galvanometer, there will be a steady deflexion γ in the latter which is connected with the instantaneous value C of the periodic current at the instant of commutation by the relation

$$\mathrm{MC}=\frac{\lambda r\mathrm{T}}{4}\gamma,$$

where λ is the reducing factor of the galvanometer and T the period.

By arranging so that the commutating brushes can be rotated on a divided circle round the drum of the commutator, commutation can be effected at any desired instant of the period, and the corresponding galvanometer reading when multiplied by the factor given in the above equation gives the ordinate of the current wave at that instant.

Take now a triad (see fig. 1, T) of coils of which p_1 and p_2 are to serve as primaries and the remaining one s placed between p_1 and p_2 to serve as common secondary. Let s be connected as before through commutator and resistance r to the galvanometer, and let M_1 be the mutual inductance of p_1 and s, and M_2 that of p_2 and s. Then, when a current C_1 circulates in p_1 and none in p_2, the galvanometer deflexion γ_1

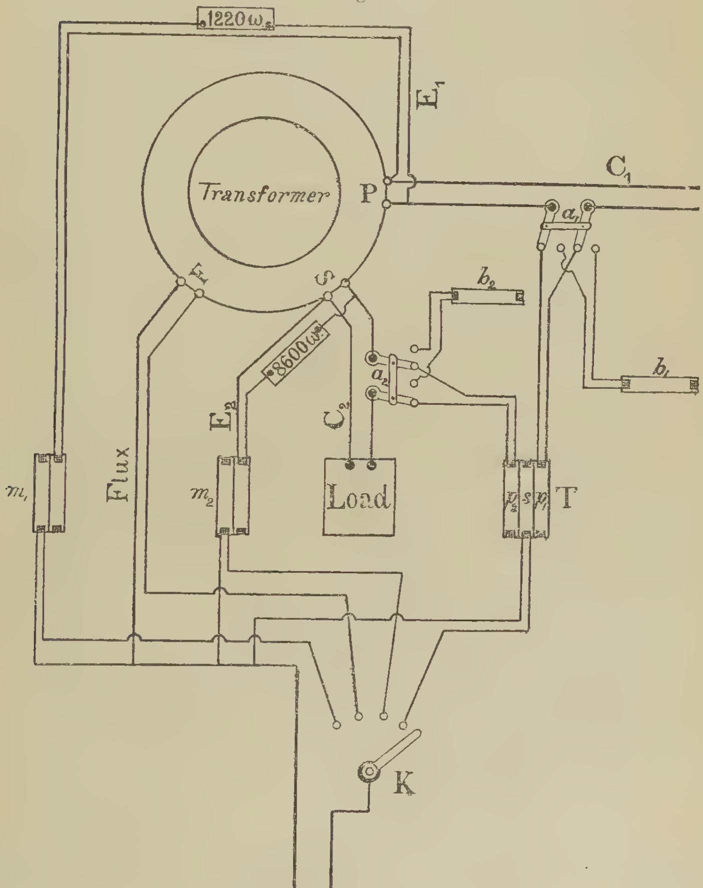
* Lyle; "Wave-Tracer and Analyser," Phil. Mag. Nov. 1903.

produced is connected with the ordinate of C_1 corresponding to the instant in the period at which commutation takes place by the equation

$$M_1C_1 = \frac{\lambda r T}{4}\gamma_1,$$

and when a current C_2, equiperiodic with C_1, circulates in

Fig. 1.



p_2 and none in p_1, a galvanometer deflexion γ_2 is produced which is similarly connected with the corresponding ordinate

of C_2 by the equation

$$M_2C_2 = \frac{\lambda r T}{4}\gamma_2.$$

When both C_1 and C_2 are flowing in their respective primaries at the same time, the galvanometer deflexion γ will be the sum of γ_1 and γ_2 produced by C_1 and C_2 separately; hence

$$M_1C_1 + M_2C_2 = \frac{\lambda r T}{4}\gamma.$$

If now n_1 and n_2 are numbers such that

$$\frac{M_1}{n_1} = \frac{M_2}{n_2} = M,$$

then $$n_1C_1 + n_2C_2 = \frac{\lambda r T}{4M}\gamma;$$

so that if γ be the galvanometer deflexion for a given position of the commutating brushes when C_1 is flowing in p_1 and C_2, correctly directed, in p_2, the ordinate of the wave which is the vector sum of n_1C_1 and n_2C_2 at the instant of commutation is the product of γ by $\lambda r T/4M$.

The same principle can obviously be extended to the determination of the vector sum of any number of equi-periodic currents each affected by an independent numerical multiplier.

In the paper already quoted it is also shown that if a coil of v turns of wire is looped on a magnetic circuit in which a periodic magnetic flux of period T is pulsating, and if the ends of the coil are connected through the synchronous commutator that changes twice per period to a high resistance r, and thence to a galvanometer as before, the ordinate F of the flux-wave and the corresponding galvanometer deflexion β for any position of the commutating brushes are connected by the relation

$$F = \frac{\lambda r T}{4v}\beta.$$

Hence any desired set of ordinates of a flux-wave can be obtained.

3. From what precedes it is obvious that in order to obtain practically the vector sum of n_1C_1 and n_2C_2, where n_1 and n_2 are numbers and C_1 and C_2 currents of equal period, it is necessary to have a triad p_1, p_2, and s of coils such that the

mutual inductance of p_1 and s is to that of p_2 and s as n_1 is to n_2. These can be wound in three grooves turned out of a circular disk of seasoned wood that has been well baked and then soaked in melted paraffin. A rough approximation to the required ratio of M_1 to M_2 can be obtained by attention to the relative numbers of turns, and a preliminary adjustment can most quickly be made by the wave-tracer as follows:—Send a periodic current of suitable value through the common secondary s. Join together one end from each of p_1 and p_2 and connect the junction to one fixed brush of the commutator. Bring the other ends of two terminals of a two-way key, from whose moving tongue a connexion is made, to the other fixed brush. Join the two movable brushes of the commutator through a suitable resistance r to the galvanometer. Thus, when a current C is flowing in s either p_1 or p_2 can be put in circuit with the galvanometer, and the deflexions γ_1 and γ_2 due to p_1 and p_2 respectively for every position of the moving brushes are in the ratio of M_1 to M_2, since

$$C=\frac{\lambda r T}{4M_1}\gamma_1=\frac{\lambda r T}{4M_2}\gamma_2.$$

Obviously in making the comparison it is desirable to place the moving brushes so that the deflexions obtained are near the maximum ordinate of the current wave.

Otherwise we might put a resistance r_1 in the galvanometer circuit when p_1 is in, and r_2 when p_2 is in, so as to obtain equal deflexions; then we would have that

$$\frac{M_1}{M_2}=\frac{r_1}{r_2}.$$

Hence the coils s and p_1 may be finished and their terminals permanently fixed, and, using the latter (equal deflexion) method, a resistance r_1 found to give a large deflexion when p_1 is in circuit and the brushes placed so as to give approximately the maximum ordinate of the current wave. Now substitute r_2 obtained by the above relation, allowing for galvanometer or other appreciable resistance that may be in circuit, and switch p_2 on to the commutator and galvanometer. An assistant can increase or reduce the turns (or fractions) until the same deflexion is obtained as was given by p_1 with r_1 in circuit. Spare wire should be left in case

the further treatment of the coils causes any change in their M ratio.

As it is necessary, if high accuracy with the wave-tracer be desired, that the M coils used should have high insulation, the triad should now be thoroughly dried in an oven to drive off moisture from the cotton or silk covering and placed in an iron dish that can be fitted with an air-tight lid that has an exhausting tube through it. A layer of solid paraffin is in the bottom of the dish, and on this the coil is placed and weighted with a piece of metal. The lid is luted on with melted paraffin, the air pumped out, and the dish then heated so as to melt the paraffin it contains. The coil now sinks in the melted paraffin, and as the air has been removed it becomes thoroughly impregnated. We can thus obtain coils of high insulation and permanent mutual inductance if, in addition, proper attention is paid to the insulation of the terminals.

A final and careful adjustment of the ratio of M_1 to M_2 should now be made with the wave-tracer.

The absolute values of M_1 and M_2 must also be known, and they can easily be obtained by either of the methods just described in terms of a known standard of mutual inductance. An alternating current is sent through the primary of the standard, and either p_1 or p_2 of the triad (the one whose M is nearest in value to that of the standard) and the secondary of the standard and s of the triad arranged so that either can be switched on to the commutator and galvanometer. Then for any position of the commutating brushes the two deflexions are proportional to the mutual inductances when the resistance in circuit is fixed, or if equal deflexions be obtained the inductances will be proportional to the resistances. Obviously it will be desirable to make a number of independent comparisons by varying the position of the commutating brushes about the place at which the maximum ordinate of the current is obtained.

The absolute value of a mutual inductance can be determined by the wave-tracer in terms of a resistance and a time, as follows. An alternating current is sent through a Kelvin balance, and at the same time a number (usually 30) of equispaced ordinates (γ) of it embracing one full wave is

obtained by the wave-tracer using the mutual inductance M to be measured; as

$$C_t = \frac{\lambda r T}{4M} \gamma$$

the square root of the mean squares of C_t, which is the Kelvin balance reading (B say), is $\lambda r T/4M$ times the square root of mean squares ($\bar{\gamma}$ say) of the galvanometer readings, hence

$$M = \frac{\lambda r T}{4} \frac{\bar{\gamma}}{B};$$

T is given by the chronograph attached to the wave-tracer, and the ratio of λ, the reducing factor of the galvanometer, to B can be obtained as follows. Send a continuous current $=B$ through the balance and through a standard resistance ρ (usually ·1 ohm). From the terminals of ρ lead a shunt circuit through a resistance R to the galvanometer and let the deflexion of the latter be d.

Then

$$\lambda d = \rho \frac{B}{R},$$

and hence

$$M = \rho T \frac{r\bar{\gamma}}{4Rd}.$$

4. The transformer, some of whose indicator diagrams will be given, was a small experimental one of the ring type of about one-half kilowatt capacity. It was used as a step-up one of ratio 1 to 6 transforming from about 40 volts (virtual) to 240 at about 50 periods per second.

Its details are as follows:—

Core:—144 annular laminæ annealed and paper insulated.

Laminæ:—Internal diameter .... 15·25 cm.
External diameter ...... 27·75 cm.
Thickness .... ·047 cm.

Primary coil:—No. of turns $(n_1) = 100$.
Resistance (warm) $= 0·0676$ ohm.

Secondary coil:—No. of turns $(n_2) = 600$.
Resistance (warm) $= 2·034$ ohms.

Before either coil was wound a single turn of well-insulated wire was looped on the core to serve as a search-coil for the determination, by means of the wave-tracer, of the magnetic

flux pulsating in the core. The secondary coil was wound next the core.

The primary current was drawn from the alternating side of a small rotary converter that was supplied with direct current from storage-cells. The spindle of the commutator and that of the converter were in line and rapidly connected so that perfectly synchronous commutation was obtained. A chronograph took a continuous record of the period, recording once every 200 alternations.

The arrangement of the transformer and the mutual inductances by means of which the different waves were determined is shown in fig. 1. The primary current C_1 from the converter enters by the two leads marked C_1. In one of these leads is placed a two-pole switch a_1, by which C_1 can be sent through the primary p_1 of the triad T, or deflected through an equal compensating coil b_1, leaving p_1 completely disconnected from the live circuit in case the triad is being used for the determination of C_2.

E_1 is determined by obtaining the trace of the current wave it sends through a non-inductive resistance of 1220 ohms. The mutual inductance m_1 of ·00061 henry is used for obtaining this trace. E_2 is similarly determined, the non-inductive resistance in circuit being 8660 ohms and the mutual inductance m_2 ·003535 henry.

The secondary current C_2 may, by means of the switch a_2, be directed through the primary p_2 of the triad T or through the equal compensating coil b_2. When $n_1C_1+n_2C_2$ is being determined, both C_1 and C_2 flow through their respective primaries p_1, p_2 of the triad in the proper relative directions, the common secondary s being joined as shown to the commutator and thence through a resistance to the galvanometer. When C_1 alone is being determined, C_2 is deflected through its compensating coil b_2; similarly when C_2 alone is being determined, C_1 is deflected through b_1. The mutual inductance of p_1 and s was ·0000485 henry, while that of p_2 and s was ·000291 henry, which bear the same ratio to each other as n_1 to n_2, that is as 1 to 6.

The points marked F in fig. 1 represent the terminals of the single loop of insulated wire wound round and close to the transformer core. When the flux-wave is being obtained

these are connected direct to the commutator, and thence through a suitable resistance to the galvanometer.

By means of the key K either the flux leads or those of any of the secondaries of the different mutual inductances can be connected with the commutator so as to obtain the corresponding wave-trace.

A glance at fig. 1 will show that the true primary current is less than the measured C_1 by the small current by which E_1 is determined. To correct for this (if correction is necessary) we subtract from each galvanometer C_1 reading an easily determinable fraction of the equi-phase E_1 galvanometer-reading. A similar correction has to be applied to the C_2 readings; but in this case the corrections have to be added. The galvanometer-readings for $n_1C_1+n_2C_2$ have also to be similarly corrected, and in this case the corrections are of great importance as the C_1 correction is affected by the factor n_1 and the C_2 correction by the factor n_2, and both corrections are approximately in the same phase.

5. For the present paper three separate sets of connected transformer quantities were determined, and they will be given below by their harmonic expressions and also represented by curves or indicator-diagrams.

These are for the transformer :—

(1) At no load.
(2) At (q.p.) full non-inductive load.
(3) At (q.p.) full inductive load.

The method by which the wave-tracer galvanometer-readings are reduced to absolute measure and the harmonic expression for the periodic quantity deduced from them has been fully explained in former papers *.

In the present experiments, for each of the periodic quantities determined, 30 wave-tracer galvanometer-readings were taken, each differing from the next in order by 12°. The full wave was thus covered. Corresponding deflexions in each half of the wave were added, that is the 1st and 16th, the 2nd and 17th, and so on, and 15 equispaced ordinates per

* "Wave-Tracer and Analyser," Phil. Mag. Nov. 1903. "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. Jan. 1905. "Expeditious Practical Method of Harmonic Analysis," Phil. Mag. Jan. 1906.

half-wave obtained. These were subjected to harmonic analysis, and the amplitudes of the different harmonics were affected by their proper factors to reduce them to flux, current, or E.M.F., as the case might be, in absolute measure.

The results obtained are as follows:—

(1) *For the transformer at no load.*

Period $= \cdot 02015$ sec.; $\omega = 311 \cdot 8$.

$$\frac{E_1}{10^8} = 57 \cdot 71 \sin(\omega t - 6 \cdot 98) - 4 \cdot 34 \sin 3(\omega t - 15 \cdot 7) + 0 \cdot 35 \sin 5(\omega t - 29).$$

$$C_1 = \cdot 1304 \sin(\omega t - 44 \cdot 23) + \cdot 0224 \sin 3(\omega t - 33) + \cdot 0057 \sin 5(\omega t - 27).$$

$$F = 184700 \sin(\omega t - 97 \cdot 11) + 4300 \sin 3(\omega t - 105 \cdot 9) - 300 \sin 5(\omega t - 84).$$

And as E_2 at no load $= -n_2 \dfrac{dF}{dt}$, we find that

$$\frac{E_2}{10^8} = 345 \cdot 5 \sin(\omega t - 187 \cdot 11) - 24 \cdot 1 \sin 3(\omega t - 195 \cdot 9) + 2 \cdot 8 \sin 5(\omega t - 210)$$

(The different quantities are in absolute units and the phase angles in degrees.)

(2) *For the transformer at (q.p.) full non-inductive load.*

Period $= \cdot 02053$ sec.; $\omega = 306$.

The load was a manganin non-inductive resistance of 104·7 ohms.

$$\frac{E_1}{10^8} = 56 \sin(\omega t - 2 \cdot 59) + 4 \cdot 18 \sin 3(\omega t - 51 \cdot 9) + \cdot 78 \sin 5(\omega t - 61 \cdot 2).$$

$$C_1 = 1 \cdot 945 \sin(\omega t - 5 \cdot 71) + \cdot 145 \sin 3(\omega t - 50 \cdot 1) + \cdot 034 \sin 5(\omega t - 33 \cdot 7).$$

$$n_1 C_1 + n_2 C_2 = 12 \cdot 6 \sin(\omega t - 45 \cdot 21) + 2 \cdot 67 \sin 3(\omega t - 33 \cdot 4) + \cdot 75 \sin 5(\omega t - 30).$$

$$F = 178900 \sin(\omega t - 93 \cdot 22) + 3900 \sin 3(\omega t - 83 \cdot 2) - 400 \sin 5(\omega t - 93).$$

$$C_2 = \cdot 309 \sin(\omega t - 183 \cdot 16) + \cdot 022 \sin 3(\omega t - 233 \cdot 5) + \cdot 005 \sin 5(\omega t - 216).$$

And as $E_2 = R_2 C_2 = 104 \cdot 7 \times 10^9 \,.\, C_2$, we find that

$$\frac{E_2}{10^8} = 323 \cdot 5 \sin(\omega t - 183 \cdot 16) + 23 \sin 3(\omega t - 233 \cdot 5) + 5 \cdot 2 \sin 5(\omega t - 216).$$

(3) *For the transformer at (q.p.) full inductive load. Power factor* $= \cdot 73$.

Period $= \cdot 01835$ sec.; $\omega = 342 \cdot 4$.

The load was a manganin non-inductive resistance in series

with a copper wire inductance-coil. The total external resistance was 76·52 ohms, and the inductance, measured independently, was ·212 henry.

$$\frac{E_1}{10^8} = 54{\cdot}13 \sin(\omega t - 10{\cdot}5) - 5{\cdot}36 \sin 3(\omega t - 12{\cdot}5) + 1{\cdot}37 \sin 5(\omega t - 13{\cdot}2).$$

$$C_1 = 1{\cdot}907 \sin(\omega t - 52{\cdot}1) - {\cdot}065 \sin 3(\omega t - 34{\cdot}7) - {\cdot}017 \sin 5(\omega t - 63{\cdot}1).$$

$$n_1C_1 + n_2C_2 = 11{\cdot}18 \sin(\omega t - 48{\cdot}37) + 1{\cdot}82 \sin 3(\omega t - 42{\cdot}14) + {\cdot}31 \sin 5(\omega t - 35).$$

$$F = 154900 \sin(\omega t - 99{\cdot}9) + 5200 \sin 3(\omega t - 101{\cdot}7) + 800 \sin 5(\omega t - 98).$$

$$C_2 = {\cdot}299 \sin(\omega t - 232{\cdot}38) - {\cdot}015 \sin 3(\omega t - 215{\cdot}4) - {\cdot}002 \sin 5(\omega t - 235).$$

$$E_2 = 312{\cdot}9 \sin(\omega t - 189{\cdot}08) - 31{\cdot}8 \sin 3(\omega t - 189{\cdot}5) - 8{\cdot}3 \sin 5(\omega t - 223)$$

6. The same wave-tracer deflexions were individually multiplied by their proper factors to reduce them to absolute measure, and the products plotted as wave ordinates against the corresponding wave-tracer divided-circle readings (*i. e.* against ωt where $\omega = 2\pi/\text{period}$) as abscissæ. Fig. 2 (Pl. X.) represents correctly in amplitude and relative phase the different periodic quantities for the transformer at no load ; fig. 3 (Pl. X.) for the transformer at (q.p.) full non-inductive load, and fig. 4 (Pl. X.) for the transformer at (q.p.) full inductive load.

Obviously in figs. 2, 3, and 4 the same periodic quantities are graphically represented as are analytically expressed in series 1, 2, and 3 respectively of the preceding paragraph.

In addition to the sets of related waves, there is, in each of the three diagrams, an area very similar to the well-known hysteresis indicator-diagram. In the present case these closed curves were obtained by plotting the flux as ordinate against the corresponding value of the magnetizing-current turns (i. e., $n_1C_1 + n_2C_2$) as abscissa for a complete period.

Thus the area enclosed (A say) is

$$\int^{t+T} (n_1C_1 + n_2C_2)\, dF,$$

where T is the period, and this integral can be shown to be equal to the total core loss per cycle due to both hysteresis and eddy currents as follows :—

Neglecting rc^2 losses, the energy entering the transformer on the primary side in any element of time dt is eC_1dt, where

e is the back E.M.F. due to variation of the flux; and as

$$e = n_1 \frac{dF}{dt}$$

this energy is equal to

$$n_1 C_1 \frac{dF}{dt} dt.$$

Similarly the energy leaving the transformer on the secondary side in the same element of time dt is equal to

$$-n_2 C_2 \frac{dF}{dt} dt;$$

hence in the time dt the transformer absorbs energy to the amount

$$(n_1 C_1 + n_2 C_2) \frac{dF}{dt} dt,$$

so that in one cycle, of duration T, the energy absorbed is equal to

$$\int_t^{t+T} (n_1 C_1 + n_2 C_2) \frac{dF}{dt} dt = \text{area A}.$$

7. It is easy to show that when $n_1 C_1 + n_2 C_2$ and F are expressed in the forms

$$n_1 C_1 + n_2 C_2 = m_1 \sin(\omega t - \mu_1) + m_3 \sin 3(\omega t - \mu_3) + m_5 \sin 5(\omega t - \mu_5) + \&c.,$$
$$F = f_1 \sin(\omega t - \phi_1) + f_3 \sin 3(\omega t - \phi_3) + f_5 \sin 5(\omega t - \phi_5) + \&c.,$$

the integral or area A and therefore the total core loss, in ergs, per cycle is equal to

$$\pi\{m_1 f_1 \sin(\phi_1 - \mu_1) + 3m_3 f_3 \sin 3(\phi_3 - \mu_3) + 5m_5 f_5 \sin 5(\phi_5 - \mu_5) + \&c.\},$$

which when divided by 10^7T, where T is the period, gives the total power lost in the core in watts.

This form of expression has the advantage of giving separately the power absorbed by the iron by means of the harmonics of different orders, and from the analytic expressions for E.M.F.s and currents we can also obtain for the different orders of harmonics the input, output, and copper losses. Hence we can draw up a debit and credit account for the individual harmonics which will afford a good test of the accuracy of the wave-tracer, as the account for each order should balance.

This has been done for the first and third harmonics of the three series given in this paper and the results shown in the following tables. The quantities for the fifth harmonics are negligible. The figures represent watts.

TABLE I.—No Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	0	0	0
Iron loss	29·93	−·28	29·65
Copper loss, I.	·06	·002	·06
Copper loss, II.	0	0	0
Sum	29·99	−·28	29·71
Input	29·95	−·30	29·65

TABLE II.—Non-inductive Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	499·80	2·53	502·33
Iron loss	25·63	·24	25·87
Copper loss, I.	12·79	·07	12·86
Copper loss, II.	9·71	·05	9·76
Sum	547·93	2·89	550·82
Input	543·8	3·02	546·82

TABLE III.—Inductive Load.

	1st Harmonic.	3rd Harmonic.	Total.
Output	340·5	·51	341·01
Iron loss	23·18	·01	23·19
Copper loss, I.	12·29	·01	12·30
Copper loss, II.	9·13	·02	9·15
Sum	385·10	·55	385·65
Input	386·0	·69	386·69

The output in Table III. is the mean value of E_2C_2. A second determination of it, obtained from the mean value of RC_2^2, where R is the external resistance (76·52 ohms), gives 342·86 watts as against 341.

8. Other tests can be applied to the harmonic expressions given in § 5, and though their results may not be so good as those in § 7, yet they will prove fairly satisfactory when the number of independent determinations involved in obtaining each series is considered.

Thus the pressure drop from E_1 to $E_2/6$ is too small in both series 2 and 3, and this is probably due to a common error or errors. Moreover the speed never remained quite constant

throughout a complete series. Again in series 3, knowing the external secondary resistance, and assuming as correct the value given for the external inductance (·212 henry), C_2 can be calculated from E_2. When this is done we find that

$$C_2 = \cdot 2966 \sin(\omega t - 232\cdot 58) - \cdot 014 \sin 3(\omega t - 213) - \cdot 002 \sin 5(\omega t - 239),$$

which agrees well with the value of C_2 obtained with the wave-tracer.

Conversely from E_2 and C_2 obtained with the wave-tracer, both the external resistance R and the external inductance L can be calculated, or if R is known two determinations of L can be obtained from each order of harmonic.

If, however, as in the present case $\omega L/R$ is considerable, the first order is the only one suitable for this purpose; for if ψ_1, ψ_3, and ψ_5 be the difference of phase of the first, third, and fifth harmonics respectively of E_2 and C_2, then, since

$$\tan\psi_1 = \frac{\omega L}{R}, \qquad \tan\psi_3 = \frac{3\omega L}{R}, \qquad \tan\psi_5 = \frac{5\omega L}{R},$$

and $\omega L/R$ is large, a small error in the observed values of ψ_3 or ψ_5 will cause a large error in the value of L deduced from them.

For the first harmonics, as $\psi_1 = 43^\circ \cdot 3$, $R = 76\cdot 52$, $\omega = 342\cdot 4$, we find from

$$\tan\psi_1 = \frac{\omega L}{R}$$

that $L = \cdot 2106$ henry; and from

$$C = \frac{E}{\sqrt{R^2 + \omega^2 L^2}}$$

that $L = \cdot 2084$ henry.

These give a mean value for L of ·2095 henry, which is as likely to be correct as the value ·212 already given and which was obtained by the Wheatstone's bridge method. As ω or $2\pi/T$ was determined by the chronograph attached to the wave-tracer, the above constitutes a new method of measuring an inductance in terms of a resistance and a time.

9. In Table I. § 7, it will be noticed that the iron loss for the 3rd harmonic is negative (−·28 watt). This means that all the watts put down as iron loss for the 1st harmonic

are not dissipated as heat, but that some of them are transformed by the iron to 3rd harmonic power; a portion of these are dissipated as heat, while the remainder ·28 watt are given out as electrical power by the iron to the copper circuits. This phenomenon has already been drawn attention to in a former paper * by the author.

10. In the diagrams, figs. 2, 3, and 4 (Pl. X.), it is worth drawing attention to the approximate permanence of form of the magnetizing current wave, and to the permanence of the relative phase relations of E_1, $n_1C_1+n_2C_2$, and F, for the three essentially different conditions of working for which the diagrams were obtained.

Discussion.

Mr. W. Duddell said the paper was an interesting and important one, and congratulated the author upon the ingenious double mutual induction method by which the vector sum $(n_1C_1+n_2C_2)$ was determined electrically. In the paper the core loss was nearly constant for no load and full load, but the author had used an E.M.F. wave which was practically a sine curve. He suggested that experiments should be carried out using distorted E.M.F. waves such as occur in many generators and transformers.

Mr. A. Campbell remarked in respect to the title of the paper that it seemed a wanton confusion of nomenclature to use in this connexion the term "indicator diagram" which had such a definite and accepted meaning in mechanics; we might as well talk of hysteresis loops as "iron indicator diagrams." Prof. Lyle's methods were interesting, particularly that for obtaining the curve of effective ampere turns $(n_1C_1+n_2C_2)$ by the use of a double mutual inductance which performed the vector addition. The author had wisely kept the mutual inductance small: without that condition the double air-core transformer might alter C_1 and C_2. It would probably be better to split it up into two separate pairs of coils. The setting of the ratio of the mutual inductances on the triple coil could be more easily done by Maxwell's null

* "Variation of Magnetic Hysteresis with Frequency," Phil. Mag. Jan. 1905.

method, which was perfectly applicable to a pair of secondaries having a common primary. If various values of n_2/n_1 had to be dealt with, the two primary coils might be fixed at right angles and the secondary mounted so as to be capable of rotation with regard to the primaries in order to allow of variation in the ratio M_2/M_1. In connexion with Prof. Lyle's methods of determining inductances by the help of the wave-tracer, the somewhat similar method of Dr. E. B. Rosa might be mentioned. It was interesting to see the actual hysteresis loops given; a comparison of these with ballistic tests on the same transformer would add valuable information.

Mr. A. Russell said that the author's results proved that to a first approximation the magnetizing force acting on the core followed the same law and had the same amplitude at all loads. This theorem was the starting point in the ordinary engineering theory of the transformer, but this was the first time that a careful experimental proof for a particular case had been given. The magnetic leakage in the transformer experimented on must have been very small. The formula $\int_t^{t+T}(n_1C_1+n_2C_2)\frac{dF}{dt}dt$ only gave the energy absorbed per cycle when the magnetic leakage was negligibly small. In many commercial transformers it would not be safe to use this formula. The core-loss diagrams were interesting, but their value would have been greater if the transformer had been so constructed that the magnetic flux density was approximately uniform over the cross section of the core. The internal diameter of the centre-hole iron stampings employed was only about half the external diameter, hence the magnetic force to which the iron was subjected was almost twice as great at the inner as at the outer circumference. As the values of the permeability of the iron at these forces might vary widely, we could not tell the distribution of the magnetic flux. This prevented us from making calculations as to the hysteresis and eddy-current losses in the core. The complete solution of the transformer problem could not be obtained until we knew more about hysteresis. Apparently there were molecular "frequency changers" in the core, and any theory that would elucidate their action would be of great value to electricians. In this connexion he referred to a paper read by Prof. J. Perry in 1892.

XXXV. *On the Ionization of Various Gases by the α Particles of Radium.*—No. 2. *By* W. H. BRAGG, *M.A.*, *Elder Professor of Mathematics and Physics in the University of Adelaide*.*

[Plate XI.]

Introduction.

IN a paper with a similar title (Proc. Roy. Soc. of South Australia, vol. xxx. p. 1, and Phil. Mag., May 1906, p. 617) I have given a preliminary account of an attempt to determine the relative amounts of ionization produced in various gases and vapours by the α particle of RaC. The present paper contains an account of the further progress of this work.

In the first place, I have here discussed the validity and the experimental details of the method used, and have brought forward evidence in favour of the hypothesis that $\delta\iota$, the ionization produced in consequence of the expenditure of a small quantity of energy $\delta\epsilon$ by the α particle, is related to the latter quantity by the equation $\delta\iota = kf(v)\delta\epsilon$, where $f(v)$ is a function of the velocity of the particle only, and k a constant for each gas.

Secondly, I have given the result of the attempts to determine for several gases the constant k, which may be called the specific ionization of a gas for α radiation, air being taken as the standard.

In conclusion, I have discussed very briefly the form of the function $f(v)$, and such conclusions as it seems legitimate to draw from the results so far obtained. Amongst these is the following:—The ionization per molecule (ks, where s is the stopping power) is closely connected to the molecular volume.

§ 1.

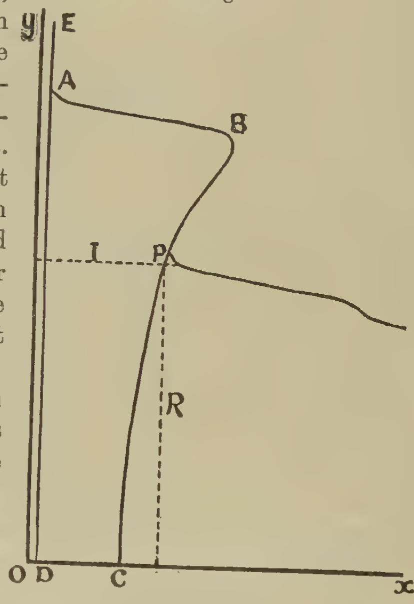
The method of this research has already been described briefly in the preliminary paper (*loc. cit.*). For the sake of clearness, however, and in order to facilitate a discussion of

* Read February 22, 1907. Part of the paper was also read before the Royal Society of South Australia, Oct. 2, 1906.

the validity and the experimental details of the method, it will be well to insert a short description here also.

A small platinum plate is coated with a very thin layer of radium bromide, and placed below a horizontal ionization-chamber of 3 mm. width, at a distance which can be altered at will. (See Plate XI.) A set of narrow vertical tubes is placed over the radium, and stops all α particles which move in any direction which is not almost vertical. Thus the particles cross the narrow chamber at right angles to its greater dimensions, and all spend 3 mm. of their paths in the air within it. The resulting ionization being plotted against the distance from the radium to the middle of the chamber, we obtain an "ionization curve," as in fig. 1, where ordinates represent distances and abscissæ represent ionization currents. Each reading of current is the difference between two others, one measured when a very thin copper screen is placed over the radium, and one when it is not.

Fig. 1.



In this curve the portion AE is due to the β rays only, and represents the effect of such β radiation as is intercepted by the screen: the chamber is out of range of α rays. Let EA be produced to meet the axis of x in D. The portion ABP represents part of the effect of the α particles from RaC. If no other radioactive substances were present, the curve would show a continuation of the portion BP down to the axis of x, in some such manner as PC.

If the ionization curve were completed in this way, the area ABPCD would represent the total ionization due to the

α radiation from RaC. If now the air were removed, another gas substituted for it, and the air re-measured, a comparison of the values obtained would give the result which this research aims at. We may call it the specific ionization of the gas. But the complete determination of the boundaries of the area is so long and complicated a process as to render this procedure impracticable. It can, however, be shown that the product of the coordinates of a certain point on the curve may be taken as a measure of the area of the curve, provided certain assumptions are made. The point is at the intersection of the top portion of the curve representing the effects of RaA with the side of that showing the effects of RaC. The coordinates of this point are comparatively easy to obtain.

Now, it might appear that it would be better to measure of one time the whole of the ionization produced by the particle, rather than to determine the ionization point by point along its path; since, if this were done, it would no longer be necessary either to find the exact form of the ionization curve, or to depend upon the validity of assumptions. We might spread a layer of radioactive material on the floor of an ionization-chamber, and so arrange the temperature and pressure of the gas in relation to the dimensions of the chamber, that all the α particles completed their paths within the gas. But the potential gradient required to separate and collect the ions made by the α particle is generally very great. For example, in ethyl chloride at 30 cm. pressure and ordinary temperatures, about 1000 volts per cm. is desirable, if saturation is to be certain. With such gas it would be necessary to make the height of the chamber about 4 cm., in order to allow all the α particles to complete their ranges; even if the radioactive material were uranium or polonium. Thus, a potential of 4000 volts would be required, and such large electromotive forces are out of the question. If the pressure of the gas were lowered, less electric force would be sufficient; but the paths of the α particles would be longer, the chamber would need to be higher, and the total potential as great as ever.

It is absolutely necessary to use a narrow ionization-chamber if sufficient electric force is to be obtained without the use of enormous battery-power. Clearly it would be no

gain to use such a chamber if the radioactive material were spread on one of its walls. For in this case some of the particles would complete their full ranges within it, others only part, and an estimate of the ionization to be expected would render it necessary to take into account the amount of the range completed by each particle as determined by the nature and physical conditions of the gas and the dimensions of the chamber, the reckoning being further complicated by the fact that the ionization produced by the particle is not constant along its path. It is possible that an experiment might be arranged in which a thin sheet of α radiation entered the chamber through a slit at the side, and spent itself within the chamber without touching the walls. It would be necessary to make sure that the same portion of the range was completed within the chamber by the particle, no matter with what gas the chamber was filled. I have not yet tried this plan.

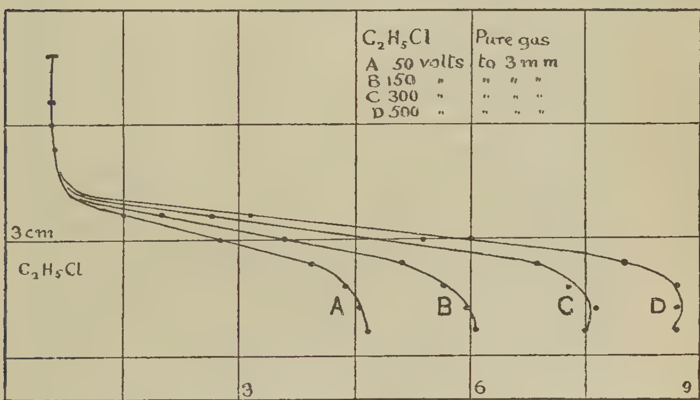
It will now be clear, I think, that the method actually used is not without its advantages. It avoids the use of very high potentials, and does not require lengthy and uncertain calculations. It has also this in its favour, that it gives the range of the particle in the gas, so that it is possible to make a sufficiently accurate estimate of the amount of any air that may be present. The presence of this air can then be allowed for.

Let us, therefore, proceed to consider the assumptions and approximations which the method requires.

In the first place it is necessary to consider whether any disadvantages are likely to arise from the use of a sheet of gauze as the lower wall of the ionization-chamber. The electric field must be distorted in the neighbourhood of the gauze; some very small portions of the chamber which are just over the openings in the gauze must be under feeble forces, and the ions made there be separated only when the potential is high. It is easy, however, to show that this effect is negligible by a consideration of the ionizations due to β rays. This ionization does not show initial recombination, as in the case of the α rays; a fact first demonstrated by R. Kleeman, formerly of this University. In fig. 2 are drawn the upper parts of the ionization curves of ethyl chloride under different potentials. It will be seen that in the portion which represents

the effects of β rays only, saturation is complete when 50 volts are applied to the 3 mm. chamber ; but the ionization due to α rays is far from being collected completely by ten times that potential. Now, if the field distortion due to the gauze were appreciably effective, we should find the β rays also producing an ionization which appeared to increase at higher potentials ; and there is no trace of any such effect. The same result shows that, although ions are very apt to be drawn through a gauze by a strong field on one side, yet in this case nothing of the sort takes place. To prevent it, a second gauze has been placed 3 mm. below the first, and earthed, so that there are strong, equally-balanced fields on both sides of the latter.

Fig. 2.



A thin uniform metal sheet might replace the gauze, but unless it were very thin it would cut off more of the range than can generally be spared ; and if it were thin it would be liable to flexure by the powerful electric forces, so that the depth of the ionization-chamber might become indeterminate.

We must now consider the assumption that the area of the ionization curve may be represented by the product of the coordinates RI as already defined. This is really equivalent to the supposition that the ionization resulting from the expenditure of a quantity of energy $\delta\epsilon$ by the α particle is equal to $kf(v)\delta\epsilon$, where $f(v)$ is a function of the velocity of the particle and k is a constant, depending on the nature of

the gas molecule. It implies in the first place that the area of the ionization curve in any gas is not dependent on pressure and temperature, and that, if the form of the curve is altered by a variation of these conditions, it is only in so far that all the ordinates are multiplied by some factor, and all the abscissæ divided by the same factor. It implies, in the second place, that the ionization curve of one gas can be made to coincide with the curve of any other gas, by multiplying all the ordinates by some factor, and all the abscissæ by some other factor. Let us examine the evidence in favour of these statements.

If the hypothesis is true, RI must be independent of pressure and temperature. As regards pressure, some results were quoted in a paper "On the Recombination of Ions in Air and other Gases" *, which showed this to be correct in the cases of air and ethyl chloride; and further evidence will be found in the results given at the end of this paper. For, without having made any exhaustive comparison of the values of RI at different pressures in each gas, I have often used various pressures in the determination of the specific ionization of a gas; and the general agreement between the results obtained is good evidence that pressure is without effect.

In the same way, since many determinations in the case of the same gas have been made at different temperatures, the close agreement shows also that temperature has no influence on the ionization. More direct confirmation can be obtained from the following results. During a number of the determinations of RI, the ionization-chamber was connected in parallel with a second chamber containing a uranium layer. The ionization currents acted against each other. Thus the values of the currents in the radium apparatus could be determined by balancing against the uranium: the latter was always at the temperature of the room, and therefore formed a fixed standard. The extent of the surface of the uranium could be varied by means of a semaphore, having a graduated circle on the same axis. It was then found that although the RI in air appeared to decrease as the temperature of the radium apparatus was raised, yet when the readings were expressed in terms of the uranium scale, the value of RI was

* Trans. Roy. Soc. S. A. 1905, p. 187; Phil. Mag. April, 1906, p. 466.

constant. The decline was merely apparent, and due to leakage through the heated glass insulators. The actual values of RI were :—

Five determinations, 20° to 60° C.: 320, 326, 318, 314, 314; mean, 320.

Five determinations, 60° to 80° C.: 296, 314, 311, 334, 327; mean, 316.

The experiments were made at various times, and some of the irregularities are probably due to slight alterations in the amount of the RaC present.

Furthermore, it has already been shown with respect to ionization in general that pressure and temperature have no effect (Patterson, Proc. Roy. Soc. lxix. p. 277, 1901, and Phil. Mag. Aug. 1903). I have thought it well, however, to reconsider the point with special reference to the circumstances of this experiment.

It is convenient at this stage to state that temperature does not seem to have much effect on initial recombination. The latter decreases rapidly as pressure is lowered. This has been shown by Kleeman and myself ("On the Recombination of Ions, &c."). But when the alteration in density occasioned by a rise of temperature has been allowed for, there appears to remain only a slight diminution in initial recombination, which can be ascribed to the direct result of the increase in temperature. This is shown with some clearness in some experiments which I have made with CO_2. They may be tabulated as follows, the ionization at an electric force of 1000 volts per cm. being taken as 100:—

CO_2.	Ionization at 1000 volts per cm.	Ionization at 333 volts per cm.	Ionization at 166 volts per cm.
(*a*) Pressure, 651 mm., 20° C. ..	100·0	95·0	90·2
(*b*) Pressure, 760 mm., 72° C. ..	100·0	96·8	94·0

A repetition of the experiment gave practically the same result. The pressures and temperatures were so arranged that the density was the same in each experiment.

I also tried the experiment with ethyl chloride, but the results were not so definite; that is to say, change of temperature produced no very obvious effect.

It is further assumed that the curves for different gases are of the same form; in other words, that the function $f(v)$ is the same for all gases.

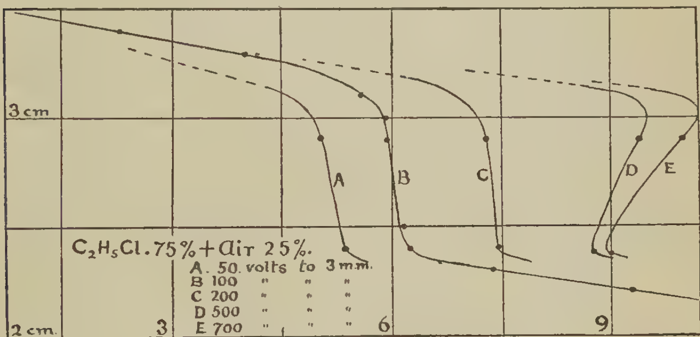
A complete test of this hypothesis would require an accurate delineation of the ionization curve in the case of each gas. As has already been said, this would be a difficult task, inclusive, indeed, of our present purpose. But a comparison of the curves in different gases, so far as they have been obtained, shows that the principle is at least approximately true. For example, the ratio of the range of RaC to that of RaA is the same in all gases within errors of experiment, and again the ratio of the maximum abscissa of the RaC curve to the abscissa I is also constant, so far as I have measured it. As examples of the constancy of the first of these two ratios, I have at different times found it to be 1·46 in air, 1·47 in pentane, 1·47 in ethyl chloride, 1·44 in carbon dioxide, 1·48 in ethyl alcohol, and 1·49 in ethylene. The differences here are probably experimental only. As regards the second ratio, I have found it to be 1·36 in air, 1·37 in ether, 1·44 in ethylene, 1·35 and again 1·41 in ethyl chloride. This ratio is much more liable to error than the former; for all ionizations are harder to measure correctly than ranges, and the peak of the ionization curve is an especially uncertain point. Also there is a special difficulty due to the existence of a peculiar phenomenon, which must now be considered.

It is to be observed that the ionization curves in different gases will not correspond unless the potential employed is enough to saturate at all points of the path of the α particle. More electric force is required as the particle slows down. This may be deduced from figures given in the paper "On the Recombination of Ions, &c." p. 196. It is there stated that the ratio of the saturated ionization current to that at 25 volts per cm. in the case of the ions made by the α particle of RaC at a distance of 6·25 cm. from its origin was found to be 1·29; whereas, when the distance was reduced to 5·05 cm., it was found to be 1·19. Each of these ratios is the mean of four determinations. (By an arithmetical error, one of the latter is incorrectly given in the paper quoted: 1·23 should be 1·20.)

Again, the effect is clearly shown by the curves of fig. 3,

which represent the results of experiments on a mixture of ethyl chloride and air. It will be seen that the curve does not show the characteristic increase of ionization with distance when the electric force is small, the reason being that it is so much more difficult to collect the ions made by the α particle at the end of its path.

Fig. 3.



It is necessary to refer to one more assumption which is made in calculating the results, viz., that the RI of a mixture can be determined from a knowledge of the RI of each component. For example, it is supposed that the RI of air being 100 and of ethyl chloride 132, then the RI of a mixture in such proportions that the α particle spends half its energy in each is 116.

For I have not been able to prevent the leakage of air into the apparatus when raised above ordinary temperatures, and it is necessary to measure and allow for the air present in each experiment. The apparatus holds very well when not heated ; but it is sometimes necessary to raise the temperature to 60° or 70° C. in order to obtain a sufficient density of the vapour under treatment. Fortunately, however, the air present may be a considerable fraction of the gas when measured by pressure, and yet be of little importance when measured in terms of the energy spent in it. Thus the correction for air present is usually quite small, as will be seen from a consideration of the numerical results in § 2.

The assumption is by no means an obvious one. If any part of the ionization in a gas is secondary, and is due to

radiation originating in one molecule and acting on a neighbouring molecule, it might well be that complications would arise in a mixture of gases. I have made several direct attempts to find whether any such effects existed: the results of some of them are shown in the following tables. Each table refers to a set of experiments carried out consecutively. The percentage of gas in each mixture is determined from the stopping power, and the percentage of energy spent in the gas is then calculated. The value of RI for the gas is calculated by multiplying the observed value for air by the specific ionization of the gas, as taken from the final tables given at the end of the paper. For example, in the first set RI for air is 198·5, and RI for ethyl chloride is taken to be $189{\cdot}5 \times 1{\cdot}32 = 262$. The RI for each mixture is then calculated. In the table the calculated and observed values are put side by side, and it will be seen that there is a good agreement:—

	Percent. of Gas.	Percent. of Air.	Percent. of Energy spent in Gas.	Pressure in mm.	Temp.	RI observed.	RI calculated.
C_2H_5Cl:							
1. ……	0	100	0	760	37	198·5	—
2. ……	88·5	11·5	94·5	421	37·5	253	259
3. ……	39·7	60·3	61	437	38	235	237·5
4. ……	17·3	82·7	33	433	38	220	220
5. ……	8·5	91·5	18	441	38·2	212	210·5
6. ……	0	100	0	760	39	198·5	—
C_2H_5Cl:							
1. ……	0	100	0	760	32	200·5	—
2. ……	91·5	8·5	96	294	34·5	258	260
3. ……	35·3	64·7	56·5	310	36·5	232	235
4. ……	0	100	0	760	37	198·5	—
C_5H_{12}:							
1. ……	0	100	0	760	41	202	—
2. ……	83·5	16·5	95	321	43	262	264
3. ……	19·2	80·8	45·5	351	45	232	229·5
4 ……	0	100	0	760	45·6	195	—

Nevertheless, in a number of cases in which I have attempted to calculate the value of RI of one gas from a knowledge of the values of RI for air, and for a mixture in known proportions, I have obtained an unexpectedly high result, and when I began some direct experiments on the question I was quite prepared to find that the ionization of a mixture of air and gas was more than the sum of the ionization of air and gas separately. Further experiment will, no doubt, make

everything quite clear. In the meantime it is sufficiently evident that the principle is at least nearly true. For the purpose of this investigation it may be taken as quite true, since the correction to be made for the presence of air is, at the most, only small.

The quantity I, as measured, includes a small proportion of β ray ionization. It must be shown that this does not harm the result.

In the form of apparatus which I use the ionization in the portion AE of the curve (see fig. 1) is nearly 6 per cent. of the ionization at P, and I have not found enough variation from gas to gas to justify an attempt at correcting for it. Of course, the quantity is only small.

The curve shows only this β ionization above A; that below is hidden. But I have found by experiment that it varies very little throughout the whole distance from the axis of x. This I did by placing over the radium just enough tinfoil to cut off all the α rays.

In the foregoing will be found, I think, sufficient justification for the choice of the method of this paper, and for the assumptions made during the calculation of the results.

In the previous paper I have already given a brief description of the process of an experiment. Some points, however, deserve reconsideration in the light of further experience, and some changes have been found convenient. These are best discussed in relation to an actual experiment; I will take a determination of RI in carbon bisulphide.

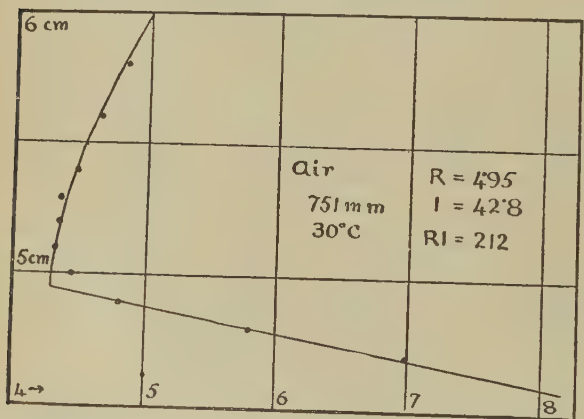
I have found it best to separate experiments whose object is to determine RI from those whose object is to find the stopping power of the gas. In the former the chief difficulty lies in overcoming initial recombination. This requires the pressure of the gas to be low, and the applied potential to be high. A little leakage of air into the apparatus, which can hardly be avoided under these circumstances, is no serious disadvantage, since the proportion of air can be found from a knowledge of R, and of the pressure and temperature at the time when R is measured; and these data are easily obtained. In the latter, any moderate voltage will do, since the range does not depend on potential; but it is desirable to have as

much gas as possible, and no leakage of air during the experiment, so that when the bulb containing a sample of the gas is taken away and weighed in order to find the proportion of the mixture, it may truly represent the condition of things during the earlier part of the experiment. It is best to work at a high temperature, if such is required to fill the chamber with gas which is nearly at atmospheric pressure.

Carbon bisulphide vapour is well superheated at a temperature of 30° and a pressure of 25 cm. The apparatus is, therefore, heated to that temperature; and RI for air is first measured.

Fig. 4 shows the readings obtained, and the curve which is drawn to find R and I. These are determined to be 4·95

Fig. 4.

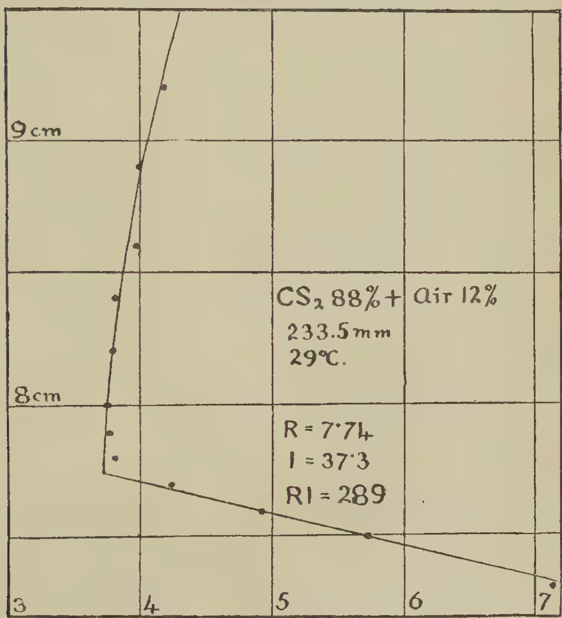


and 42·8 respectively, so that RI = 212, the temperature being 30° C., and the pressure being 75·1 cm. The apparatus is then exhausted and filled with CS_2 vapour to a pressure of about 24 cms. It is known from a separate experiment that 1000 volts per cm. is a saturating potential gradient, and a battery of 300 volts is therefore put on to the 3 mm. chamber.

The readings then taken, and the curve drawn are shown in fig. 5. It appears from these that R = 7·74, I = 37·3, so that RI = 289. The pressure has altered about 1 cm. during

the determination of the curve, but was found to be 23·35 at the moment when the corner (R, I) was passed. The temperature at the same time was 29°. Now the stopping-power of

Fig. 5.



CS_2 is 2·20, and the stopping-power of the mixture is (comparing with the previous experiment)

$$\frac{495}{774} \cdot \frac{751}{233 \cdot 5} = 2 \cdot 06.$$

Hence:

If x be the percentage of gas, we must have

$$x \times 2 \cdot 2 + 1 - x = 2 \cdot 06, \quad \therefore \; x = 88 \cdot 5.$$

The vapour is then cleared out of the apparatus, and dry air admitted. The value of RI for air is again determined, as shown in fig. 6, and found to be 214.

In the second experiment the proportion of gas to air, by pressure, is as 88·5 to 11·5; but, according to the energy spent by the α particle, as 88·5 to 11·5/2·2, *i. e.*, as 94·5 to 5·5. Hence

the true value of RI is found from the equation

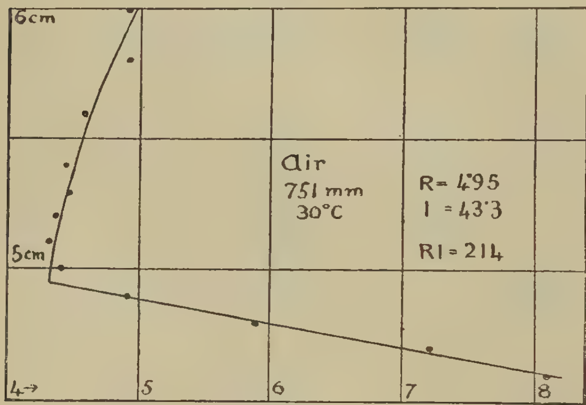
$$\cdot 945\,\mathrm{RI} + \cdot 055 \times 213 = 289;$$

whence $\mathrm{RI} = 294.$

Hence specific ionization of $CS_2 = 294/213 = 1{\cdot}38$.

The results of this particular experiment are recorded in the second line of the results for carbon bisulphide in § 2.

Fig. 6.



It seems probable that the determination of the ionization in various cases due to the α rays may be of considerable importance, and I therefore attach a drawing of the apparatus which I have used (Plate XI.) in the hope that it may save the time of any other workers in this direction.

In the figure, P is one of the three glass pillars which support the high potential plate. I have also used a glass plate, as shown by the dotted surface, to insulate the upper plate of the ionization-chamber. Sulphur and ebonite do not stand the heat. The upper gauze, *gg*, is the lower wall of the chamber, *g′g′* is the lower gauze and is earthed ; it is supported by three brass pillars, only one of which is shown. The vertical tubes are shown at TT, and the radium plate at RR. The semaphore, *ss*, is made of thin sheet-copper, and can be turned round so as to uncover the radium. It may be worth while mentioning that I have found it better to keep the plate, QQ, "out of sight" of any insulating material connected with the high-potential plate; and if this is not done

then the creep of electricity over the insulators, which is apt to occur when the potential is changed, exerts a troublesome electrostatic effect. DEFG is the outline of the electric oven. The tube A goes to the manometer, B to a bulb used in the determinations of stopping-power, and C to a bulb which contains the liquid whose vapour is being treated.

§ 2.

The following tables contain the results of the determination of the constant of specific ionization due to α rays. These have all been made recently, except when the contrary is stated. I have rejected a large number of earlier measurements. In the case of each experiment with a gas, the value of RI for air was found immediately before and immediately afterwards. It varies somewhat from day to day, and generally increases during any long series of experiments, since the warmth and dryness are conducive to the better retention of the emanation. The radium plate is not quite in so good a condition as it used to be, being covered with a very thin film of dirt and grease (mainly from the taps). This could, no doubt, be removed by a red heat; but I am unwilling to handle the radium film so roughly just now. The presence of the film slightly lowers the ranges, about ·5 to 1 mm. in air; and rather blurs the corners of the ionization curves.

Volts per cm.	Pressure in cm.	Temperature.	Percent. of Gas by Pressure.	Percent. of Gas by Energy.	RI.	RI (corrd.)	RI (air).	Ratio.
	Pentane.							
1670	25	44·5	83·5	95	262	265	198	1·34
,,	32·7	43	79·5	93·5	261	265	198	1·34
,,	23·75	35	91·5	97·5	261	263	197	1·33
,,	29·4	37·8	88·5	96·5	265	267	197	1·35
,,	38·5	40	81	94	271	275	200	1·37
,,	39·6	37·5	91	97	277	279	208	1·34
1000	32·6	39	88·5	96·5	281	284	208·5	1·36
,,	25·6	40	88·5	96·5	282	285	209	1·36
							Mean	*1·35*
	Carbon Bisulphide.							
1670	25·8	40	91	96	282	285	209·5	1·36
1000	23·35	29	88·5	94·5	289	294	213	1·38
,,	30·1	40	91	96	284	287	209·6	1·37
							Mean	*1·37*

Volts. per cm.	Pressure in cm.	Temperature.	Percent. of Gas by Pressure.	Percent. of Gas by Energy.	RI.	RI (corrd.).	RI (air).	Ratio.
	Ether ($C_4H_{10}O$).							
1000	24·2	29·7	82·5	94·5	274	277	214	1·295
1670	24·7	30	86·5	95·5	277	280	214	1·31
,,	26·2	49·5	87	96	270	273	207	1·32
								Mean 1·31
	Chloroform.							
1670	26·1	50·5	86	95	262	265	207	1·28
,,	23·5	50	86	95	262	265	207	1·28
,,	27·4	54	83	94	260	264	202	1·31
								Mean 1·29
	Ethyl Chloride.							
1670	23·3	14	89	95	244	247	191	1·29
3000	38	16	93	97	270	272	204	1·33
,,	58	26	91·5	96·5	266	268	203	1·32
1670	26·2	72	81	91	202	211	166	1·27
,,	31·8	60	85	93	238	242	186	1·30
,,	32·8	60	85	93	239	243	186	1·30
,,	29·4	34·5	91·5	96	258	261	199·5	1·31
,,	42·1	37·5	88·5	94·5	253	256	198·5	1·29
								Mean 1·30
	Carbon Tetrachloride.							
1670	25·0	60·7	89	97	257	259	197	1·315
,,	22·8	61	72	91	259	265	199	1·335
,,	27·9	61	83	95	253	256	199	1·28
,,	25·7	54	81	94·5	264	267	204	1·31
,,	26·4	53·5	90	97·5	264	266	202	1·32
								Mean 1·31
	Ethyl Iodide.							
1000	29·2	65	75·5	91	223	228	177	1·29
1700	25·2	68·5	90·5	97	261	263	207	1·27
								Mean 1·28
	Ethyl Alcohol.							
1000	34·8	72	85	98	213	216	174	1·24
,,	33·0	67	73	85	217	224	184	1·22
								Mean 1·23
	Methyl Alcohol.							
1000	35·9	65	92	94	217	219	179	1·22
	Methyl Iodide.							
1000	35·35	47	88	95	210	213	160	1·33
	Benzene.							
1670	27·4	62	71	89·5	226	232	181	1·28
,,	27·2	61·5	77·5	92·2	235	240	185	1·30
1000	25·6	67	89	96·5	251	253	194	1·30
1670	29·3	67	88	96	248	250	198	1.26
,,	27·0	67	83	94	252	256	198	1·29
								Mean 1·29

Volts per cm.	Pressure in cm.	Temperature.	RI.	RI (air).	Ratio.
	Acetylene.				
1000	Atmo.	54	274	223	1·23
,,	,,	37·5	298	232	1·28
,,	,,	30·5	294	229	1·28
,,	,,	70	252	201	1·25
					Mean 1·26
	Ethylene.				
1000	Atmo.	34·5	290	227	1·28
	Carbon Dioxide.				
1000	Atmo.	20	235	215	1·09
,,	,,	72	192	176	1·09
,,	,,	31	237	225	1·05
					Mean 1·08
	Nitrous Oxide.				
1000	Atmo.	29	240	229	1·05
	Oxygen.				
1000	Atmo.	20	247	226	1·09
	Nitrogen.				
1000	Atmo.	19	214	224	·96

Of the measurements recorded in the above table, those for acetylene, ethylene, carbon dioxide, and nitrous oxide were made some time ago. But they are probably quite correct enough to rank with the rest, which have for the most part been made recently, since they are not affected by temperature and initial recombination difficulties. The measurements most likely to contain error are those of the alcohols and methyl iodide, the latter because I have been unable from lack of material to repeat the one somewhat ancient determination, the former because for some reason the alcohols are very difficult to manage in my apparatus. They are apt to cause—particularly methyl alcohol—very large normal leaks, though other vapours, such as benzene, have no such effect I believe the cause to be connected with the presence of minute particles of fluff, which bridge across the walls of the ionization chamber, being stretched along the lines of force Although the apparatus is guarded with plugs of glass-wool, yet things of this sort seem to find their way into the chamber at times, and it is possible that the methyl alcohol sets them

free from the sides or base of the apparatus to which they are fastened by traces of grease. I have only once had the apparatus in perfect working order with methyl alcohol: at that time I had gone over the working parts with a magnifying glass to find and remove every foreign particle, and had washed the whole apparatus out with methyl alcohol itself. These good conditions lasted only a short time, and unfortunately a second cleansing process was not equally effective.

I must point out that the results for benzene and acetylene are now close together. In the preliminary paper I believed them to differ considerably, and used them as an illustration of the want of direct connexion between the energy spent and the ionization produced. It will be seen later that this effect is now clearly shown, but I was unfortunate in using a comparison of benzene with acetylene as an illustration.

§ 3 *.

Though our knowledge of the process of ionization by the α particle is as yet only small and imperfect, it does not seem out of place to draw together what facts we do know, and to endeavour to connect them by some thread of argument, which may be useful for a time.

In the first place, there is the fact that the ionization produced by the α particle increases as its velocity diminishes. Now, Rutherford has recently shown (Phil. Mag., Aug. 1906) that the particle spends energy at a uniform rate along its path. It follows, therefore, that the ionization produced is not proportional to the energy spent. In my preliminary paper I have already given a reason for supposing that the energy spent and the ionization produced are not directly connected, viz., that the former is related to the atomic weight by a simple law and the latter is not.

As a temporary hypothesis let us suppose that there is an intervening link; that the α particle produces a primary effect A, which in turn produces a secondary effect B. The

* The greater part of § 3 has been written since the above was read before the Royal Society of South Australia.

latter consists of ionization, the former may or may not do so. It is in the production of the primary effect that the energy of the particle is spent.

Since the energy spent is related to the atomic weight by a simple law, since it is independent of velocity, and since there is a critical speed at which all ionization ceases, which speed is the same for all atoms, it appears clear that A is a sub-atomic effect. It consists in the performance of some act which always involves the expenditure of the same amount of energy; and the stopping-power of an atom is proportional to the number of times that the act is performed within it. The effect might consist, for example, in some operation upon a common constituent of all atoms, such as an α particle. The critical speed might be that at which the moving α particle failed to penetrate, or, more generally, act upon the α particle of the atom.

In the next place, consider the effect B. The proportion of ionization to energy spent varies from molecule to molecule, and is dependent on the velocity of the α particle. The results described in this paper show that, as already said, $di = kf(v)de$. The nature of the function $f(v)$ is of great interest. In two previous papers I have made attempts to find it. In the first (Phil. Mag. Sept. 1905) I showed that if we assumed the ionization produced to be proportional to the energy spent, and both to v^n, and also assumed all the energy to be spent on ionization, then the form of the curve was most readily explained by taking $n = -\frac{1}{2}$. Later Rutherford showed that the energy of the α particle was not all spent on ionization, but that much still remained when ionization ceased. Using his figures, I then pointed out that with this modification of the hypothesis it seemed probable that $n = -2$ (Phil. Mag. Nov. 1905). But Rutherford's recent work shows that the hypothesis is still fundamentally wrong, because the ionization is not proportional to the energy spent. His results settle the whole question.

If $v =$ the velocity of the particle, r the range yet to be run, d a constant, which Rutherford estimates at 1·25 cm., then his conclusion is that v is proportional to $\sqrt{(r+d)}$. Now I have shown (Phil. Mag. Nov. 1905) that the ionization

produced by the particle during the last r cm. of its path is proportional to $\sqrt{(r+d)}-\sqrt{d}$ where $d=1\cdot33$. The two values of d may be taken to be the same. Hence di/dr is proportional to $1/\sqrt{(r+d)}$, *i. e.*, to $1/v$; which means that $f(v)=1/v$, or that the ionization produced at different points of the path in any gas is proportional to the time spent by the α particle in crossing the atom.

The formula which I have used here for the ionization was calculated on the hypothesis that the α particle lost its ionizing power abruptly, and that the slope of the top of the ionization curve was due to the effects of the thickness of the Ra film. Bronson's results (Phil. Mag. June 1906) seem to show that the loss of ionizing power is not quite so sudden as I supposed it to be. But I find that this does not affect the calculation of the form of $f(v)$. For we may take an extreme view and suppose the whole of the top slope to be due to a gradual decay of the α particle's powers, and none to the thickness of the radium layer. In that case the form of the ionization curve represents the effects of one particle. Now, the ionization at 6·5 cm. (in air) for RaC is nearly 4/3 of the ionization at 5 cm. At the former distance $r+d=\cdot5+1\cdot25=75$, and at the latter $2+1\cdot25=3\cdot25$. But $\sqrt{3\cdot25}/\sqrt{1\cdot75}=1\cdot36$; which is very nearly 4/3. Thus the ionization on this hypothesis also is inversely proportional to $\sqrt{(r+d)}$, and the true explanation of the top slope must lie between the two extremes.

It seems clear, then, that the ionization in the molecule is proportional to the energy spent in it (*i.e.*, to the stopping-power, or the amount of the effect A), to the velocity of the α particle inversely, and to a quantity k, constant for any one gas, but varying from gas to gas. It is this quantity which is given in the last column of the tables above.

The velocity of the α particle might enter into the formula because A is effective in producing B in proportion to the derangement of the atom or molecule consequent on the presence of the particle within it, and therefore to the time during which the intrusion lasts. There is something odd about this conclusion, which suggests a reconsideration of the position.

At this stage, therefore, it is natural to raise the question

whether the effect A really is the cause of the effect B, whether, that is to say, the energy spent by the α particle goes to the production of ions, or the ionization energy comes from some other source and the α particle merely pulls the trigger in its passage through the molecule. The fact that the ionization produced varies as the time of passage is certainly indicative of the truth of the latter hypothesis; whilst the occurrence of the stopping-power in the expression for the ionization is not necessarily evidence against it, because the factor k might be taken in conjunction with s, and ks might be found to represent not some derivative of the energy spent by the particle within the molecule, but some inherent property of the molecule which determined the ionization produced in consequence of the pulling of the trigger.

The quantity ks represents in the first place the specific ionization of the molecule; that is a relative measure of the ionization produced in a molecule when an α particle passes through it at a given speed. Now, it is an extraordinary thing that the values of ks which I have obtained for different molecules prove to be nearly related to already well-known molecular constants, such as the molecular volumes, molecular refraction constants, and so on.

In the following table the values of k, s, and ks of a number of substances are given in the first three columns; the fourth contains the values of the molecular volume v, and the fifth the ratio v/ks. The values of the volumes were for the most part taken from the tables in Ostwald's *Lehrbuch der Allgemeinen Chemie*, 2nd edition, p. 356, &c., but those of C_2H_2 and C_2H_4 were calculated from the general equation for obtaining the molecular volumes of organic compounds, and the values for CO_2, O_2, and H_2 were adopted on the assumption that they fell into line with the same equation. This is justifiable since my immediate object is to show a relationship between ks and the atomic volume in combination. As a matter of fact, the molecular volume of O_2 *per se* has been found by Dewar to be 27·4 (Chem. News, June 1898). This is close to the value in the table, viz., 24·4. But Dewar also finds H_2 to be 28, which is much larger than the value used in the ordinary formula.

	$k \times 10^2$.	$s \times 10^2$.	$ks \times 10^2$.	v.	$v/ks \times 10$.	B.	$ks/B \times 10_2$.
C_6H_6	129	333	430	96·0	223	75·5	5·8
C_5H_{12}	135	359	485	117·0	242	87·5	5·5
C_2H_4	128	135	173	44·0	254	43·0	3.9
C_2H_2	126	111	140	33·0	236	36·0	3.9
$C_4H_{10}O$	132	333	440	106·0	241	83·0	5.3
C_2H_6O	123	200	246	62·0	252	47·0	5.2
CH_4O	122	143	174	42·5	244	...	...
CCl_4	132	400	528	104·0	197	80·0	6·8
$CHCl_3$	129	316	408	85·0	208	72·0	5.7
C_2H_5Cl	132	236	312	71·0	227	55·5	5·6
CH_3I	133	258	343	66·0	193	52·0	6·6
C_2H_5I	128	312	400	86·0	215	68·8	5·8
CS_2	137	218	299	62·0	207	50·0	6·0
CO_2	108	147	159	35·4	222	30·0	5·3
N_2O	105	146	153	...	...	29·0	5·3
O_2	109	105	115	24·4	212	19·0	6.1
N_2	96	96	94	...	...	...	...
H_2	100	24	24	11·0	460	8·6	2·8

The value of k for H_2 is set down as 100. This is only approximate, and is probably too high. Its accurate determination will require the construction of special apparatus.

The agreement between the ratios v/ks in the fifth column is not such as to show that v and ks are directly proportional; but it is good enough to suggest strongly that they both rest immediately on some more fundamental property. The case is even a little stronger than appears at first sight, since it is clear that H_2 contributes an abnormal amount to the molecular volumes; the ratio v/ks is high whenever H preponderates in the molecule. Moreover, the molecular refractions also run closely parallel, as is well known, with the molecular volumes, and in general the connexion between the various physical properties of the molecule and its volume is more obvious than any connexion with its molecular weight. Consequently the quantity ks is closely related to most of the physical properties of the molecule. As a second instance, I have put in the sixth column of the above table the respective values of Sutherland's molecular volume B (Phil. Mag. Jan. 1895), and shown in the last column that this also is closely connected to ks. According to Sutherland, B tends to be proportional to the electric moment of the molecule. In this case also the variations in the ratios (see the last column) seem to be due to abnormalities in B rather than in ks; *e.g.*, C_2H_2 and C_2H_4 would fall into

line if the values of B for these substances were more in keeping with those for C_6H_6 and C_5H_{12}.

Since k is nearly the same for a number of gases, v/s is also nearly the same. Thus the molecular volume is connected, not very distantly, with the sum of the square roots of the weights of the atoms which make up the molecule.

Each of these physical properties which are so nearly related is partly additive, partly constitutive. For example, the molecular volume of an organic molecule depends in part on the sum of the volumes of the constituent atoms, and in part on the mode of constitution. This suggests that there is some fundamental and purely additive property of the atom itself, on which various semi-additive properties are based. For this reason it appears to be of great interest that the stopping-power of the atom has shown itself to be simply additive, so far as experiment has tried it; and at the same time to be closely connected with the atomic volume, the atomic refraction, and the rest. The additive nature of the constant may be seen from the following table, in which the observed stopping-powers of a number of gases are set alongside those calculated from assumed values for H, C, O, and Cl.

Assume $H_2 = \cdot 24$, $C_2 = \cdot 85$, $O_2 = 1\cdot 03$, $Cl_2 = 1\cdot 78$, the air molecule being taken as standard :—

	C_2H_2.	C_2H_4.	C_6H_6.	C_5H_{12}.	CH_4O.	C_2H_6O.
Calculated	1·09	1·33	3·27	3·56	1·41	2·08
Observed	1·11	1·35	1·33	3·59	1·43	2·00

	$C_4H_{10}O$.	CO_2.	CCl_4.	$CHCl_3$.	C_2H_5Cl.
Calculated	3·41	1·47	3·98	3·21	2·34
Observed	3·33	1·48	4·00	3·16	2·36

It is of course too early to say that the stopping-power has been proved to be a perfectly additive property of the atom, yet it is clear enough that it is more so than any other known property, except one. The more nearly experiment shows it to be strictly additive, the greater will be its title to rank with mass itself. I hope to begin soon a fresh and more accurate set of experiments in the endeavour to find to what extent the additive law holds.

The near proportionality of the stopping-power to the atomic square root is an effect which is quite apart from its additive nature. Its existence is a connecting link between the atomic weight on the one hand and the atomic volume, refractive power, &c., on the other. The preliminary paper on this subject contained a table of stopping-powers as found up to that time. I have made several new measurements of these constants, which are, I believe, an improvement on the old. This is particularly the case with the metals Au, Pt, Sn, Ag, Cu, and Al, since the specimens used were obtained as pure from Messrs. Johnson, Matthey & Co. I find that if the stopping-powers of S, Cl, and I are calculated from those of molecules containing them, on the assumption that the additive law holds, then these fit in very well with the metals. So also does Br fit in very well; it is quite possible that the divergence is due to experimental error, since the only measurement on a molecule containing Br was made at a very early stage of this inquiry. The divergence from the exactness of the square-root law, which I have previously pointed out, seems to occur only in the molecules whose weights are below 30; these have an abnormally low value, as may be seen from the table below, in which the "air-atom" = 1. It is curious that a similar effect should occur in the case of the atomic heats:—

	H.	C.	N.	O.	Al.	S.	Cl.	Fe.	Ni.
s	·24	·85	·94	1·05	1·495	1·76	1·78	2·29	2·44
$\sqrt{\omega}$	1·00	3·47	3·74	4·00	5·20	5·65	5·96	7·48	7·65
$s\sqrt{\omega}\times 10^3$	240	246	251	262	287	312	299	307	319

	Cu.	Br.	Ag.	Sn.	I.	Pt.	Au.	Pb.
s	2·46	2·60	3·28	3·56	3·44	4·14	4·22	4·27
$\sqrt{\omega}$	7·96	8·93	10·37	10·9	11·2	13·95	14·0	14·35
$s\sqrt{\omega}\times 10^3$	309	291	316	326	307	297	301	298

[*Note added Nov.* 8.—The figures in this table will no doubt be somewhat modified by more accurate measurements, and I am revising my methods for this purpose. The values for C and N are certainly, perhaps 4 per cent., too low. As regards C, see also the accurate measurements of Kučera and Mašek (*Phys. Zeit.* 1906, p. 634). The value of H may be in error as this gas really requires a special apparatus.

Good sheets of Fe and Ni are hard to obtain. The value for Br requires redetermination. I think the rest are fairly accurate, and I trust the average error in this table will not in the end be found to exceed one or two per cent. The values of s for Sn and Ag certainly show some variation from the square-root law as indicated by the difference of the ratios 316 and 326.

The results of Kučera and Mašek are a very welcome confirmation of the square-root law. Only on one point there seems to be still an obscurity. These authors claim to have shown that the stopping-power of a metal depends on the velocity of the α particle, and certainly some effects observed by them and Kleeman and myself would be explained if this were true. On the other hand, this hypothesis is contradicted by Rutherford's direct measurements of the velocity after passing through various sheets of foil. It might be argued that Kučera and Mašek measure the *relative* stopping-powers of metal and air, and that their results really show that the stopping-power of air decreases with increasing velocity of the α particle, not that that of the metal increases. But this is inconsistent with their own measurements in the earlier part of the paper, where they show that two equal Al foils cause twice as much drop as one; and also with the similar and fuller measurements of this effect made by Levin. I may add that in comparing their own results with those of Kleeman and myself, they make a natural mistake in supposing that we placed our foils in contact with the Ra. But as a matter of fact they were about 1·5 cm. away, so that the average speed of the particle in going through a metal equal to 3 cm. of air was about the same as the average speed in a gas.]

One other point invites some consideration. Whilst the saturated ionization curve seems to be the same for all gases, yet the effects of initial recombination vary from gas to gas and from point to point on the curve. This fact can be explained by the consideration that the amount of the ionization produced is an intramolecular effect, and is therefore independent of the physical conditions of the molecule and of the relations of one molecule to another, whilst the amount of initial recombination depends on extramolecular relations,

on pressure, perhaps on temperature, and so on. The increase of initial recombination towards the end of the path of the particle may be due in part to the existence of a greater number of molecules that have lost more than one ion, since in such cases recombination would be harder to prevent. This raises the question as to how the ionization is distributed between the molecules which the α particle traverses. There does not appear to be any evidence, as yet, that the chance of an ion being formed from a molecule is dependent on whether the molecule has already lost one or more ions: rather the contrary. If this is the case, occasional molecules must lose several ions. Nor is it yet clear in what mode ionization occurs. Does the α particle simply cause the removal from the molecule of one or more electrons? May there not possibly be a more complete disruption of the molecule, or even the atom? There is one curious parallelism in numbers which may have a bearing on this question. Ramsay and Soddy (Proc. Roy. Soc. lxxii. p. 204, 1903) found that 50 mmg. of radium bromide in solution evolved gases at the rate of ·5 cc. per day—*i. e.*, 2×10^{19} molecules per day. Now, Rutherford has shown that one gram of radium bromide, without its radioactive descendants, produces $3{\cdot}6 \times 10^{10}$ α particles per second. Each α particle makes 86,000 pairs of ions. Hence the number of ions made in one day by 50 mmg. is

$$3{\cdot}6 \times 10^{10} \times {\cdot}05 \times 60 \times 60 \times 24 \times 172{,}000 = 2{\cdot}7 \times 10^{19}.$$

This number is an inferior limit. A superior limit is found by considering all the radioactive products of radium to be present in full, in which case the number will be between five and six times greater. The close agreement of these numbers certainly fits in with the hypothesis that an actual disruption of the water molecule takes place in consequence of the passage of the α particle through it.

I owe my thanks to my assistant, Mr. A. L. Rogers, for the great care and skill with which he has made the apparatus used in this work, and drawn the plate illustrating this paper.

Nov. 14.—The very interesting experiments described by Kučera and Mašek in recent numbers of the *Phys. Zeit.* afford

a very welcome confirmation of the square-root law of absorption.

On one point only there is still some obscurity. It was pointed out by Kleeman and myself (Phil. Mag. Sept. 1905) that the α particles from Ra A were less stopped than those from Ra C by a sheet of metal placed at a given distance from the Ra. (The distance was 1·5 cm. in our experiments, not zero as Kučera and Mašek suppose; the point is of some importance in view of their comparison of our results with their own.) This could be explained by supposing that "the slower α particles were a little less affected than the swifter" (Phil. Mag. Sept. 1905, p. 337). Kučera and Mašek come to the conclusion that they have established this fact from their own experiments. The trouble is that this is hardly consistent with Rutherford's experimental determination that successive equal layers of Al foil absorb equal energies from the α particle's motion. Unless indeed it is supposed that metals absorb to an extent which is independent of speed, and that it is the air which is at fault; for the experiments of Kučera and Mašek and those of Kleeman and myself relate to the relative absorption of metal and air. But this again is opposed to the experiment of Levin (*Phys. Zeit.* Aug. 1906), who found that equal layers of foil dropped the ionization curve by equal amounts; and to a similar experiment described by Kučera and Mašek themselves (*Phys. Zeit.* xviii. p. 630).

The following figures are taken from measurements by Kleeman and myself:—

Drop of curve when sheet of metal is placed 1·5 cm. above Ra.

	Drop for RaC.	Drop for RaA.	Drop for Emanation.	Drop for Ra.
1. Al	2·31	2·25	......	......
2. Sn	1·92	1·77	1·67	......
3. Sn	1·91	1·81	......	......
4. Ag	·86	·81	·50	·73
5. Au	1·63	1·46	1·36	......

Discussion.

Mr. F. Soddy, in a letter read by the Secretary, drew attention to the parallelism between the conclusion of the author that the same molecule may become ionized more than once, and that of Rutherford that the α-particle itself is doubly charged. The view that the charge carried by the gaseous ion was the same in all cases seemed to have been definitely rendered untenable by the results of the author. So also was the view that the energy expended in the production of an ion was independent of the nature of the gas and the same in all cases. The theoretical consequences of the work of the author were of the greatest importance, for it is inconceivable that the α-particle should be able to pass through matter in the manner it has been proved to do, if the matter in question and the α-particles consisted substantially of electrons. The suggestion made finally in the paper of the relation between the ionization and production of hydrogen and oxygen by radium had a'so been made independently by Rutherford.

Dr. R. S. Willows said he could not agree with Mr. Soddy that on the evidence at present available the paper rendered untenable the hypothesis that all ions carried the same charge. He pointed out that the stopping-power of a gas obeyed a certain law down to atomic weights about 30, and a different law afterwards, and said that this might have some relation to the fact discovered by Barkla that the secondary radiation produced by Röntgen rays followed a definite law down to atomic weights about 40, and then a different law. The numbers would depend, no doubt, to some extent on the character of the radiation used.

XXXVI. *The Rate of Recovery of Residual Charge in Electric Condensers.* *By* Fred. T. Trouton, *F.R.S.*, *and* Sidney Russ, *B.Sc.**

[Plate XII.]

The analogy between the phenomenon of residual charge in Leyden jars and the recovery from elastic overstrain in solids has often been pointed out. The theoretical investigations of Boltzmann and the experiments of Kohlrausch and Hopkinson have thrown considerable light on the subject.

In view, however, of recent work done by Mr. A. O. Rankine †, and subsequently confirmed by Mr. Phillips ‡, on the recovery of solids from overstrain by a method in which the strain was kept constant while the stress disappears, it seemed desirable to experimentally examine the relations governing the rate at which residual charge makes its appearance in a discharged condenser under conditions analogous to those of their experiments.

The explanation of the phenomenon of residual charge has been referred § to a heterogeneity in the structure of the dielectric due to its consisting of parts having diverse conducting and dielectric properties, all however subject to simple linear laws. This hypothesis leads to an exponential expression for the rate of recovery of the residual charge. This explanation is on somewhat the same general lines as and leads to analogous results to that which has been advanced in the case of recovery from elastic overstrain, and which is based on the assumption of the material consisting in part of viscous and in part of purely elastic constitutents. This theory of elastic overstrain leads to an exponential expression for the recovery from the strain in terms of the time. Now no such expression fits experimental evidence, but on the other hand the recovery can be well represented by the logarithm of a quantity proportional to the time of recovery. Thus in

* Read March 8, 1907.

† Phil. Mag. [6] vol. viii. 1904, p. 538.

‡ Phil. Mag. [6] vol. ix. 1905, p. 513.

§ Maxwell, vol. i. p. 414.

Rankine's experiments where a constant strain is maintained, the stress while decreasing is given by

$$S=S_0(1-K\log(pt+1)),$$

where S represents the stress at any moment required to preserve a constant strain in the stretched substance.

Our work was undertaken in the first place to examine whether the exponential law of the recovery of the Residual Charge was justified by experimental data, and failing this, to attempt to find a law of the above type agreeing with observation.

The examination of the rate of recovery was carried out in two ways.

Where the Residual Charges were great enough the current as it came out of the recuperating dielectric was simply passed through a sensitive galvanometer and the rate of recovery thus noted.

This method was found quite feasible in the case of a large condenser, the dielectric of which was celluloid, but was unsuited for mica condensers of the ordinary standard type.

For such, a method was devised in which the plates of the condenser were allowed to charge up to a certain difference of potential (due to the Residual Charge coming out of the dielectric) which was measured by an electrometer, and kept at that difference of potential by inserting a variable resistance in parallel with the condenser.

The difficulty of getting a resistance as high as that required, and one which at the same time could be easily adjusted, was surmounted by adopting ionized air as the material of which the resistance was made. The cross section of this resistance was arranged so as to be easily and quickly varied as alteration in the resistance was required.

Rate of Recovery, or Current at Constant Difference of Potential.

In these experiments a constant difference of potential was maintained between the two coatings of the condenser, and the rate at which the Residual Charge leaked out under these conditions was found.

It is clear that if a constant deflexion on the scale of the electrometer can be maintained while the Residual Charge is appearing, then a constant difference of potential exists between the coatings of the condenser.

This constant potential-difference was maintained by the use of a variable resistance obtained by connecting the two coatings of the condenser to two parallel plates of tin, on one of which a thin layer of uranium fluoride, for the purpose of producing the necessary ionization of the air between the plates, was uniformly sifted. Owing to the difference of potential between these plates there will be a gradual leak of the residual charge across the air-gap between them, and if the effective area of the uranium surface can be adjusted the value of this current can be controlled as required. The effective area of the air conductor was regulated by a movable shutter which was made to slide in between the two parallel plates, thus curtailing the active area of the uranium fluoride surface and thereby the cross section of our conductor.

The mode of carrying out an experiment was to allow the difference in potential between the plates of the condenser to rise through the development of residual charge, to a certain selected value, and then to prevent further increase by connecting them through the variable ionized air-resistance. To preserve this constancy in potential the shutter must be continuously pushed in so as to diminish the current at the proper rate, namely, that at which the residual charge is developed by the unstraining dielectric.

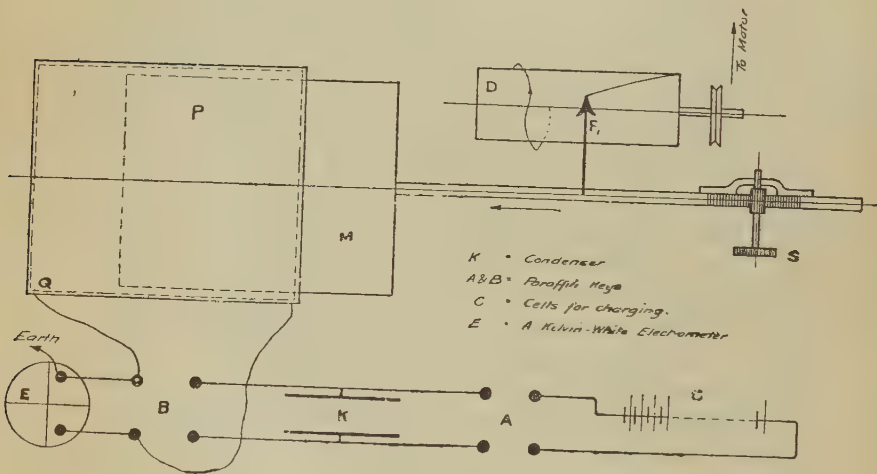
The position of the shutter at any moment while being pushed in so as to keep the deflexion of the electrometer constant, could be readily found: this is a measure of the current or the rate at which the charge comes out of the condenser when a constant potential-difference is maintained between its coatings.

A "hand" method was first tried for closing the shutter. The apparatus used is indicated in fig. 1. K a condenser of mica plates was first charged by a series of cells C for a definite interval of time ($1\frac{1}{2}$ mins.) discharged through the key B, and its two coatings were then connected to the electrometer E, one quadrant of which was earthed. P and Q

are the two parallel plates, the uranium fluoride being spread on the bottom one Q. M the movable shutter was initially drawn right back, thus exposing the whole of the uranium fluoride surface.

On connecting K to E the deflexion increased at first, but owing to the leak introduced the turning point was soon reached; at this instant the shutter M was moved in by means of the wheel S, which engaged in the milling of a brass strip connected to M. The motion of the shutter was recorded by a pointer P (attached to the brass strip), which traced a path over

Fig. 1.

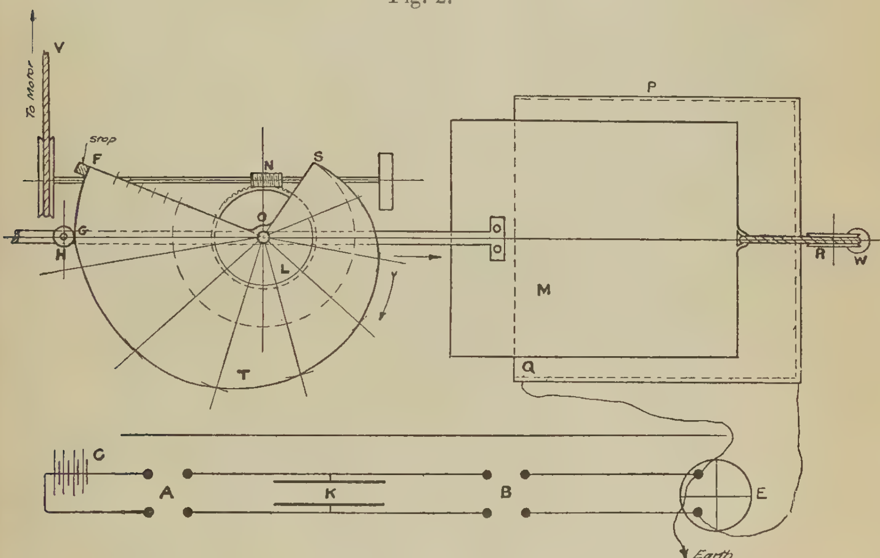


a blackened surface of paper on a cylindrical drum D which revolved at a known speed by means of a motor, to which it was connected by a system of gearing-down countershafting. In this way the amount of active area exposed at each moment was recorded. This was proportional to the conductivity, and consequently to the current developed at each moment by the recovering dielectric.

In controlling the motion of the shutter by hand one is very apt to overshoot the mark, due no doubt to air remaining ionized for a short interval after the shutter has covered the

ionizing surface, this making the apparatus slow in responding to any alteration in area. If S is turned too quickly, the leak is being cut too rapidly and the deflexion on the scale increases, so that S has to be stopped for a short interval in order that the deflexion may return to its former value. On account of this the curve traced out by P on the drum was unduly irregular. After many trials made with this apparatus it was decided, for the above reasons, to move the shutter mechanically. The best of the tracings obtained by the previous experiments was utilized for constructing a

Fig. 2.



cam for this purpose. The cam, which was cut out of wood, was made to close the shutter by a uniform rotation about its axis. Fig. 2 illustrates the way in which this was done.

H is a small brass wheel fixed to a brass strip which passes underneath the cam T and is joined to the shutter M; a cord fixed to the other end of the shutter passes over a pulley R and carries a weight W which keeps the wheel H tightly pressed against the cam. As the cam rotates

H is pulled forward and the shutter moves in at a varying speed.

The uniform rotation of the cam was obtained by means of the toothed wheel L, which engaged in an endless worm N cut in a steel shaft, the other end of which carried a pulley-wheel V which was driven by a small motor highly geared down. The portion FG on the cam entails no forward motion of H, and is to allow the speed of the motor to become constant after starting. The method of carrying out an experiment was as follows:—

The condenser was charged for $1\frac{1}{2}$ minutes to 20 volts, discharged through the key B, and the coatings then connected to the quadrants and the parallel plates P, Q. The cam was timed so that the point G on it was just opposite H when the coatings of the condenser were connected to P, Q.

The desired result was that the deflexion of the electrometer remained constant while the shutter was being moved in. How nearly this was attained will be seen from the figs. 3, 4, 5 (Pl. XII.), in which the deflexions are plotted against time, the data being given in Tables I., II., III. In each figure the top curve was obtained by allowing the Residual Charge to come out of the condenser in the ordinary unrestrained

Table I.

$\frac{1}{2}$ Partition exposed.

Time in Minutes.	*Deflexions of Electrometer.*		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
$\frac{1}{4}$	15·3	14	13·4
$\frac{1}{2}$	17·5	14·8	13 9
$\frac{3}{4}$	18·5	15	13·6
1	19	15	13·2
$1\frac{1}{4}$	19·3	15	12·6
$1\frac{1}{2}$	19·5	15	11·8
$1\frac{3}{4}$	19·6	14·9	11·1
2	19·5	14·8	10·4
$2\frac{1}{4}$	19·4	↑	9·7
$2\frac{1}{2}$	19·3	|	9·0
$2\frac{3}{4}$	19·2	|	8·4
3	19·1	↓	7·8

TABLE II.

2 Partitions exposed.

Time in Minutes.	Deflexions of Electrometer.		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
¼	16·5	15·3	14·9
½	18·0	15·5	14·3
¾	18·5	15·6	12·8
1	18·85	15·6	11·1
1¼	18·95	15·6	9·4
1½	18·95	15·65	7·8
1¾	18·82	15·7	6·4
2	18·65	↑	5·1
2¼	18·50		4·1
2½	18·3		3·1
2¾	18·1		2·3
3	18·0	↓	1·5

TABLE III.

4 Partitions exposed.

Time in Minutes.	Deflexions of Electrometer.		
	Shutter kept closed.	Shutter moved in by cam.	Shutter kept open.
¼	14·5	12·1	10·3
½	15·9	12·1	8·05
¾	16·5	12·05	6·0
1	16·8	12·1	4·0
1¼	16·9	12·3	2·5
1½	16·92	12·3	1·45
1¾	16·88	↑	·60
2	16·80		·01
2¼	16·7		↑
2½	16·5		
2¾	16·35		
3	16·2	↓	↓

way, no correction being made for surface leakage. The middle curve was obtained with the cam in action, and the bottom curve with the plate Q fully exposed the whole

time, thus showing how the Residual Charge immediately begins to leak across the gap between the two parallel plates P, Q.

The plate Q, upon which the uranium fluoride was spread, was divided into six partitions which could be separately covered with metal sheets, thus varying the surface over which the leak took place.

Fig. 5 was obtained with $\frac{1}{2}$ partition exposed.
Fig. 6 ,, ,, 2 partitions exposed.
Fig. 7 ,, ,, 4 partitions exposed.

From the foregoing curves it was seen that the shape of the cam was approximately correct as it maintained the deflexion of the electrometer constant.

Now since the difference of potential between the coatings of the condenser was constant, the rate the Residual Charge was coming out of the condenser at any moment (and passing across the space between the two parallel plates) was proportional to the area of the bottom plate exposed. This, in turn, was proportional to $(L_t - OS)$, where L_t is the length of the radius vector of the cam which at any moment passes through H, and OS is the length of the radius vector when the shutter is closed and therefore no current passing across the air-gap.

If, then, we plot the values of $(L_t - OS)$ against time, we obtain the rate at which the Residual Charge came out of the condenser under a constant difference of potential.

In all the experiments, however, surface-leakage was present. Its value in terms of the uranium-fluoride leak was always found after the main experiment was finished. Owing to this surface-leakage a certain quantity (which admits of easy calculation) must be added to the value of $(L_t - OS)$ in order to give the rate of appearance of Residual Charge.

This has been done, and the corrected values of $(L_t - OS)$ which represent current have been plotted against time in figs. 6, 7, 8, where ordinates represent current and abscissæ time. The data corresponding to these figures will be found in Tables IV., V., and VI.

TABLE IV.—$\frac{1}{2}$ Partition exposed.

Time. 1 unit=7·5 seconds.	*Current* (in arbitrary units). Observed.	*Current* (in arbitrary units). Calculated from equation $C=\frac{d}{t+b}-a$.
0	6·79	6·79
1	5·29	5·35
2	4·19	4·36
3	3·47	3·64
4	3·09	3·10
5	2·79	2·66
6	2·47	2·31
7	2·09	2·05
8	1·84	1·78
9	1·61	1·57
10	1·49	1·40
11	1·29	1·23
12	1·09	1·10
13	·89	·97
14	·79	·81
15	·65	·77
16	·49	·68

Values of constants in the above equation:—

$a=1{\cdot}01$, $b=4{\cdot}44$, $d=34{\cdot}63$.

TABLE V.—2 Partitions exposed.

Time. 1 unit=6·5 seconds.	*Current* (in arbitrary units). Observed.	*Current* (in arbitrary units). Calculated from equation $C=\frac{d}{t+b}-a$.
0	6·48	6·48
1	4·98	5·04
2	3·88	4·06
3	3·16	3·33
4	2·78	2·78
5	2·48	2·34
6	2·16	1 99
7	1·78	1·70
8	1·53	1·46
9	1·30	1·25
10	1·18	1·07
11	·98	·91
12	·78	·78
13	·58	·65
14	·48	·55
15	·34	·45
16	·18	·36

Values of constants in the above equation:—

$a=1{\cdot}33$, $b=4{\cdot}44$, $d=34{\cdot}68$.

TABLE VI.—4 Partitions exposed.

Time. 1 unit=5 seconds.	*Current* (in arbitrary units).	
	Observed.	Calculated from equation $C=\frac{d}{t+b}-a$.
0	6·42	6·42
1	4·92	4·98
2	3·82	3·99
3	3·10	3·27
4	2·72	2·72
5	2·42	2·29
6	2·10	1·94
7	1·72	1·68
8	1·47	1·41
9	1·24	1·20
10	1·12	1·03
11	·92	·86
12	·72	·73
13	·52	·60
14	·42	·44
15	·28	·40
16	·12	·31

Values of constants in the above equation:—

$a=1{\cdot}38, \quad b=4{\cdot}44, \quad d=34{\cdot}63.$

Fig. 6 corresponds to the case of $\frac{1}{2}$ partition exposed.
Fig. 7 ,, ,, ,, 2 partitions exposed.
Fig. 8 ,, ,, ,, 4 partitions exposed.

The amount of surface-leakage can be seen from Table VII.

TABLE VII.

Partitions exposed.	Ratio: $\frac{\text{Surface Leakage}}{\text{Uranium Leakage}}$.	Percentage.
$\frac{1}{2}$	$\frac{\cdot 0137}{\cdot 175}$	7·8
2	$\frac{\cdot 015}{\cdot 52}$	2·9
4	$\frac{\cdot 0118}{\cdot 646}$	1·8

The corrected values of the current when plotted against time were found not to agree with an equation of the exponential type, with which they should, were the actions going on in an unstraining dielectric in agreement with the theory of heterogeneous structure as given by Maxwell.

An equation of the form

$$C=\frac{d}{t+b}-a$$

was found to fit the experimental curves, figs. 6, 7, & 8 well. In these figures the continuous lines are obtained from the above equation, and the experimental values appear as circles.

Galvanometer Method.

The condenser used in these experiments was of about 5 microfarads. It was found to afford a very large residual charge. Celluloid sheets, in conjunction with layers of paper, laid on either side, were employed in its construction. This gave a dielectric of avowedly heterogeneous structure, and therefore might very well be expected to act in accordance with the exponential law; nevertheless it will be seen, from the character of the curves obtained with it, that this is not the case.

The method of experiment was to charge the condenser up to about 20 volts, discharge, and connect with a low-resistance D'Arsonval galvanometer which was practically dead-beat. A sufficient current was then obtained, which was read at frequent intervals until too small to be observed. Two persons were required for this—one to note when the spot of light passed the divisions on the scale, the other to take the time of doing so.

It was always found possible to fit a hyperbola to the observations so obtained. One of the series (Table VIII.) of observations is shown plotted in fig. 9 (Pl. XII.). The curve

$$C=\frac{2304 \cdot 4}{t+8 \cdot 94}$$

was found to suit these observations, and is there shown. The experimental points are seen to fit the curve in a most satisfactory way.

TABLE VIII.

C.	t.	C.	t.
258	0	25	79
208	3	23	87
158	7	21	97
108	13	18	115
78	21	16	134
58	30	14	155
48	38	12	184
43	43	11	203
38	49	10	222
33	58	9	251
28	69		

In these experiments, since the condenser was practically short-circuited, there was comparatively speaking no difference in potential between the plates throughout the recovery; so that the dielectric recovered at its maximum or normal rate.

These and the experiments in the first part of the paper are in accordance with the analogy of recovery of elastic solids from overstrain. The quantity of electricity recovered up to any time t is given by an expression of the type

$$Q = a \log (t+b),$$

which is the expression found by Rankine and others to fit the recovery from overstrain in elastic bodies.

DISCUSSION.

The CHAIRMAN (Prof. PERRY) said that together with Prof. Ayrton he had carried out a series of experiments on this subject which were described in vol. 30 of the Proceedings of the Royal Society. Observations were made upon strained materials, Leyden jars, and voltameters, with the object of testing Maxwell's Law. The current flowing into or from the condenser was kept constant and equal to zero, and under the conditions of the experiments the solution of Maxwell's differential equation gave for the potential v between the surfaces of the condenser the equation

$$v = A + Be^{-\beta t} + Ce^{-\gamma t} + \ldots$$

If β was less than γ, γ less than δ, and so on, then after

certain intervals of time certain of the terms became unimportant, and if the time were sufficiently large the equation for v reduced to

$$v = A + Be^{-\beta t}.$$

He was of opinion that if the higher terms in the expression for v were maintained, it would be found that the results obtained by the authors could be made to fit a formula deduced from Maxwell's theory. It was necessary to exercise caution in the selection of an empirical formula to represent the results of experiments. Two very different formulæ would often over a moderate range fit a series of results equally well.

Mr. A. Campbell said that the method of using ionized air for a variable and adjustable resistance was very ingenious. A somewhat similar use of ionized air as a constant high resistance was made by Prof. McClelland (Royal Dublin Society, Feb. 1904), who found such a resistance most useful in comparing very small capacities. In the case of the celluloid and paper condenser the effects must be partly electrolytic, for moisture is present as an important factor. This might even be true in the case of a mica condenser in which the tin-foil does not make extremely good contact with the dielectric and where creeping charges might occur. Thus the effects described in the paper may not be truly dielectric.

Mr. Rollo Appleyard remarked that in the absence of details regarding the condenser, there was considerable difficulty in interpreting the results obtained by the authors. They inform us that it was constructed of "celluloid sheets, in conjunction with layers of paper, laid on either side," but we are left in doubt as to the material of the conducting surfaces, and as to the nature of the contact between the celluloid and those surfaces. As the object of the authors is to investigate electrical stresses in a manner analogous to that employed by Prof. Rankine for mechanical stresses, it would seem to be as important to ensure that the electrical data relate to the substance of the dielectric and not to the contact-surface, as it is to premise that in the mechanical case the movements of the relative parts of the material are

measured according to a plan which avoids the errors due to slip of the clamping devices.

If the contact is imperfect, the results relate not so much to what transpires within the dielectric as to what takes place at the surfaces, and such results are more nearly analogous to those representing mechanical slip at the clamps than to those representing changes of stress or strain of a test-bar. It happens that with celluloid it is comparatively easy to eliminate the effects due to imperfect contact. That the residual-charge effects for celluloid are small may be predicted from general principles. It is convenient to take the case of a sheet of dielectric placed between two contact-discs making intimate electrical contact with it except at the edges of the sheet. If the circuit is completed through a battery and a galvanometer, there is usually a considerable initial deflexion, which becomes less, minute by minute. Let the reading (A) be taken after the first minute, and let a second reading (B) be taken after say the second minute. Then, with proper precautions, let the battery be switched out of the circuit, and let the circuit be again closed. The deflexion will now be in the reverse direction, and at the end of another minute let its value be (C). It is common knowledge that, for most dielectrics, approximately, $A=B+C$. The deflexion C is the measure of the residual effect after one minute from the moment of switching out the battery, and it is obvious that C can be evaluated as being approximately the difference $(A-B)$. But $(A-B)$ is what is known as the "electrification." It follows that where there is no "electrification" there is no great residual effect. Thirteen years ago * he (Mr. Appleyard) showed that, when mercury electrodes are employed with celluloid sheets, $(A-B)$ vanishes; in this case the residual effect very rapidly disappears. But he also showed that when mercury electrodes are not used, $(A-B)$ may be considerable, and that this is entirely due to surface causes †. Since reading the proof of the authors' paper, I have obtained the following figures :—

* Proc. Physical Soc. vol. xiii. pp. 155–169, January 1895.

† Proc. Physical Soc. vol. xix. pp. 724–739, December, 1905.

Celluloid Sheet.—(Mercury Electrodes.)

Volts.	Deflexion after		Deflexion after 3 mins. (Battery removed at 2 mins.)	Resistance between electrodes (megohms).
	1 min.	2 mins.		
	(A)	(B)	(C)	
176...	118	118	−0·5	80·5
289...	232·5	232·5	−0·5	67·0
442...	348	348	−1·0	68·5
600...	464·5	464·5	−1·0	69·7

Celluloid Sheet.—(Tin-foil Electrodes.)

Volts.	Deflexion after		Deflexion after 3 mins. (Battery removed at 2 mins.)	Resistance between electrodes (megohms).
	1 min.	2 mins.		
	(A)	(B)	(C)	
176...	28	31	0·0	339
289...	134	143	0·0	116
442...	293	302	0·0	81
600...	471	480	+1·0	69

In the above tests the same sheet of celluloid was used. Its thickness was 0·177 cm., and the diameter of the circular electrode surface was 14 cm. The total pressure on the upper tin-foil electrode was 3170 grammes. Temperature 20 ° C.

In comparison with these figures, the results arrived at by the authors seem to break new ground, and a study of our differences should lead to a better understanding of these phenomena. As the authors worked with only 20 volts the attractive force between the electrodes and the dielectric would be less than in his case. The steadying effect of the higher voltage is seen in the above table, where the resistance gradually diminishes with the volts; finally it has about the same value as that measured with mercury electrodes. In Table VIII. the authors have expressed their results in an arbitrary unit of time, so that it is impossible to determine whether in this respect our results are in agreement. They have also adopted an arbitrary unit of current, so that again it is impossible to reproduce their conditions.

Mr. W. A. Price pointed out that there were at least two

possible theories to explain absorption or residual charge. Maxwell explained it by supposing that different parts of the dielectric differed in conductivity and in their dielectric constants. It might, however, be due to the slow change of orientation of electrically polarized elements of the dielectric, called by Heaviside *electrets*, moving in a viscous medium under the action of the electric force. Either hypothesis might give the same differential equations, and observations on the rate of accumulation or dissipation of charge could not distinguish between them.

Prof. TROUTON said, in reply to the Chairman, that they had tried to represent their results by an exponential expression, but the result was not nearly so good as that obtained from the formula given in the paper. Of course with more constants an exponential expression could be used. He had found that their formula fitted the results of the Chairman and Prof. Ayrton. In reply to Mr. Campbell and Mr. Appleyard, he said that possibly there might be electrolytic action, but most of the phenomena described were due to residual-charge effects. With reference to the units, he said the times were expressed in seconds but the unit of current was arbitrary.

XXXVII. *Experimental Mathematics.* By E. A. N. POCHIN, *B.A.**

"MATHEMATICS is an experimental science, just as much as Chemistry or Physics."

This very suggestive statement was made, about a year ago, by a friend who held that everything on earth was done the wrong way, and the teaching of mathematics in particular.

Being greatly struck by so original a view, I resolved to give it a trial, and selected e for my purpose, as being the first thing to which I could not attach some visible, material meaning. The method of investigation I now place before you, in the hope that it will be of interest, not only in itself, but especially as affording a graphical treatment of logarithms, which may be of use to those engaged in teaching.

Here is an instrument, for drawing logarithmic spirals,

* Read March 22, 1907.

which is shown diagrammatically in fig. 1. It consists of a metal boss, with a compass-point at its centre O, and has a

Fig. 1.

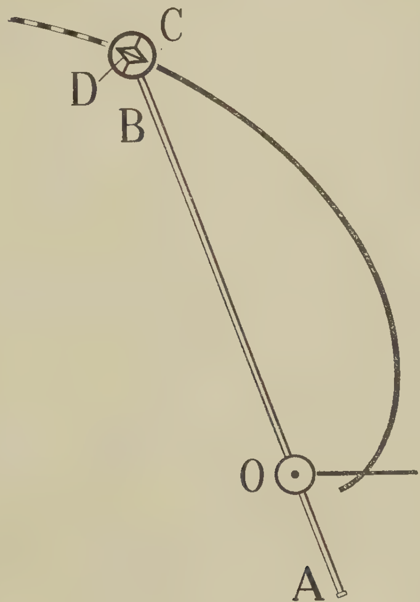
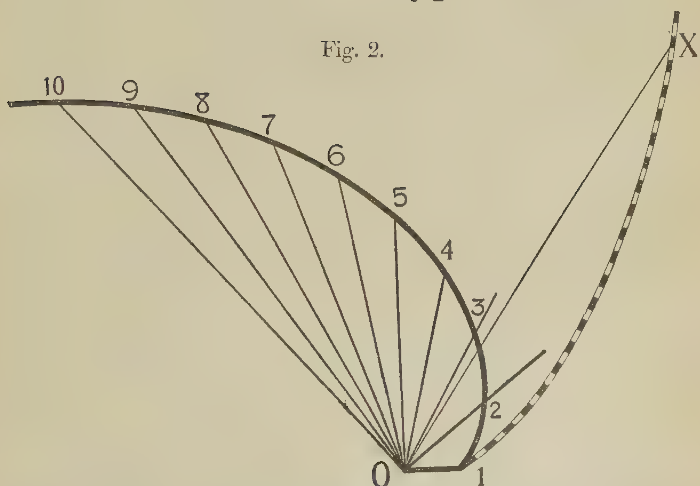


Fig. 2.



smooth steel rod AB sliding freely through it. To one end of this rod is connected a sharp-edged roller D, running in a carriage C, and capable of being clamped at any desired

angle with AB by means of a milled nut. Two small guides are added, to keep the carriage upright. By way of illustration, I will clamp the roller at 45° to the rod, place it at unit distance—say one inch—from the compass point O, and pressing it firmly into the paper, push it forward, thus describing the curve which will form the main subject of our consideration. This curve is, of course, a particular case of the equiangular, or logarithmic, spiral, and is commonly known as $r=e^{\theta}$.

The spiral is again shown in fig. 2, starting from the point 1, and passing through the points marked 2, 3, 4, &c., at which the distance from the pole O is respectively 2, 3, 4, &c. inches. Here also are several sectors of cardboard which have been cut out to fit the various angles 102, 103, 104, &c., and are therefore the Naperian logarithms of the natural numbers.

By means of these sectors, it may be shown experimentally that if we add to the angle 103 the angle 102, we arrive at the angle 106—($3 \times 2 = 6$), and, conversely, if we take away from the angle 108 the angle 102, we shall be left with the direction 04—($8 \div 2 = 4$). Without further examples, it may be demonstrated in the most general way, that by adding the angle under any one value of the radius vector to that under any other value, we obtain a direction giving their product: or by subtraction their quotient. Evolution and involution follow naturally. For instance, to find the value of $5^{1\cdot 6}$: we divide the angle 105 into ten equal parts, and then by taking 16 such parts we get a direction which at once gives us the desired result. By ordinary arithmetic even this simple calculation is quite impracticable, and it is given to show how the real utility and importance of logarithms may be impressed on the student.

The properties mentioned above are common to all curves drawn with this instrument, and not merely to the special case in which the roller is set at 45°.

Instead of working direct from the curve, by means of cardboard sectors, or compasses, the values of the various angles may be expressed in terms of some standard angle, such as the radian, and the results printed, in tabular form, for future reference.

Here is a protractor, graduated in radians, which we will apply to fig. 2, and by means of which we can read off the values given below.

Length of radius vector in units of 1 inch.	Corresponding angle in radians.
1	0
2	·693
3	1·098
4	1·386
5	1·609
6	1·791
7	1·946
8	2·079
9	2·197
10	2·302
&c.	&c.

This constitutes a table of natural logarithms, which may be used in the ordinary way for making calculations. The highly accurate values, usually published in book form, are not obtained by direct measurement, but by an indirect process which does not at present concern us; and their use is chiefly restricted to those who desire great precision. For the most part, however, we do not require this extreme accuracy, and continue to work direct from the spiral. The radial lines are cut off to a circle about O, and the appropriate length of each is written against it. Two such circles are connected by a pivot through the poles, and constitute one form of the familiar "watch calculator," by means of which the desired angles may readily be added or subtracted:—here is a model of this useful appliance. It should be noted that the graduations are not necessarily made from the spiral in fig. 2, but are, like this model, more usually derived from a spiral drawn with the roller set at about $68\frac{1}{2}°$—$(\tan^{-1} 2\pi/\log_e 10)$. In this position, the roller will be ten inches from O, after sweeping through 360°. An important advantage arises from this change, to which we shall refer.

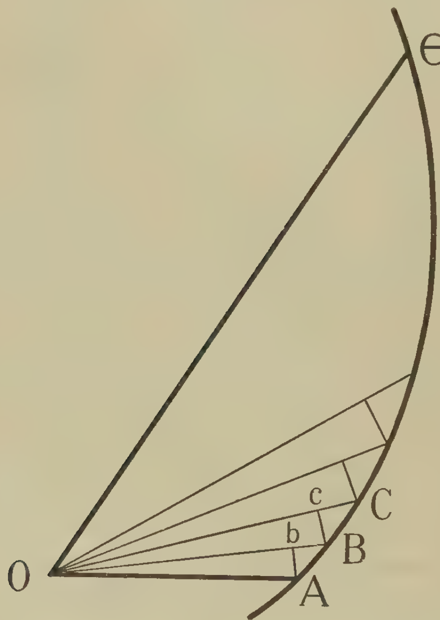
By rolling one of the dials along a straight piece of wood and copying off the graduations, as each comes into contact, we obtain a straight logarithmic scale; and two of these, when combined, form the device known as a slide-rule.

Analysis.

So far, the accuracy of our conclusions has a purely constructional basis, and it is very desirable to apply the crucial test of analysis, and satisfy ourselves that no residual errors exist, which might have eluded our most careful measurements.

Fig. 3 is a reproduction, on a larger scale, of the spiral shown in fig. 2. The angle AOe has been drawn equal to

Fig. 3.



one radian, OA being one inch, or unity. This angle is supposed to be divided into one million equal parts by lines OB, OC, OD ... which meet the curve at B, C, D .... Also Ab is drawn perpendicular to OB, Bc to OC, &c.

Now, from the mechanical construction of the curve, it is clear that the spiral is, at all points, uniformly inclined at 45° to the radius vector; for the tracing roller was clamped at that inclination to the steel rod. It is also evident that we may, without material error, regard the portions of the spiral AB, BC, &c. as straight lines, and also consider OA equal to Ob, &c.

Accordingly $\qquad OA = 1$

$$OB = Ob + bB$$
$$= OA + Ab.$$

But $Ab = OA$ multiplied by the circular measure of the angle AOB, which is 1/1000000. Hence writing m to denote one million, we have

$$OB = 1 + 1/m.$$

Similarly $\qquad OC = OB + Bc,$

But $\qquad Bc = OB \times 1/m.$

Therefore $\qquad OC = OB(1 + 1/m) = (1 + 1/m)^2.$

In this way we can write down the values of the radius vector as it proceeds from OA to Oe:—

The radius vector is	When the angle is
1	0
$(1+1/m)$	$1/m$
$(1+1/m)^2$	$2/m$
$(1+1/m)^3$	$3/m$
. . . . .	. . . . .
$(1+1/m)^m$	m/m, or 1

By calculation, the value of $(1+1/m)^m$ is found to be 2·7 . ., and by direct measurement with a foot-rule we arrive at an identical result. This expression is usually denoted by e, and accordingly $(1+1/m) = e^{1/m}$, $(1+1/m)^2 = e^{2/m}$, &c.

The previous list may therefore be written as follows:—

The radius vector is	When the angle is
1	0
$e^{1/m}$	$1/m$
$e^{2/m}$	$2/m$
$e^{3/m}$	$3/m$
. . . . .	. . . . .
$e^{m/m}$, or e	m/m, or 1

We have thus secured a most important quantity e, which affords a simple index relation between the radius vector and its angle.

Hence we can, as before, multiply together any two values, such as $e^{a/m}$ and $e^{b/m}$, by adding the angles a/m, b/m, and reading off the value of the radius vector $e^{a/m+b/m}$, which is both appropriate to the angle $a/m + b/m$, and at the same

time, the obvious value of the product. And since this analysis is applicable throughout the spiral, we have shown both experimentally and analytically that

$$\log A + \log B = \log AB.$$

"*e*."

Perhaps we may with advantage pause one moment to consider the full meaning of *e*. It is usually defined as the sum of $1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}$ ......, and though undoubtedly correct, I would suggest that this is not really *e*, but only a method of calculating *e*. There are, for example, many ways of evaluating π; but surely our conception of π must always be the visible circumference over the diameter, and not an infinite series. In the same way the student might, I think, get a clearer view of *e* by regarding it as the result of unity growing, in a special manner, through unit angle: as the amount of £1 at compound interest after one year, interest being paid continuously at the rate of one millionth of the capital per one millionth of the time, or even as the first great milestone along the 45° spiral, with "2·7 miles from London" painted on it in big letters.

Change of Base.

Possibly you have never seen a modulus. Here is one made of cardboard. It is called "the modulus of the common system," and is a sector having an angle of 2·302 radians, divided into tenths. If we apply this modulus to the natural logarithms in fig. 2 it will transform them into common logs, and we are able to read off their values, just as we did with the protractor graduated in radians. In adopting a unit angle 2·3 times as big as before, we make all angles measured by it appear 1/2·3 of their previous value. This affords a visible explanation of the method given in text-books for changing from the base *e* to the base 10.

The beginner will doubtless ask why we take the trouble to convert natural into common logs, and the model of the watch calculator, which you have seen, provides a satisfactory answer. After making half a dozen calculations (2×3,

4×5, 6×7, ...) it will easily be grasped that

$$\log \ 2 = \ldots\ldots\ldots\ldots\ldots \cdot 301 \text{ of a revolution.}$$
$$\log \ 20 = 1 \text{ revolution } + \cdot 301 \quad ,, \quad ,,$$
$$\log 200 = 2 \text{ revolutions } + \cdot 301 \quad ,, \quad ,,$$

Had the dials been derived from fig. 2—the natural system—there would be no recurrence, and we should be unable, in this simple manner, to adapt a small range of logarithms to cover an unlimited range of values. The state of affairs might be compared to a clock in which the small hand moved over 2·302 divisions during an hour.

Instead of obtaining common logs by transformation we may entirely discard the natural system, strike out a new course, and with a fresh setting of the roller describe the spiral, shown by the dotted line, in fig. 2. This curve meets unit angle—one radian—in the point X, distant 10 inches from O, and yields logs to the base 10, by direct measurement with the standard protractor graduated in radians. Along this curve, e occurs at an angle of 1/2·302 radians, thus proving experimentally that $\log_e 10$ is the reciprocal of $\log_{10} e$.

The value 10 may be reached by growing from unity, either along the e spiral at the standard rate for 2·3 radians, or along this new curve at 2·3 times that rate through one radian. We may regard this alternatively as the proof, or the consequence, of the new curve being drawn with an angle whose tangent is 1/2·302. In any case it is clear that $\tan \alpha = \mathrm{M}$, and that all equiangular spirals transform from one into the other, by a suitable change in the unit angle.

Owing to the rather prevalent idea that common logs can be calculated only by derivation from the natural system, I may perhaps be excused for alluding to the independent, though obvious, method of extracting square roots.

This concludes the graphic treatment of logarithms. Some portions have been made very elementary, and must necessarily appear prolix. On the other hand, I have entirely omitted such questions as the limiting value of $\left(1 + \frac{1}{m}\right)^m$, negative characteristics, and other points which are fully explained in the ordinary books. The short descriptions of

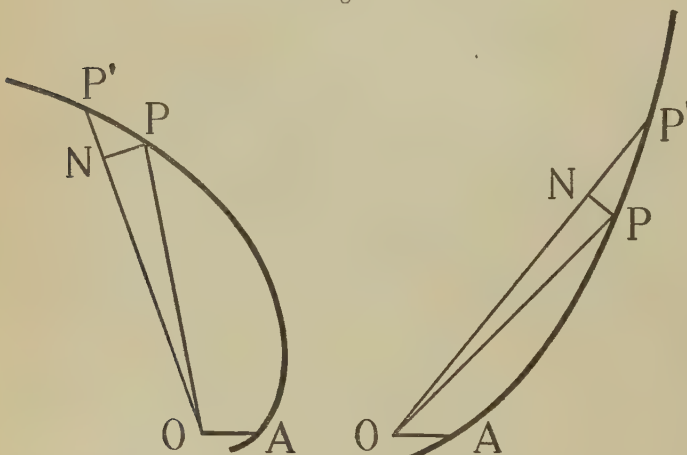
the watch calculator and slide-rule, though not essential, have been deliberately introduced, in the belief that such examples are beneficial.

I will terminate by mentioning, very briefly, two other properties of the spiral which admit of illustration.

Differentiation.

On the left of fig. 4 is the e spiral; the remainder of the construction being self-evident.

Fig. 4.



The angle POP′ is the increment of the natural log, corresponding to the increment NP′ of the radius vector. Therefore

$$\frac{d \log_e r}{dr} = \text{limit of } \frac{\text{PN/OP}}{\text{NP}'} = \frac{1}{r}.$$

On the right of fig. 4 is the "10" spiral, drawn, as we have seen, with the roller at an angle whose tangent is 1/2·302 .... Therefore in this case

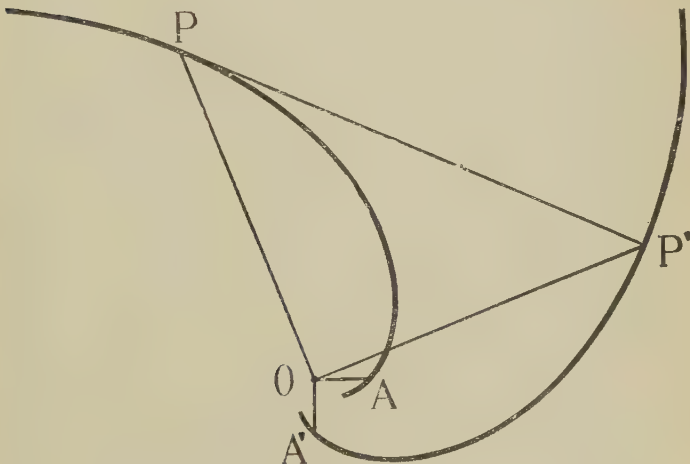
$$\frac{d \log_{10} r}{dr} = \frac{1}{2{\cdot}302\,.\,r} = \frac{\text{M}}{r}.$$

Evolute and Involute.

It is clear from the nature of the spiral that its evolute must be some form of spiral which also merges ultimately in the pole.

Here is a logarithmic spiral cut out of wood, from which the curve AP, fig. 5, has been drawn. An adjustable string is attached, and we find that of all the involutes which can be drawn to this curve, with varying lengths of string, there is one, viz. A′P′, which exactly fits the wooden model. Therefore the spirals AP, A′P′ are reciprocally evolute and involute. And we have seen that they must have a common

Fig. 5.



pole; therefore the evolute of A′P′ is the same curve turned through an angle about O.

Again, let PP′ be any position of the string; then the angle OPP′ is complementary to OP′P, and therefore POP′ is a right angle. But the unwound portion of the string PP′ is obviously the length of the spiral from the pole to P. Accordingly the length of the spiral at any point is the intercept on the tangent between the radius vector and its normal through the pole. This of course agrees with

$$s = \int \sec \alpha \, dr = r \sec \alpha,$$

α being the angle of the spiral.

Conclusion.

In the report of a recent Educationa Conference, I noticed that whilst differences arose on all other points, there was

absolute unanimity regarding the excellence of Euclid I.–III. Now Euclid is, above all, an experimental science—the experiment is first performed, then analysed to confirm its accuracy—and the vitality of Euclid is, beyond doubt, due to this pre-eminently rational treatment.

This plan is the one I have tried to follow—with what success you must yourselves decide. In more competent hands it might, I am certain, be improved and extended to other problems, with great benefit to all concerned. The larger part of mathematics has arisen in the consideration of practical questions; and by divorcing the reasoning from its original significance, we rob it both of its visible explanation, its interest, and its application. The earlier portions of arithmetic, algebra, trigonometry, &c. are clearly experimental, and describe, in symbols, the results of definite operations which any boy can readily understand. As we advance, however, the real meaning gradually becomes obscured, till we are finally left with little more than a notation, intelligible only to those possessing special aptitude.

When the extensive and growing equipment of our laboratories and technical schools is contrasted with the solitary blackboard, it must surely be admitted that the disparity is excessive. If we will but show diagrams, models, and actual measurements—alluding constantly to the practical applications of the matter in hand, and pointing out its real utility—we shall by such means do much to awaken interest, stimulate intelligence, and discountenance the idea that this oldest, and most important, of all the sciences is merely a collection of tricks with symbols, culminating in a calf-bound volume, neatly embossed with the school arms.

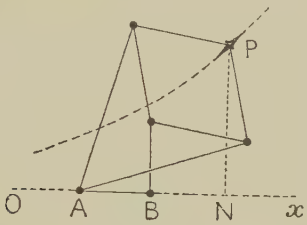
Discussion.

Mr. W. A. Price described an instrument, constructed somewhat on the lines of that described by Mr. Pochin, which may be used to draw the exponential curve $y=ae^{x/b}$.

A Peaucellier cell (fig. 1) carries a wheel at P directed so that its plane always passes through A. The link AB is moved along a straight line Ox. P describes the exponential

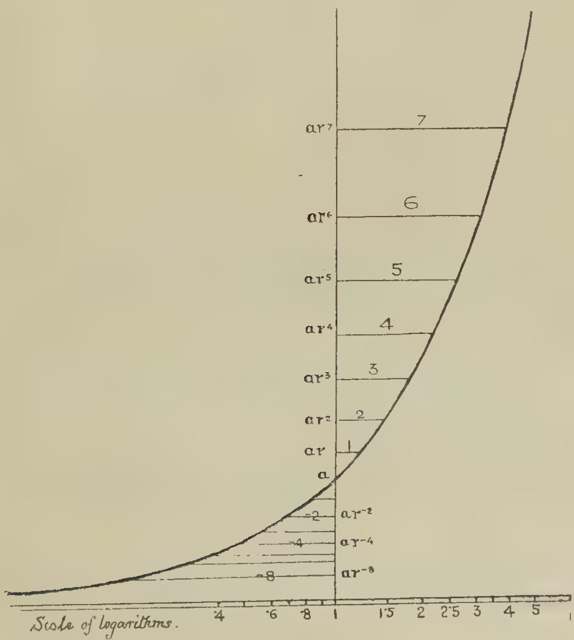
curve. That this is so is seen from the fact that the subtangent AN is constant.

Fig. 1.



Any exponential law is a relation between two quantities changing in arithmetical and geometrical progression respectively. Fig. 2 is a diagram of the exponential curve

Fig. 2.



in which the ordinates change in G. P. and the abscissæ in A. P. It shows the formation of the logarithmic scale constructed by marking on the scale of abscissæ the corresponding values of the ordinates. Mr. Price suggests that

logarithms should form part of the subject of arithmetical and geometrical progressions: the conception of continuous growth being added by interpolation of terms in the usual series.

The CHAIRMAN (Prof. PERRY) said the paper was interesting to those acquainted with logarithms and logarithmic spirals, but in his opinion the methods described by the author were not the best for beginners. He pointed out that the early geometers used practical methods to check the results of their theoretical reasoning.

XXXVIII. *Logarithmic Lazytongs and Lattice-works.* *By* THOMAS H. BLAKESLEY *.

THE point of view in which the Equiangular Spiral is usually regarded is that implied in its name, viz., the curve which makes the same angle with its radius vector,

$$r\frac{d\theta}{dr} = \tan\alpha.$$

It is rather from what I may perhaps call its polygonal character that I shall present and apply it in this paper. By this I mean that it is a circumscribing curve to polygonal figures following simple laws.

If a series of equal straight lines form a consecutive number of the sides of a regular polygon, the circumscribing circle is absolutely determined.

But if those straight lines, still maintaining the equality of the angles between any consecutive two, in magnitude form a geometrical series, the circumscribing curve will be the equiangular spiral.

According to the value of the angle between consecutive lines, one may speak of the figure as a regular logarithmic pentagon, hexagon, octagon, &c., and more generally as a regular logarithmic polygon. The regularity consists in the

* Read March 22, 1907.

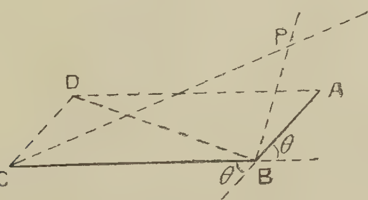
equality of the angles between the lines, and in those subtended by them at the pole of the spiral.

Consecutive chords are those straight lines which form consecutive sides of a logarithmic polygon.

Some geometrical matters more immediately arising from this view of the curve may be introduced.

The problem of finding the pole when two consecutive chords are given is solved thus:—Let AB, BC, be the consecutive chords given. Complete the parallelogram, and let BD be the diagonal through B. Make the angle BCP

Fig. 1.



equal to the angle DBC, and make the angle ABP equal to the same angle. Then P is the pole of the spiral in which AB, BC are consecutive chords.

As an alternative to the setting off of one of the angles BCP or ABP, the angle BAP may be made equal to DBA.

Or, as it may be shown that the product of PB . DB is equal to that of AB . BC:

BP may be easily calculated from the data, viz., the values of AB and BC, and the angle between them.

If one of the two chords (say BC) is maintained in position, but the other BA is made to turn round B, so as to vary the angle between the chords, the pole P will describe a circle whose centre is in the line CB, produced if required.

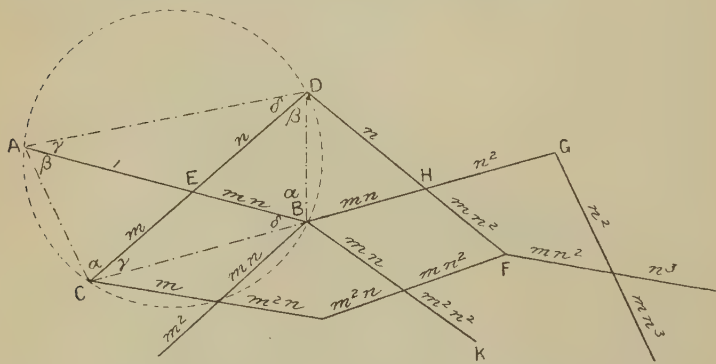
If θ is the angle between the two consecutive chords externally, the characteristic angle of the spiral (α) will be given by

$$\tan \alpha = \frac{\theta}{\log \frac{BC}{AB}}.$$

In any cases therefore in which $\dfrac{\theta}{\log \dfrac{BC}{AB}}$ has the same value, the spirals are similar.

It follows that, in any mechanical construction of linkages &c., if we can keep $\dfrac{BC}{AB}$ constant, but can at the same time cause θ to vary, we have the power of changing α, that is to say the one thing which settles the character of the equi-angular spiral.

Fig. 2.



If now two straight rods or lines, AB, CD (fig. 2) are taken in one plane, and meeting in E (whether in their actual lengths as shown, or in their geometrical productions, is immaterial), and so conditioned that A, C, B, D lie in one circle, then the products of their segments are equal or CE . ED = AE . EB.

It follows that E remaining the same for both rods A, C, B, D will always lie on a circle, whatever be the angle between the rods.

If DE $= n$.AE, and CE $= m$.AE, then the condition is fulfilled if EB $= mn$.AE ; n and m may have any values whatever.

It will be convenient for geometrical reasoning to imagine or describe the straight lines AC, CB, BD, DA. Then the triangles DEB, AEC, are similar, and

$$DE : EB : BD :: AE : EC : CA :: n : 1.$$

Similarly regarding the triangles CEB, AED, they are similar, and CE : EB : BC :: AE : ED ; DA :: m : 1.

Call the angles ECA, EAC, EAD, EDA, α, β, γ, and δ respectively. Then also the angles EBD, EDB, ECB, EBC are α, β, γ, and δ respectively.

Now DB may be derived from AC, as regards direction and magnitude, by allowing AC to revolve first through ACE or α in one direction, and then through EDB, or β in the opposite direction, and by reduction in the ratio $1 : n$.

Thus DB makes with AC the angle $\overline{\alpha-\beta}$, and $\mathrm{DB}=n\,.\,\mathrm{AC}$.

Similarly CB makes with AD the angle $\overline{\gamma-\delta}$ and $\mathrm{CB}=m.\mathrm{AD}$.

Now suppose another pair of rods DF, BG jointed at H, and similar to the first pair in all respects but bearing the ratio to them of $n : 1$, jointed on to the first pair at B and D.

All the lines in this, which may be called the second cell in the direction n, including those of the circumscribing circle are homologous in a ratio $n : 1$ with the corresponding lines of the first cell, and the angular displacement relatively to the first cell is $\overline{\alpha-\beta}$.

To the points FG may now be connected a third cell, constituted in a similar manner to the second, and so on indefinitely.

To AC may be connected a similar cell bearing to the first cell the linear proportion of $1 : n$ and to this another, and so on indefinitely in the direction $\frac{1}{n}$.

Such a line of cells may be called a logarithmic lazytongs.

It is clear that all such points as A, D, G lie upon an equiangular spiral, the tangent of whose characteristic angle is equal to $\frac{\overline{\alpha-\beta}}{\log n}$.

As all the cells of a series are similar, any motion involving the increase or decrease of the angle between the bars of one cell, will be accompanied by the same change in angle in all the cells.

The sides AD, CB may also have cells attached to them, the same rules as before being observed. m will take the place of n in the change of scale and $\overline{\gamma-\delta}$ the place of $\overline{\alpha-\beta}$ in the change of direction.

It is to be pointed out that if two cells in the m direction

be applied to BC and BF, the other *adjacent* points of the two cells will coincide. In other words, the cell on BF may be considered as derived from the first cell either by one move in the n direction followed by another in the m direction, or by one in the m direction followed by a second in the n direction. Whence it follows that the whole of a plane surface may be occupied by a plenum of cells forming an infinite lattice-work, in which, if the angle between the cross-bars of any one cell is changed, an equal change takes place in that between the cross-bars of any other cell.

In the m direction the tangent of the angle characteristic of the spiral through such points as A and C is equal to $\frac{\overline{\gamma-\delta}}{\log_e m}$.

There is a common pole for both the m and the n spirals.

If for the sake of easy description we liken the plenum of cells so obtained to a chessboard, the rooks' moves would take place either in the n direction or in the m direction, or in the $\frac{1}{n}$ direction or in the $\frac{1}{m}$ direction.

But the bars AB &c. which lie in what would then be called a bishop's move, would also have their extremities in an equiangular spiral having the same pole as the other sets of spirals. The angle between successive chords (external) will be in this case $\overline{\alpha-\beta}+\overline{\gamma-\delta}$, and the change ratio of the chords mn, so that the angle characteristic of the spiral will have for its tangent

$$\frac{\overline{\alpha-\beta}+\overline{\gamma-\delta}}{\log_e mn}.$$

Circles and straight lines are only limiting cases of equiangular spirals. They may therefore be awaited among the cases arising from changing m, n or the angle between the cross-bars.

If $n=1$ the n spirals become circles, and since in that case $\gamma=\delta$, the m spirals become straight lines.

If $m=1$ the m spirals become circles and the n spirals straight lines.

If both m and n are equal to unity, both systems are straight lines.

If $mn=1$, in which case AB is bisected in E, there is no change of scale along the bishop's move in the direction AB. In consequence the spirals through AB are circles.

Similarly if $m=n$ CD is bisected, and such lines lie on circles.

Such lines as AB, BK, &c. may under some circumstances lie in a straight line.

The displacement in angle of such lines is $(\alpha-\beta+\gamma-\delta)$.

If this is equal to zero

$$\alpha+\gamma=\overline{\beta+\delta}$$

$$\therefore \qquad \overline{\beta+\delta}=\frac{\pi}{2}.$$

Thus the circumscribing circle must have AB for a diameter. Therefore as D and C are both on that circle CD must either be equal to AB, in which case the two chords bisect each other ($m=n=1$), or CD is less than AB. It is therefore only the longer of the cross-bars which, by the variation of the angle between them, can come into a straight line. The longer cross-bar is also that one which is divided most unequally, since the product of the segments is the same in the two bars.

Discussion.

Mr. W. A. Price pointed out that the principle of extending a quadrilateral by constructing similar and similarly disposed figures on the sides, and thus covering an indefinite area, was not confined to inscribable quadrilaterals nor even to plane ones. A gauche quadrilateral with its diagonals is a tetrahedron and may be regarded as contained by three pairs of opposite edges. Any two of these pairs may be used to originate a set of quadrilaterals on the author's plan, which will lie on the surface of a helicoid, and whose sides and angular points will describe helical spirals. There will be three sides each of two pairs of edges from each of which may be developed a helicoidal surface with its families of helical spirals; so that from a single gauche quadrilateral three helicoids may be developed.

XXXIX. *Electrical Conductivity produced by Heating Salts.* *By* A. E. Garrett *.

The experiments described in the following paper form a continuation of those conducted by Dr. R. S. Willows and myself which were published in the Phil. Mag. for October 1904.

The experiments there described showed that increased electrical conductivity took place when the halogen compounds of zinc were heated to a temperature of about 360° C. This led us to think that other inorganic chemical compounds might possibly behave in a similar manner, so, in the first series of the present experiments, I tested a number of compounds to see if they increased the electrical conductivity when their temperatures were raised to about 360° C. In the second series of experiments I have confined my attention to a few special cases in which increased conductivity had been found to occur, in order to find out what are the causes of increased conductivity (more particularly so in the cases where the presence of the positive ions could be detected), and to find out the nature of the positive ions.

First Series.

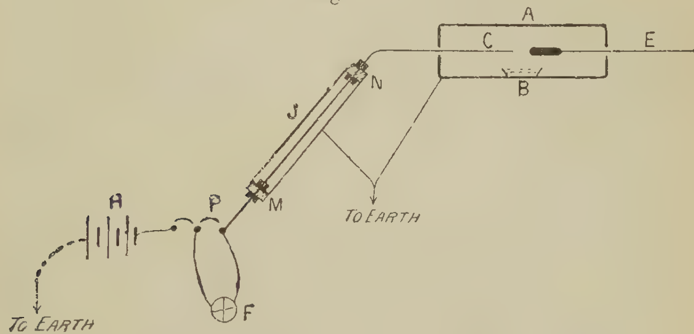
For these preliminary experiments the apparatus used is shown in figure 1.

A is an iron cylinder about 12 cms. in length and 5 cms. in diameter; caps of the same material partially close its ends. C is a platinum wire, connected to a wire passing through the centre of the metal tube J, and carefully insulated by means of ebonite plugs passed through the corks N and M. F is a quadrant electrometer; both pairs of its quadrants were first connected together and to the insulated pole of the battery H, as in figure 1. The quadrants were then disconnected, one pair being kept on the battery, the other pair to C. A leak of electricity from C is then shown by a constantly increasing deflexion of the electro-

* Read April 26, 1907.

meter-needle. The material to be tested was placed in the platinum dish B, which was usually placed in an iron dish so as to prevent those compounds which became liquid at the higher temperatures escaping on to the tube A.

Fig. 1.



The temperatures were ascertained by means of a nitrogen-filled mercury thermometer E reading to 360° C. The tube A and its contents were heated by a current passed round the tube through wires insulated from the tube and wound non-inductively.

In these first experiments the conductivity in cylinder A was measured at atmospheric pressure.

The following compounds when heated to temperatures not higher than 360° C. increased the rate of leak of both negative and positive charges from C except in the four cases noted.

(a) *Chlorides* of iron (Fe_2Cl_6), aluminium (Al_2Cl_6), ammonium, magnesium, tin ($SnCl_2 + 2H_2O$), manganese, and cadmium (C negatively charged).

(b) *Fluorides* of calcium, and aluminium (C negatively charged).

(c) *Iodide* of cadmium.

(d) *Nitrates* of ammonium, cadmium (C negatively charged), and cobalt (C negatively charged).

The following substances were tested, but no increased conductivity could be detected when using them:—

(a) *Metals with low melting-point.*—Tin, lead, powdered bismuth.

(b) *Chlorides* of copper, calcium, barium, strontium, lithium, potassium, and antimony.
(c) *Iodides* of potassium, lead, and silver.
(d) *Bromide* of potassium.
(e) *Fluoride* of sodium.
(f) *Oxides* of copper, zinc, tin, iron, calcium, and magnesium.
(g) *Sulphates* of zinc, iron, copper, and magnesium.
(h) *Carbonates* of zinc, magnesium, potassium, and sodium.
(i) *Bicarbonate* of soda.
(j) *Nitrates* of lead and barium.

In these experiments the conductivity was measured by means of a quadrant electrometer which gave a deflexion of 50–60 divisions for one volt. Hence it is quite possible that some of those cases in which no increased conductivity could be detected by such means, did produce an increase in the conductivity, but it was too small to be detected. *E.g.*, in some later experiments, when using a Dolezalek electrometer giving a deflexion of 600 scale-divisions for one volt, a considerable leak was obtained by heating lead iodide. It will be seen from the preceding list that the halogen salts of some metals when heated make the surrounding gas a conductor of electricity.

As no attempt at this stage was made to obtain anhydrous salts, the presence of water may have altered their ability to undergo chemical change, and in so doing may have altered their powers of producing conductivity.

Many of the chlorides which were found to be able to produce a leak of electricity are known to undergo chemical change when heated. It is probable that, at the moment chemical change takes place, an emission of an electrified particle occurs. In the case of the nitrates, the clean iron of the tube in which these compounds were heated was oxidized, and Beattie * and Schuster have both shown that this process causes the surrounding air to be positively electrified. As, however, in the case of ammonium nitrate an increased conductivity was observed when the electrode C was

* South African Association, vol. i. April 1903, p. 4.

positively charged, this does not account for all the results obtained with those compounds.

As a general rule, the leak was larger when due to positive ions, and that leak could be detected in most cases at lower temperatures than when due to negative ions.

Later, some zinc phosphate was tested by heating up to 290° C. at a pressure of 10 mm. of mercury. Although the electrometer gave a deflexion of 700 divisions for one volt, no leak was detected. Potassium iodide was also tested up to temperature 260° C. at a pressure of 10 mm. of mercury. The electrometer gave a deflexion of 600 for one volt, but no leak was detected.

Thallium iodide was heated up to 210° C. at a pressure of about 15 mm. of mercury. The electrometer gave a deflexion of 350 for one volt, but no leak was detected.

The negative result with potassium iodide is of interest, as H. A. Wilson in his papers * on conduction by salt vapours when sprayed in solution into a hot tube, found conduction, even when measuring currents with a galvanometer, at temperatures near 270° C. (It is shown below that the leaks are not obtained in the absence of water vapour in some cases. If this is always so then dry KI would not give any leak.)

We hence conclude that in most cases definite chemical changes take place when increased conductivity is brought about.

Second Series.

I. *The Effect of Low Temperature on the Conductivity produced by Zinc Chloride.*

The zinc chloride was heated to 360° C. and the tube in which it was heated could be put into communication with the testing apparatus in two distinct ways, by means of glass tubes.

One of these glass tubes A passed direct from the part containing the heated zinc chloride into the testing vessel, the other B was bent, in between the two parts of the apparatus, to form a flat spiral. This spiral was immersed

* Phil. Trans. A. p. 296 (1901).

in solid CO_2. The air was drawn over the zinc chloride and through A or B into the testing apparatus by means of a large glass aspirator so arranged as to produce an air current of about 10 cms. per second in the tube containing the electrode C (fig. 1).

(a) *Effect upon the positive ions.*—The rate of leak was first tested without aspirating, then when aspirating through A, and lastly when aspirating through B which was roughly equal in length to A. When the air was drawn through the flat spiral of glass immersed in the solid CO_2 contained in a beaker, the rate of leak was reduced to one-eighth the value it had when drawn through A. When drawn through A, a fog was produced in the aspirator like that previously noted by Dr. Willows and myself *, but when the air was drawn from the heating tube through B hardly any fog could be noted.

(b) *Effect upon negative ions.*—In this case the leak was reduced to $\frac{2}{5}$ the value when aspirating through B that it had when aspirating through A. The fog was much reduced.

The electrode C (fig. 1) was charged to 190 volts in both cases.

It is probable that the reason why more effect is apparently produced by the low temperatures on the positive ions is that the rate of leak obtained depended largely upon the velocity of the ions, as it was not possible under the conditions of the experiment to obtain a saturation-current.

The production of fogs would lead us to think that the ions are bodies of considerable size; in the case of the positive ions other evidence points to the same conclusion.

II. *Variation of the Leak with Changes of Temperature.*

Richardson has found † that the formula $I = a\theta^{\frac{1}{2}}e^{-\frac{b}{\theta}}$ can be used to express the relation between the saturation-current I and the absolute temperature θ in the case of a hot platinum wire and a metal cylinder surrounding it in a high vacuum.

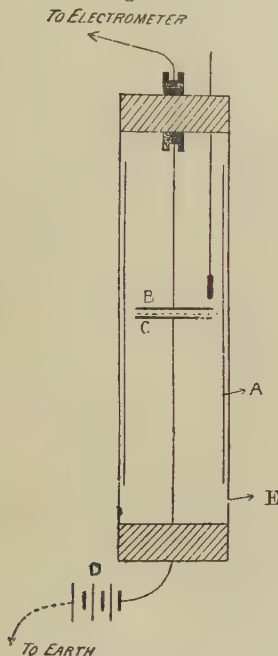
I find in all the cases tried that a similar formula represents the relation between I and θ.

* Phil. Mag. Oct. 1904, pp. 452–3.

† Proc. Camb. Phil. Soc. xi. p. 286 (1902).

If logs. are taken of both sides of the equation, since a and b are constants it will be seen that if $\frac{1}{\theta}$ is plotted for abscissæ and $\frac{1}{2}\log_e\theta - \log_e I$ for ordinates, the result obtained should be represented by a straight line.

Fig. 2.



These earlier results were obtained with voltages which were too small to produce saturation-currents, although they were sufficiently large for the rate of leak to depend rather upon the total number of ions present than their velocity. In order therefore to ascertain whether Richardson's formula held for results obtained with sufficient voltage to produce a saturation-current, some zinc iodide was heated up and tested in apparatus in which saturation could be obtained. This is shown in fig. 2.

A is a brass tube earthed. B and C are brass plates, the material to be tested being placed on the lower of the two plates; the plates were separated by a few millimetres. D is a battery of storage-cells. E a glass tube fitted with corks and made air-tight.

The following results were obtained by this means for negative carriers :—

Temperature in degrees Centigrade.	Leak with voltage sufficient to produce saturation.
180	25
190	70
202	130
204	170
214	400
222	500

These are found when plotted with $\frac{1}{\theta}$ for abscissæ and

$\frac{1}{2}\log_e \theta - \log_e I$ for ordinates to give points which lie practically along a straight line. Zinc iodide was tested for negative ions up to temperature 255° C. with similar result.

From the formula $C = A\theta^{\frac{1}{2}}e^{-\frac{Q}{2\theta}}$ in which Q represents the amount of energy in calories associated with the production of one gramme molecular weight of the ions, it readily follows that $\log_e \frac{C_1}{C_2} = \frac{Q}{2}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)$ where $\theta_1 - \theta_2$ is only a small interval of temperature.

Making use of this relation I obtained the following values of Q for ions obtained from the substances named. The average temperatures are stated in each case.

For positive ions.

Calcium fluoride (297° C.)	$2{\cdot}6 \times 10^4$,
Aluminium fluoride (330° C.)	$2{\cdot}9 \times 10^4$,
Ammonium nitrate (312° C.)	$3{\cdot}3 \times 10^4$.

For negative ions.

Zinc iodide (241° C.)	$2{\cdot}9 \times 10^4$,
Iron chloride (355° C.)	$6{\cdot}1 \times 10^4$,
Ammonium chloride (352° C.)	$5{\cdot}0 \times 10^4$,
Calcium fluoride (346° C.)	$6{\cdot}0 \times 10^4$,
Ammonium nitrate (343° C.)	$4{\cdot}3 \times 10^4$,
Magnesium chloride (326° C.)	$2{\cdot}4 \times 10^4$.

It will be seen that the values of Q obtained for the negative ions are in general larger than those obtained for the positive ions. A similar result for the ions produced by hot platinum has been noted by Richardson *, and the values here obtained are of the same order of magnitude as those obtained by him.

I found also on heating zinc iodide at a pressure of 2·5 mm. of mercury that the escape of positive ions could be represented by a similar formula, a saturation current being obtained.

* Phil. Trans. A., Nov. 1906, p. 61.

The following were the results obtained :—

Temperature in degrees Centigrade.	Leak with voltage sufficient to produce saturation current.
230	22
242	35
248	52
252	67
254	80
256	110
258	150
259	180

When these are plotted with $\frac{1}{\theta}$ for abscissæ and $\frac{1}{2}\log\theta - \log_e I$ for ordinates, it is seen that a decided break occurs in the straight line in the neighbourhood of temperature 250° C.

The probable reason for this is that some fresh source of ionization is tapped at this temperature. It was subsequently found that at reduced pressures, negative ions as well as positive could be detected at that temperature.

III. *Experiments on Halogen Salts of Zinc at reduced pressures.*

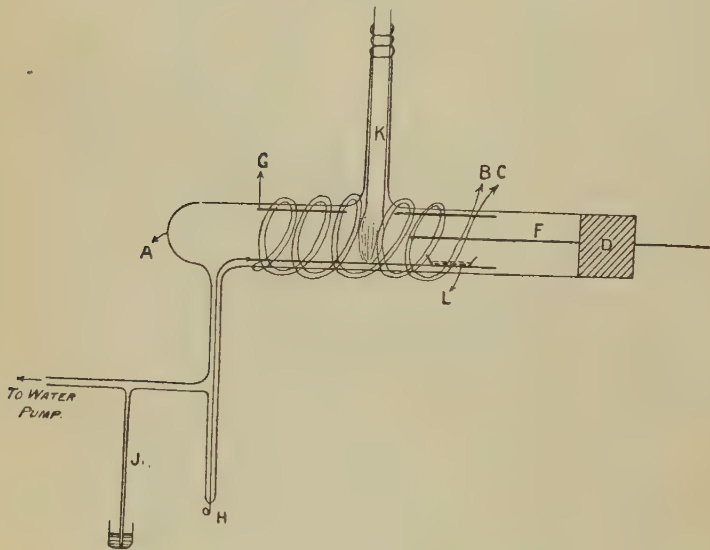
The apparatus now used is shown in fig. 3. A is a glass tube ; B, C, wires for heating ; D, ebonite stopper made airtight ; F is the electrode ; G, iron tube earthed by means of H ; J is a gauge ; K, thermometer ; L, iron dish containing a small platinum dish in which materials to be tested were placed. Experiments with zinc bromide were tried with this apparatus. The pressure was reduced to 200 mm. of mercury, and it was then found that the leak with positive ions could be detected at somewhat lower temperatures than when the substance was heated at atmospheric pressures. The following is a typical set of readings obtained when the electrode F was charged negatively to a potential of 80 volts :—

Temperature, in degrees Centigrade.	Leak (scale-divisions).
195	7
207	17
216	33

From this it appears that extra leak can already be detected before 200° C. is reached.

With a pressure of 220 mm. of mercury and an average temperature of 190° C., a saturation-current was obtained.

Fig. 3.



The readings were all taken over an interval of one minute, and the temperature was nearly constant. Saturation was obtained with F charged to 160 volts when testing for positive ions. At still higher voltages a large increase in the leak took place. This is probably due to fresh ions being formed by collisions, since the voltages then used were nearly sufficient to produce sparking.

The apparatus was next arranged so that the temperature could be kept at a constant value during the observations, and the pressure was still further reduced to 20 mm. of mercury. The temperature was now 198° C., and varied less than 1° C. A aturatsion-current was now obtained in the case of the positive ions with a difference of potential of

110 volts. Hence at lower pressures a saturation-current is obtained with lower voltages.

It was found that the voltage required to produce saturation was greater in the case of the positive ions than of the negative ions—a result which can be explained if the positive ions move at a slower rate in the same electric field than the negative ions. This was afterwards proved to be the case.

In some later experiments with pure zinc iodide, it was found that at temperature 248° C., at a pressure of 2·5 mm. of mercury a saturation-current could be obtained with a difference of potential of about 260 volts in the case of the positive ions, and with about 180 volts for the negative ions. The brass tube used in place of iron tube F (fig. 3) had an internal diameter of 1·4 cm., and the central electrode a diameter of ·25 cm.

IV. *The Effect of Moisture.*

On one or two occasions, when a series of readings were required and the time at disposal was limited, some fresh material was placed in the apparatus while the latter was at a temperature between 250° C. and 260° C. On each occasion when using zinc iodide, it was noted that the rate of leak obtained was extremely small compared with previous results. From this it appeared that either the method of heating or the presence of water-vapour had an important effect on the rate of leak. Some zinc iodide was therefore prepared synthetically by heating pure zinc and iodine in distilled water until the colour of the solution disappeared, then evaporating to dryness, and leaving the residue in a desiccator for a week.

An addition was now made to the apparatus shown in fig. 3, in the form of a trap of concentrated H_2SO_4 between the tube A and the water-pump.

When the zinc iodide was placed in the apparatus and heated up slowly at a pressure of 30 mm. of mercury, the leak obtained was similar to that when commercial iodide of zinc was used under similar conditions.

When, however, some of this zinc iodide was placed in the apparatus while the latter was at 256° C., no extra leak

could be detected with pressures as low as 30 mm. of mercury.

The material was left in the apparatus for a week, care being taken to exclude moisture as far as possible. At the end of that time, with the temperature raised to 263° C. and the pressure reduced to 30 mm. of mercury, no extra leak could be detected. Hence moisture is necessary to make these salt vapours conductors of electricity.

Other effects of moisture were noted in the later experiments made to determine the velocities of the ions, and also in the experiments on the rate of decay of zinc iodide. These will be referred to in connexion with the parts named.

V. *Effect of the nature of the surrounding Gas upon the rate of Leak.*

The air was pumped out of the apparatus until the pressure was reduced to 20–30 mm. of mercury. The gas was then introduced, and the apparatus again pumped down to 30 mm. of mercury. More gas was then admitted, and the pressure again reduced ; the process being repeated so as to get rid of as much of the air as possible.

The gases experimented with were hydrogen, oxygen, and carbon dioxide.

In the case of hydrogen the results obtained were somewhat irregular, although in the majority of cases a decreased rate of leak occurred in the presence of this gas.

No certain change in the previous rate of leak was noted in the case of either of the other gases tried.

VI. *Effect of strongly heating the Zinc Iodide previously.*

Some zinc iodide was placed in a piece of platinum-foil, and was then heated in the Bunsen flame until it was just melted.

After this, it was placed in the apparatus and tested as in the other cases. The rate of leak which was so produced was found to be of the same order as that produced by zinc iodide which had not been previously heated.

VII. *Experiments carried out with the Zinc Iodide at the ordinary temperature of the laboratory.*

These experiments required the use of an instrument of small capacity. A Wilson* sulphur-bead gold-leaf electroscope was therefore used. The gold-leaf system was enclosed in a brass tube, which was itself placed inside a glass vessel which could be evacuated at will. The pressure could be ascertained by means of a Macleod gauge.

(*a*) *At ordinary atmospheric pressure* an extra leak due to the positive ions was noted when zinc iodide was placed inside the electroscope. A small leak was also noted when the leaves were charged positively. On the average, it was found that with positively charged leaf the rate of leak was increased 30 per cent. when zinc iodide was placed in the electroscope.

(*b*) *At reduced pressures. Effect of a strong magnetic field.*—A series of observations was taken with zinc iodide in the apparatus, the leaves being positively charged and the pressures ranging from 70 mm. of mercury to 15 mm. of mercury. In all cases it was found that the time taken for the leaf to move over one scale-division could be increased in the ratio 8 : 10 by the application of a strong magnetic field, showing the emission of negatively charged particles from the zinc iodide to be taking place.

When the leaf was negatively charged, no alteration in the rate of leak could be detected (for pressures as low as 15 mm. of mercury) when the magnetic field was applied.

In these experiments with the magnetic field the part below the sulphur bead was shaped like L. The leaden box containing the zinc iodide was made long and narrow, and was placed with its long axis parallel to the lower limb of the L and at a distance of 1–2 mm. from it.

At pressures 2·54 mm. and ·1 mm., when the leaves were positively charged, the time taken for the leaf to move over one division with the magnet off, to the time taken with the magnet on, was in the ratio 7 : 10. Practically the same results were obtained for both directions of the magnetic

* Proc. Roy. Soc. vol. lxviii. p. 152 (1901).

field ; therefore they were not due to any dissymmetry in the apparatus.

It was noticed on some occasions that putting on a magnetic field caused a sudden discharge, such as would be caused by a spark. A similar effect with a Wehnelt cathode was shown by Prof. J. J. Thomson in a lecture at the Royal Institution a year ago.

E. g., when the pressure was reduced to ·015 mm., and the magnetic field was put on, the leaf was instantly discharged if it were positively charged above a certain potential.

When the leaf was negatively charged under the same conditions, a partial discharge takes place when the magnetic field is first put on, and then no further discharge takes place when the magnetic field is again put on.

There also appears to be a distinct lag when the leaf is negatively charged between the putting on of the field and the partial discharge of the leaf.

In the case of both positively and negatively charged leaf, there was found to be a critical potential to which the leaf must be charged in order that a discharge may take place when the magnetic field is put on. This critical potential was much lower in the case of the negatively charged leaf.

The discharge effect with the magnetic field became apparent with pressures as high as ·05 mm. of mercury. It was found that when the leaf was charged below the critical potential [so that discharge did not take place when the field was put on] and the pressure was ·01 mm. of mercury, the magnetic field had not such a marked effect as at higher pressures and higher potentials. Under these conditions, with leaf positively charged, the time with magnet off to the time with magnet on was in ratio 17 : 20. With negatively charged leaf, under similar conditions, the magnet produced no effects.

In the case of the sudden discharge, which was brought about by the application of the magnetic field, it is possible that the field produces a concentration of the paths of the ions ; and hence a much more marked effect, owing to the formation of ions by collision, is the result. The effect on the positive ions may be a secondary one, since it is

known that very strong magnetic fields are required to deflect positive ions.

(c) *Leak stopped by aluminium-foil.*—Some zinc iodide was placed in a small copper dish, and covered with two layers of aluminium foil, each ·0004 cm. in thickness. Some wax was placed round the edges of the dish, and the foil was placed on the wax so as to reduce the amount of diffusion to a minimum. This was placed in the electroscope, and the pressure reduced to 2·8 mm. of mercury. No leak could now be detected, when the leaf was either positively or negatively charged. Hence it appears that the ions producing the leak must be very easily absorbed, as their action could be stopped by ·0008 cm. of aluminium.

(d) It was repeatedly found that if zinc iodide was left in the apparatus (fig. 3) for two or three days, a very large leak due to negative ions occurred when the apparatus was first connected with the battery, but that the leak quickly died away. In four minutes the leak was less than one-half its original value. In order to see if the presence of the zinc iodide was the actual cause of this increased conductivity of the enclosed air, the tube was left empty for five days under similar conditions. At the end of that time no large leak was noted.

Hence it appears that the zinc iodide has the power of emitting negatively charged particles at ordinary temperatures.

VIII. *Experiments on the Velocity of the Ions.*

In the conditions in which the ions are produced, the choice of the method used to determine their velocity was limited.

Prof. J. J. Thomson has shown, in his 'Conduction of Electricity through Gases,' first edition, pp. 74–78, that if the ionization is confined to a very thin layer compared with the total distance between two parallel plates, the current is given by the following formula:

$$V^2 = \frac{32\,\pi}{9\,R} i l^3,$$

where V = difference of potential between the plates in electrostatic units,

R = velocity of ion in cms. per second for unit difference of potential per cm.

i = the current obtained due to the ions per unit of surface measured in electrostatic units,

l = distance between the plates, measured in cms.

In deducing this equation it is assumed

(*a*) That the layer in which ionization occurs is very small compared with l.

(*b*) That the current i is only a very small fraction of that obtained with voltages large enough to produce a saturation-current ; *i. e.*, so that for small changes of voltage, the current varies as the square of the voltage.

In my investigations of the velocities of the ions from the iodides which I tested, I arranged my apparatus so as to be able to make use of the above formula. The apparatus was that shown in fig. 2. The glass tube E was about 50 cms. in length and was heated in the centre. The whole was made air-tight.

The ions were driven from the lower electrode C, which was connected to one pole of a battery of small storage-cells, the other pole of which was earthed, on to the insulated electrode B, which was connected with one pair of the quadrants of a Dolezalek electrometer, the other pair of quadrants being earthed.

The capacity of the whole system was measured so that the value of the current could be obtained in absolute units.

It will be seen from the equation that the value of the current i does not depend upon the number of ions present but only upon their velocity, so long as the conditions specified above are fulfilled. In order to do this a very thin layer of the salt to be tested was placed on C, and the distance between B and C was made equal to 2 cms. The variation of the current with the potential was also tested from time to time, in order to see whether the voltage applied was of the right order of magnitude.

The following results were obtained. They refer to the positive ions obtained from the various materials named ; the salts in each case being dry.

Bismuth iodide, lead iodide, barium iodide, and calcium iodide were anhydrous as supplied by Kahlbaum ; the others were kept in a desiccator for some time.

The velocities are given in cms. per second for one volt per cm. No great accuracy for the absolute values of the velocities given is claimed. Their relative values are nearer the truth, since as far as possible the conditions for each set of observations were the same.

Temp. degree Cent.	Pressure in mm. of mercury.	Zinc iodide.	Bismuth iodide.	Lead iodide.	Cadmium iodide.	Pressure × Velocity for Bismuth iodide.
215	10	·055	·06	·061	·12	·600
,, ...	15	......	·055	·054	......	·825
,, ...	20	·044				
,, ...	25	......	·041	·036	·087	1·025
,, ...	30	·035	......	......	·073	
,, ...	35	......	·032	·032	......	1·120
,, ...	40	·031	·03	......	......	1·200
,, ...	45	......	·027	·025	·06	1·215
,, ...	50	·024	·023	......	......	1·150
,, ...	55	......	......	·021		
,, ...	60	·022				
,, ...	70	......	·014	......	·05	·980
,, ...	90	......	·009	......	......	·810

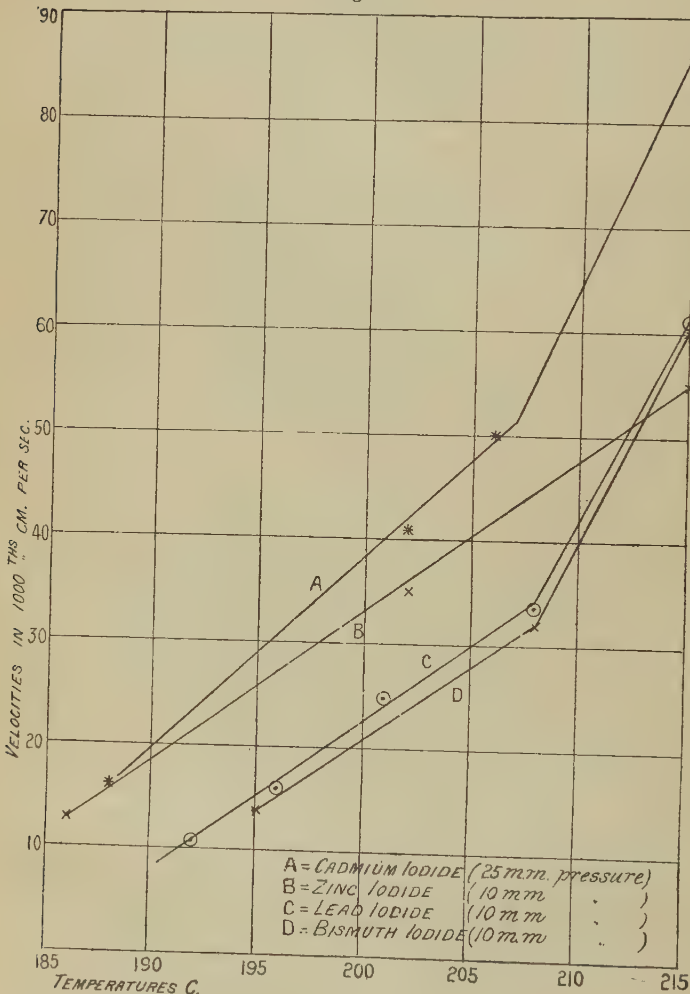
A number of determinations at other temperatures and pressures were also made with the same compounds. In all cases I found that the positive ions produced by heating zinc iodide, lead iodide, and bismuth iodide move with practically equal velocities under the same conditions ; while the positive ions from cadmium iodide move with just about twice the velocity of the positive ions from the other three compounds.

Quite recently Gehrcke and Reichenheim * have found that when various salts, chiefly chlorides, are made the anode in a vacuum-tube, and when this anode is heated, positively charged particles, which they call anode rays, are emitted. These effects seem to be the same as those studied here, except that the temperatues in their case are higher. They find the anode rays give the metal lines in the spectrum, from which it seems that the positive ions are metallic. They may, however, have attached to them, combined or

* *Deutsch. Phys. Gesell. Verh.* viii. 21, pp. 559–566, Nov. 1906.

otherwise, products of the chemical change. If this were not so, we should expect their velocities to be more unequal and to be roughly inversely proportional to their atomic

Fig. 4.



weights. Zinc iodide very quickly and readily absorbs water, and it was found that the velocities obtained for positive ions from this salt were made much smaller by moisture. Bismuth

iodide and lead iodide, on the other hand, do not become loaded in this manner.

The cadmium iodide, which does not absorb moisture, gives off positive ions which move with a velocity just about double that of the positive ions from lead iodide or bismuth iodide.

The influence of pressure upon the velocity of the ions can be seen by comparing the products obtained of velocity and pressure as given in the last column of the table. These are for the ions from bismuth iodide. It will be seen that between pressures 50 and 25 mm. of mercury the product does not vary much. There is evidence of simplification taking place between pressures 90 mm. and 50 mm. Langevin * has shown that $P \times V$ remains very nearly constant for positive ions produced by Röntgen rays between pressures 75 mm. and 1435 mm. of mercury, thus showing that between those pressures the constitution of those positive ions remains the same. The change in P.V at the lower pressures I cannot account for. The manner in which the velocity changes with the temperature at low pressures when the pressure remains constant is shown in fig. 4, in which the temperatures are plotted for abscissæ and the velocities for ordinates.

It will be seen that in all cases, except that of zinc iodide at a pressure of 10 mm. of mercury, there is a very rapid rise in the velocity at a temperature between 205° C. and 210° C., indicating that a simplification of the ion has taken place at that temperature.

Phillips † has found a linear relation to hold between the velocity and temperature for ions produced by Röntgen rays between −64° C. and 138° C.

Velocities of the negative ions.—The velocity of the negative ions was measured in the case of those from barium, calcium, and zinc iodides; and the results obtained from all these sources under the same conditions of experiment were practically identical. These are shown in the following tabulated list. The velocities are all for temperature 215° C., and are given in cms. per second for one volt per cm.

* *Thèses Université de Paris*, p. 190.

† Proc. Roy. Soc. vol. lxxviii., 1906.

Pressure in mm. of mercury.	Zinc iodide.	Calcium iodide.	Barium iodide.	Pressure × Velocity in case of Zinc iodide.
10	1·05	1·10	1·11	11·05
20	......	......	·74	
30	56	·56	......	16 80
35	......	......	·45	
40	·42	·33	......	16·80
50	·35	·22	......	17·50
60	......	·2	·29	
80	·21	......	......	16·80

In this case also it will be seen that, with the exception of the velocity at 10 mm. pressure, the values of (Pressure × Velocity) are fairly constant.

The emission of the negative ions from the salts of calcium and barium is interesting, since this has been made a special study in the case of calcium oxide by Wehnelt * and also by Dr. Horton †. The latter finds that the decreasing resistance of calcium oxide and barium oxide to the passage of electricity with rising temperatures is partly due to the increasing number of free corpuscles contained by those oxides. Under the conditions of experiment, no negative ions were detected in the cases of cadmium, bismuth, and lead iodides.

In the cases of the iodides of calcium and barium, no positive ions could be detected while the salts were dry; but if the salts were left in the apparatus and then tested, it was found that the velocity of the negative ions had been reduced to about one-third of their original value, and the presence of positive ions was detected in the case of calcium iodide.

It will be seen that the velocity of the negative ions is many times greater than that of the positive. This being the case, the nature of the ions must be different, since the conditions under which their velocities were obtained were similar. It was noticed during the experiments that in every case in which the positive ions were detected a deposit (apparently due to sublimation) could be seen on the upper electrode B, and the relatively slow movements of the positive ions made it probable that the ions in this case might be

* *Annalen der Physik*, Band xiv. (1904).
† Phil. Mag. April 1906.

charged centres surrounded by particles of the salt. Bloch * came to the conclusion that dust particles were intimately connected with the conductivity of recently-prepared gases, and the conductivity produced by phosphorus.

In order to ascertain whether the conductivity could be due to such a cause, I heated some of the salts in a large tube which was fitted with two corks, through which passed two equal-sized, insulated, thick copper wires, which were connected to the opposite poles of a battery of 200 storage-cells.

The material was heated slowly up to about 250° C. by means of a Bunsen flame, which was kept moving so as to ensure more regular heating. The electrodes were first carefully balanced on a pair of scales.

With iodides of bismuth and cadmium it was found that, after heating up, the negative electrode weighed one or two milligrams more than the positive each time the experiment was tried.

With lead iodide the negative electrode was always heavier than the positive, although the difference was not so marked as in the above cases.

With zinc iodide no difference in the weight of the electrodes was noticed.

Hence it appears that some of the products in the first three cases become positively charged, and so tend to accumulate on the negative electrode ; and it is probable that a good deal of the salts sublime without undergoing chemical decomposition, the particles being positively charged.

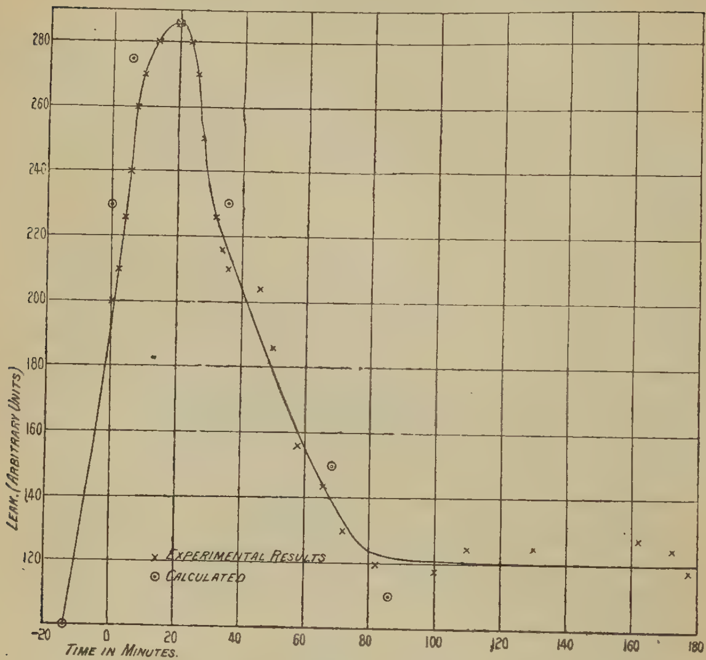
IX. *Rate of Decay of Activity with the Time.*

In our first experiments with zinc iodide (*loc. cit.*) Dr. Willows and I found that the activity first increases to a maximum, then gradually decays with the time, following an exponential law if the temperature is kept constant. I have made some further experiments on the decay of zinc iodide under different conditions of temperature and pressure.

* *Ann. de Chemie et de Physique*, Tome iv. 1905, pp. 127 *et seq.*

A typical decay-curve is shown in fig. 5.

Fig. 5.



In this the times are plotted as abscissæ and the rates of leak as ordinates.

If the ascending branch is produced backward and the point where it cuts the axis is taken as the zero of time, it is found that the curve can be represented by an equation of the form

$$I = A(e^{-\lambda_2 t} - e^{-\lambda_1 t});$$

where I = leak at time t, A is constant, and λ_2 and λ_1 are constants ($\lambda_1 = 25 \times 10^{-5}$, $\lambda_2 = 157 \times 10^{-5}$).

The ⊙'s represent points calculated from this formula on the assumption that the curve produced backward meets the axis 14 minutes behind the origin.

If we may reason from the analogy of radioactive change, such a curve would mean (see Rutherford, 'Radioactivity,' Ch. ix.) that two successive changes are produced. The first

of these produces no ions, but the second does. The presence of water is necessary to start these changes. These successive changes are of interest when taken in conjunction with the fact observed by Allen *, that the decay of photo-electric activity of zinc can be represented as the sum of two exponential terms.

Fig. 5 also shows that after the material has been heated for some time the rate of decay becomes very much slower; in fact, for the last 90 minutes the rate of leak undergoes very little decrease. For the part of the curve which can be represented by the formula, the activity declines to one-half its value in about 46 minutes.

Conclusions.

(1) Many inorganic compounds when heated up to temperatures below 360° C. have the power to produce electrical conductivity.

(2) That in the case of some of these compounds (*e. g.*, halogen salts of zinc) chemical changes take place, and the conductivity produced has its origin in these changes. The ions so produced are similar to those which have been observed in the cases of recently prepared gases, and the conductivity produced by phosphorus.

(3) That in all cases the ions are of a large size, as is shown by their relatively small velocities. They probably consist of a charged centre surrounded by particles produced from the heated compound, and in some cases water-vapour. Their nature could be best demonstrated by determining $\frac{e}{m}$, but means for producing a strong enough magnetic field were not available.

(4) The presence of water is in some cases intimately connected with the conductivity produced, as is shown by (*a*) the difference in behaviour of ordinary and carefully dried compounds, (*b*) the change in the rate of decay after heating for some time.

(5) In some of the more active substances, *e. g.* halogen compounds of zinc, increased conductivity can be readily detected in air at ordinary temperatures and pressures.

* Proc. Roy. Soc., Feb. 1906.

(6) The ions, both negative and positive, move with a very slow velocity, similar to those which have been observed in the cases of phosphorus, recently prepared gases, and gases from flames. This velocity is greatly increased by a rise in temperature. For certain ranges a straight line relation holds between velocity and temperature, but sudden changes in the direction of the line occur probably owing to simplification of the ions. Such a change occurs in the compounds tested at about 210° C. and the pressure reduced to 25–10 mm. of mercury.

The product (Pressure × Velocity) is constant for certain ranges of pressure.

(7) The rate of leak at different temperatures can be represented in all the cases tried by a formula of the form

$$I = a\theta^{\frac{1}{2}} e^{-b/\theta},$$

in which a and b are constants, θ is the absolute temperature, and I is the saturation-current.

(8) The ions produced are not able to penetrate a layer of luminium ·0008 cm. in thickness.

In conclusion I should like to thank Dr. R. S. Willows, in whose laboratory these experiments have been carried out, for much valuable advice throughout the course of the research, and Mr. F. C. G. Bratt for help in the construction of the apparatus used.

Cass Technical Institute,
Jewry Street, E.C.

Discussion.

Dr. R. S. Willows expressed his interest in the paper, and said that experiments similar to the author's had recently been described by J. J. Thomson in the Cambridge Phil. Soc. Proceedings. In the author's experiments there was a chance of chemical decomposition, but this had been guarded against by Prof. Thomson. Electrification could also be obtained by crushing salts in a mortar.

Mr. A. Russell asked if cooling salts produced any electrification.

Dr. Willows said the rate of leak increased with temperature, and he should therefore expect it to be very small at low temperatures.

XL. *On Stereoscopy with a Long Base Line.* By T. C. Porter, *M.A., D.Sc., F.R.P.S., F. Phys. Soc. of London, &c.**

T. J.

THE original communication of the suggestions and extensions of stereoscopic work given in the present paper was made to the Royal Photographic Society on January 29th of the present year; but the lecturer had no opportunity of showing his then audience the results he had obtained, save through a number of small stereoscopes kindly lent to the Society for the occasion. He therefore asked a few members of some of the learned societies to come and see the illustrations for themselves in the author's own laboratory at Eton. A few came, and amongst them the Hon. Sec. of the Physical Society of London, who urged the exhibitor to show the same things to the Physical Society in London, and promised that he would be responsible for the rather considerable preparations involved; moreover, the Council were kind enough to place practically the whole of this evening at the lecturer's disposal. Hence the present paper, which is very largely the same as that read to the Photographic Society; and the illustrations, which contain many shown at that time, and also in the lecture room at Eton. The author would at the very outset express his warm thanks to M. Selb of Brussels, for the magnificent mountain photographs, taken with a long base-line as long ago as 1903, a sufficient proof that the present lecturer is not the first to whom the idea of this application of tele-stereoscopy occurred, though apparently M. Selb did not think of publishing any account of his photographs. So far as the other applications are concerned:—for military and meteorological purposes, the suggestions seem to be really new, though they are of so obvious a character, that it is the more surprising if no one has tried them before †. One more word of preface: the

* Read May 10, 1907.

† Since reading this paper, the author has learnt that Mr. John Tennant took several very successful photographs of distant buildings and of clouds, using a long base-line, some years ago. Prints from the negatives were published in the 'Photogram' at the time.

author has deemed it well to give a brief sketch of the principles of stereoscopy, and, so far as he knows, the attempt to measure the limit of stereoscopic vision for the unassisted eye is new, as well as some other minor points, which seem scarcely worth separate mention.

The power we possess of judging of the difference in distance, of two objects in our field of view, depends first, upon various properties of those *objects* which vary with the said distance ; variations to appreciate which we require strictly but one eye, though it may well be that we appreciate them better with two. One of these properties is *the apparent size*. This is one of the most useful properties, if not altogether the most important. The *clearness or haziness of an object* in an atmosphere which is not perfectly transparent is another property of almost universal application—at any rate in our land. A third property, which depends upon the bodies themselves and also on the source of light by which they are illuminated, is *the arrangement of light and shade* they present, and *the shadows they cast*. A fourth, and perhaps it should be termed an accidental property of objects, is their *motion*. As the apparent motion of a body diminishes as the distance increases, its apparent motion gives some clue to its distance, and this is very noticeably the case if it is *we* who are moving, and not the objects which form the landscape. This is well-seen in the cinematograph pictures taken from a train, the more rapid the motion the greater is the distance from which the effect may be observed ; and this suggests a very curious method of exhibiting the true distance between very distant objects, *i. e.* that of exposing a cinematograph film rather slowly, or more correctly with rather long intervals between the different exposures, whilst the whole instrument is moving as fast as is practicable, and then, in showing the result on the screen, practically increasing the speed of the locomotive, by running the pictures through the lantern as fast as possible.

It may be that the well-known rapid side to side motion which birds, and some other animals, make when inspecting the ground, or their food, before pecking, enables them to form a precise idea of the distance of the object at which they are about to aim. The fact that their eyes lie on different

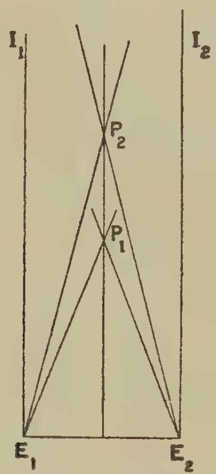
sides of their heads necessarily deprives them of certain powers of estimating distance which are possessed by animals like ourselves which have their eyes on the same side of the head.

The next means of telling the different distances of objects depends, not on properties of the objects themselves, but on one of the eye itself, namely the effort, made quite instinctively, by which we bring the images of these objects to a sharp focus on the retina, *i. e.* by the effort made in so-called *accommodation*. The power of accommodation is, however, comparatively limited in its scope, far more so than seems to be commonly thought, my experiments on many people putting the limit for ordinary sight at about six feet from the eye. On these five methods, AND ON THE MENTAL DEDUCTIONS FOLLOWING FROM THEM and founded upon previous experience, depend the whole of the powers of appreciating form and distance, so far as each eye is considered *separately*. They probably constitute the most important part of the sum of the powers we have of this kind; and to illustrate how far they are capable of giving us correct ideas of the form and distance of objects, we may present to both eyes precisely the same view, as in the case of a photograph projected on a screen (photograph projected). Our sensation of the forms and distances of the different objects are so nearly those we receive when the *two eyes* regard the *actual view* from which the photograph was taken, that we are driven to the conclusion that any means, apart from motion, of judging of form and distance which are wanting in the viewing of the picture on the screen, must either be quite insignificant in its action, or only act through a range less than that of the observer from the screen; and yet, in thus viewing a photograph, we are completely deprived of the sense of "relief" conferred upon us, under ordinary circumstances by the possession of *two* eyes, which can both be turned so as to converge their lines of sight on each object in turn, thus giving rise to the sensation properly known as stereoscopic relief.

The impressions received by the two eyes when regarding any near object are not the same, because each eye sees from its own position, which differs from that of the other eye,

by, let us say, a distance of i inches, i is equal to about 2·5 in most cases. Let E_1 E_2 (fig. 1) be the eyes, distances i inches apart, and let the lines E_1 I_1, E_2 I_2 be their respective lines of sight when looking at a very distant object, so that E_1 I_1, E_2 I_2 are practically parallel. Let P_1 be a near object: then when the observer wishes to examine P_1, he must turn each eye inwards through a certain angle, which we will call θ. This he does by making a definite effort, and it is this effort which gives the sensation of stereoscopic relief.

Fig. 1.



It is true that in order to see P_1 distinctly, at least one eye must focus P_1 by "accommodation" (in people of normal sight, both eyes focus P_1), but this accommodation-effort, which has already been discussed, is altogether different from the effort of convergence, and the stereoscopic relief will be perceived by the convergence effort alone, even if the eyes be unable to focus clearly. This I have proved quite conclusively in more than one way, which I shall describe later. To return to fig. 1. Suppose P_2 be another point, rather further from the eyes, it is clear that the angle of convergence will be smaller, the corresponding effort of convergence less, and that too just in proportion to the distance of P_2 from P_1: thus as the difference in the magnitudes of θ implicitly

measures the difference of the distances of P_1 P_2 from the eyes base-line in the figure, so the corresponding efforts of convergence measure the stereoscopic sensation of the difference of distance of P_2 and P_1 from the observer. P_1 P_2 may be any points on a given surface, and the notion of the form and distance of the points which make up the surface, *i. e.* of the surface itself, is got by the variation of θ, as the eyes converge on the different points in succession.

Suppose that two photographs of a view be taken from points the same distance apart as the eyes, and the positive prints from these be mounted side by side and then inspected, each view by its appropriate eye through a suitable instrument, we may arrange (1) that each view shall be in focus ; (2) that each shall be viewed at such a distance from each eye, that each object in each photograph shall subtend the same angles at the eyes, as those which it subtended in the actual view from which the photograph is taken, and therefore that it shall look the same size as it did in nature.

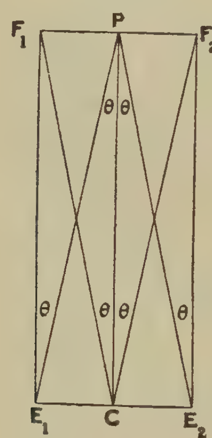
This assumes the use of certain kinds of lenses, both in taking the photographs, and also in viewing them, and in practice these conditions are only approximately realized. That this is so may be proved by taking a photograph of a scene with a fairly long focus lens, and making a positive on glass from it by contact. Now place one eye near the spot where the lens was when the negative was taken, and the positive, as far in front of the eye as the negative was behind the lens when the photograph was taken, and view the landscape through it, moving the eye, if necessary, so as to make the picture of an object just cover the corresponding real object in the view ; it will be found that on keeping the eye in this position, and regarding the rest of the view, that the other objects in the transparency do not as a rule "fit" the corresponding objects in the view. This by the way.

(3) We may arrange so that the lines of sight of the two eyes when regarding corresponding points of a very distant object in the two views are parallel. Then, if these three conditions are fulfilled, the efforts of convergence which the two eyes must make to regard the corresponding points of the two views, will be the same as they were in looking at

the corresponding points in the original view, and the stereoscopic effect will be the same. Now in this case there is no *accommodation* necessary to focus different parts of the picture, nevertheless the stereoscopic effect is obtained. This fact therefore constitutes proof positive that variation in the convergence of the eyes is *alone* sufficient to produce stereoscopic effect, apart from "accommodation."

To return to the theory: it can be seen at once from fig. 2 that the total angle ($2\,\theta$) of convergence of the eyes

Fig. 2.



for a point P at any given distance PC is exactly equal to the angle subtended at the eyes E_1 E_2 by a line equal to the distance between them, F_1 F_2, placed at that point, as in the fig., parallel to the line joining the eyes. Now in practice, when P is a certain distance from C, the angle θ, the convergence for one eye, becomes too small to be appreciated, and therefore also the *effort* necessary for the convergence through this small angle is also inappreciable, and the stereoscopic sensation will fail: in other words, stereoscopic sensation will be limited to objects nearer the observer than this distance, which is thus the stereoscopic limit, and it is obviously of interest to find what this distance is for ordinary eyes. The average distance between the eyes may be taken as 2·5 inches, that is $E_1\,E_2 = 2{\cdot}5$ inches, therefore

$E_2C = 1 \cdot 25$ inches, and the question may be stated thus :— At what distance does the eye fail to appreciate the *effort* made in turning it from F_1 to P ?

To find the average value, I drew two black lines on white paper, five-sixteenths of an inch apart, and placing them in bright daylight, measured the distance for each of eleven persons at which the effort to look first at one and then at the other became inappreciable; the mean result was 128 inches, the actual values being those on the accompanying table :—

ft.	in.		ft.	in.		ft.	in.		ft.	in.
5	7	...	9	4	...	11	8	...	14	0
8	1	...	9	6	...	12	0	...	15	8
8	10	...	10	0	...	12	0	...	—	

This shows, by a simple proportion, that the effect to look from F_1 to P becomes inappreciable *for the average person* at about 43 feet distance ; this may therefore be taken as the average distance at which stereoscopic relief ceases, though some lose it about 20 feet, and others retain the sensation up to 64 feet.*

It should be particularly noticed that the validity of this proof rests on the assumption, tacitly made, that when θ is very small, *i. e.* near the stereoscopic limit, the effect to turn one eye through a very small angle, θ, is the same, whether the other eye turns with it in the same direction, or in the contrary. This would probably not be true for a large value of θ, but this does not affect the argument. But, whether the proof holds exactly or not, there is no doubt that the limit indicated is not far from its true value : this appears, both by observation of ordinary views, when, however, it is exceedingly difficult to discriminate between the true stereoscopy and the sensations of relief and distance produced by the other causes already enumerated, so that we should naturally infer that the limiting distance for true stereoscopic effect was *greater* than it really is : and a second, and far

* With the "balloon" range-finger, and good eyes, it is certain that the stereoscopic limit is many times greater than that assigned in this paper; but the writer believes that his estimate is not greatly in error for ordinary sight under ordinary conditions.

better proof that the limit established here is fairly correct, is afforded by measurements of the distances to which stereoscopic relief is perceptible in pairs of photographs taken as described later in the present paper, and to compare the results thus obtained with the distances of the limit calculated by the method we are discussing. So far as my work in this direction has gone the values agree as well as I could expect.

It is worth noting that both by theory and experiment, the range for stereoscopic relief is *not* the same as the range for "accommodation," *i. e.*, for focussing. In my own case, everything is equally, and well in focus, at distances greater than 5 feet 8 inches, whereas I can perceive stereoscopic effect to a distance of 24 feet: nor is it necessary for the sensation of stereoscopic relief that the objects should be sharply focussed. Even if the images of an object formed by the two eyes seem more or less blurred, so long as the effort of convergence is sensible when the eyes are turned from a more distant object to the nearer, the stereoscopic sensation will be received. This explains why even very short-sighted people possess a tolerably long range for stereoscopic vision, and also why, in cases where one eye is long-sighted, and the other short, the range of stereoscopic vision is still very considerable; but the stereoscopic range is undoubtedly shortened by short-sighted vision, either of one, or of both eyes, for it is obvious that the blurred images of an object formed by each eye are wider than those sharply focussed, and hence the distance at which the images subtend the limiting angle for stereoscopic effect is less with blurred than with sharp images, and therefore the stereoscopic range is reduced.

A corollary of this is also worth noting, namely, that when the focussing has been done for all distances, as by a lens in a pair of stereoscopic photographs, such as have been already described, and in which objects appear of the natural size, the stereoscopic effect reaches its maximum theoretical value, so that a person whose stereoscopic limit is, say, only 20 feet, when looking at a view in nature, may be able to perceive it for 43 feet, the average, or even 64 feet, or more, in the same view, when looked at through the stereoscope.

So far I have stated the fundamental bases of the stereoscopic sensation as simply and shortly as I could, and it is necessary thus to clear the ground before considering the extension of the range of stereoscopic vision, by increasing the distance between the two simultaneous points of view. That it must be so increased is evident from fig. 3, where E_1 and E_2 stand for a pair of eyes 1 inch apart, and E_3 and E_4 form a pair 3 inches apart. The angle θ_1 is the angle which measures the converging effort, and the stereoscopic effect

Fig. 3.

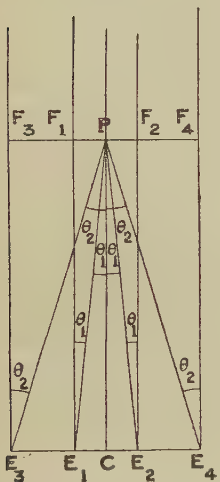
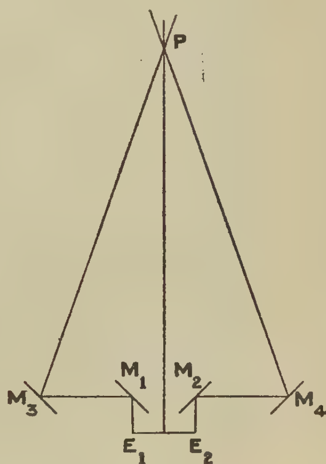


Fig. 4.



for the first pair, when viewing the object P, and the angle θ_2 the similar effort for the second pair, and it is plain that θ_2 is greater than θ_1, and that if P were at such a distance that θ_1 were too small to produce any stereoscopic sensation, θ_2 would still be large enough to produce it, so that the greater the *base-line*, as E_1 E_2 may be called, the greater the range.

The exact connexion between the ranges d_1 d_2 with base-lines i_1 i_2 in length being given very simply thus:—

Let θ be the critical value of the angle F_1 E_1 P (fig. 3), so that for any smaller value of θ the stereoscopic sensation fails, then

$$\tan\theta = \frac{E_1C}{CP} = \frac{\frac{1}{2}i}{d_1}, \quad \ldots \ldots \quad \text{(i)}$$

and since θ will be the same for the other base-line when stereoscopic sensation vanishes,

$$\tan\theta \text{ also} = \frac{\frac{1}{2}i_2}{d_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii})$$

$$\therefore \text{ from (i) and (ii) } d_1 : d_2 : : i_1 : i_2,$$

or in ordinary language :—

The range of stereoscopic effect varies directly as the length of the base-line. If then the distance between the eyes could be increased, say ten times, *i. e.* to 25 inches, the stereoscopic effect would be perceptible to a distance of 430 feet for an average person. Now such an increase of distance can be made to all intents and purposes by means of reflectors M_1 M_2 M_3 M_4 placed as in fig. 4, the eyes at E_1 E_2 seeing the object P as reflected in the mirrors M_3 M_4, *i. e.*, as viewed from the ends of a base-line extending from M_3 to M_4 instead of only from E_1 E_2. By suitably inclining the mirrors M_1 M_2, the two images formed by M_3 and M_4 can be brought into apparent coincidence, and the stereoscopic effect witnessed. This experiment was, I believe, first made by Von Helmholtz, and he thus extended the base-line to many inches, and called the instrument the telestereoscope. The larger base-line is also one cause of the increase of range of stereoscopic effect in many modern field-glasses, but it is not the *only* cause, for the magnification of a distant object is equivalent to viewing those parts of it which are visible, at a shorter distance, and this means an increase in the angle θ, but, as will be shown presently, magnification does not act in the same way as actually bringing the object itself nearer, a fact which should be carefully remembered. The stereoscopic effect is also greatly increased in some modern forms of telemeter, and from the same causes. But in all such instruments, the extension of the stereoscopic effect soon reaches a limit. This is due to difficulties of construction felt both in making plane mirrors, and lenses, and also troubles arising from the want of rigidity in bars, and the loss of light in reflexion and by absorption, not only in traversing the lenses, but also any considerable thickness of ordinary air. I cannot find any

published account of an extension of base-line beyond that of Helmholtz.

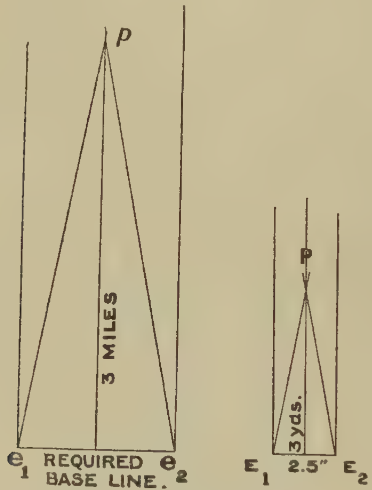
But it is at once evident that no such limitations affect the two photographs, which may be made of a distant object, and afterwards viewed in the stereoscope. However great the base-line, so long as its two ends are about the same distance from the objects to be photographed, the absorption of light caused by the air and the camera may be, and generally will be, the same. Hence, so long as the objects themselves are clear enough and large enough to yield a picture, we have the means of recognizing their forms and relative distances, however far off they may be. I pass at once to the few practical examples, the only stereoscopic photographs of the kind, which I have had time and opportunity for taking, since the thought of the possibility of an enormous practical extension of the field of stereoscopy first occurred to me.

Suppose, then, we have in the distance the confused collection of buildings constituting a fortified city; the nearest points we will say are two miles away, the furthest four. We wish to see at a glance the relative distances between the different objects as plainly as we should see them if an exact small model of the city were made, and viewed by the unassisted eyes, 2·5 inches apart, from a distance, let us say, of three yards. We wish also to see the same parts of the model as we could see of the real city, and moreover the model is to be on such a scale that the parts of the model at a mean distance of three yards shall look the same size as the same parts of the real city at the mean distance of three miles.

To fulfil these conditions, (1) we must use a camera lens and stereoscopic lenses, such that when the photographs are viewed through the stereoscope each object depicted subtends the same angle at the eye as the corresponding part of the original: this can be done without any great difficulty, and since this is not the main question so far as we are concerned, I shall not enter into further details. (2) To get the required stereoscopic effect: if p and P are corresponding points in the real city and in the supposed model respectively. Fig. 5

will show that the base-line for the two photographs must be as many times 2·5 inches as there are three yards in three miles, *i. e.* 122 yards nearly; and if we act accordingly, we shall get the required result, as I think the photograph of Windsor Castle, viewed from a mile's distance, and taken from the ends of a base-line of forty yards, sufficiently shows. This photograph reveals plainly the general arrangement of the Castle buildings round the escarpment of the chalk hill on which it is built. One can see quite plainly the

Fig. 5.



relatively central position of the Round Tower, and also how far behind it the Courtyard side of the Eastern Buildings lies, points which no ordinary photograph or actual view (unless taken from a balloon) could possibly show. The stereoscopic analysis, as we may call it, in this particular case is the more surprising because both the points of view, *i. e.* the whole base-line, lies well below the level of all the Castle buildings. Views of this kind may often prove exceedingly valuable in time of war. There are no "rights" for a government to purchase: anyone armed with an ordinary snapshot camera and a stereoscope, such as may be purchased for a few pence, has it in his power to unravel the tangled arrangements of forts and country from a long distance off,

for in clear air the two photographs may be taken at five or six miles distance, requiring for a horseman only a few seconds, dismount; and after development, a good rinsing, fixation, and another good rinsing, two prints may be taken without drying the negative, on such a paper as Velox, fixed, and placed in position on a card in the stereoscope, without waiting for them to dry, and the riddle of the strategical position read, as it could be read in no other way.

However hazy the objects may appear, as long as they can be photographed at all. they can be seen by this method with any amount of "relief" desired. It seems to me that for scouts of all kinds the extra kit involved, *i. e.* a single ordinary $3\frac{1}{4} \times 3\frac{1}{4}$ inch camera, with films for lightness, or two plates, if more cannot be carried, would be well worth the extra trouble. It is clear that the presentation of objects by such photographs is quite different from those obtained in any other way, for to revert to the "model city" already mentioned seen from three yards, which expresses well the effect witnessed on looking through the stereoscope at the photographs of the real city three miles off:—it is clear that if an attempt were made to view the real city from three yards distance, only a very small part of it could be seen at all; possibly, at the only points accessible in time of war, a few stones, or a part of a wall, and closed gates. I should like to add that, whilst writing this paper (on the high seas), I showed some of these photographs to Mr. Lloyd Griscom, the Ambassador of the United States at Rome. He informed me that during the late Russo-Japanese war, a man whose name I much regret to say I have forgotten, had made for Admiral Togo some naval photographs, which the ambassador thought must have been taken on much the same principles as these of mine, but he added that no account of these had been published, though he thought they had proved of real value to the Japanese. Mr. Griscom's remark as to the photographs I showed him was that he had never seen anything like them before.

An important point to be noticed is the very great magnitude of the area which may thus be seen at a glance in strong "relief." For, suppose that the stereoscopic view embraces an angle of 25 degrees width of scenery, and suppose that

the objects to be examined are very large, and lie at a mean distance of fifty miles, we then have a stereoscopic presentation of a tract of country of more than twenty-one miles from end to end, stretching away to a distance only limited by the curve of the earth, the altitude of the objects, and the clearness of the air; and thus we are directly led to the second great field of stereoscopic work, which this investigation seems to me to open, I mean that of exploration, and in particular the exploration of mountain ranges. Suppose then that the explorer is anxious to study the "lie" and relative heights of peaks standing up in the distance behind the part of the range facing him. Let us take an extreme case, and assume that the highest peaks visible in the distance may be as much as 100 miles off, he wishes to see them as he would see a model of the ranges in which the model of the furthest mountain was at a distance of 30 feet, the model being inspected with the unassisted eyes, say 2·5 inches apart, and the 30 feet lying well within the limit of their stereoscopic vision. We have at once the simple proportion :—

As 30 feet is to 2·5 inches, so must 100 miles be to the base-line required; and on working this elementary sum out, we find for the necessary base-line a length of $\frac{2}{3}$ of a mile, a perfectly manageable distance in many cases, for note well, the observer does not require a level plane $\frac{2}{3}$ of a mile long and parallel to the range. He only needs two spots, somewhere about the same level above the sea, the higher the better, at $\frac{2}{3}$ of a mile minimum distance. If his base-line is perforce longer, the result will only be to view the "model" mountains nearer; and if shorter, further off. It will be noticed that the experiment will only fail for any particular mountain, if it should happen to be hidden from one of the points of view: or, if the base-line is not long enough, and the "model" mountains are, some of them at any rate, viewed at a distance beyond the range of stereoscopic effect. It seems therefore worth while to show how we may, by means of what has already been stated, find the smallest base-line which will give stereoscopic effect to any limit, say the 100 miles required in the present example. I purposely give it in a form which all can easily understand.

For the average person with eyes 2·5 inches apart, the stereoscopic sensation fails after 43 feet, thus the "model" mountains must not be viewed at a distance from the furthest of them greater than 43 feet, *i.e.* we have the simple proportion :—As 43 feet is to 2·5 inches, so is 100 miles to the distance required, and this gives the answer as half a mile very nearly.

This length of base-line, then, would be sufficient to bring all the mountains visible into stereoscopic relief, but it would, no doubt, in all cases be advisable to take a base-line considerably larger than the minimum.

There is one point in particular which should be carefully remembered in connexion with viewing these stereoscopic photographs. Hitherto I have supposed in every case that the objects "looked as big in the stereoscope as they did in nature." I put the words in inverted commas because this is the simplest way of saying that when viewed in the stereoscope, the objects in the photographs subtend the same angles at the observer's eye as their originals did at the observer's eye in the natural scene; and the ordinary expression "look as big as" is a convenient abbreviation, when its exact meaning is understood. Generally, however, the objects as seen in the stereoscope do not look the same size as they do in nature: they are magnified, *i. e.*, subtend greater angles, and at first sight it would perhaps seem as if magnification must increase the stereoscopic effect in exactly the same way as taking a correspondingly longer base-line. This is, however, not the case, for a moment's reflection will show that, however much we may magnify two views A and B, taken 2·5 inches apart, say, we can never make them the same as two views—C and D, say, taken, let us suppose 2·5 yards apart, for the views themselves are different in the two cases: even if A and C are taken from the same spot, and are therefore the same, B and D cannot be the same; and the stereoscopic result of magnifying A and B thirty-six times (the number of inches in a yard) will not be the same as that of C and D. The stereoscopic effect of what objects there are on the photographs A and B is undoubtedly multiplied thirty-six times, when A and B are so magnified, and so far as these photographed objects go, it is equivalent to viewing them at one-

thirty-sixth the distance, *i. e.*, the stereoscopic effect for the photographed objects is thirty-six times as great; but there are actually objects on the photographs C and D, taken together, which are not to be found on the other pair, whilst some of the objects which appear in A and B will not be found on C and D. The greater base-line used for C and D makes us see, as it were, further "round" the objects than we did in A and B, and supplies us with fresh information for the stereoscopic estimation of the form and distance of each object. One curious effect of the magnification of any pair of stereoscopic photographs must be to increase unduly the apparent depth of the objects themselves, and also the apparent distance between them:—in short to produce what may be called "stereoscopic distortion."

The last, and perhaps the most interesting application of this extension of the base-line from the ends of which the two necessary photographs are taken, is to the sky, i. e., *to the analysis of the distances at which the different layers of cloud lie.*

If we take the greatest vertical height at which any cloud is seen to be as great as 20 miles—and this is undoubtedly of rare occurrence—the smallest base-line necessary will be for the average person about one-tenth of a mile; but in practice, it will be better to use about twice this distance, or say 350 yards. This is for clouds overhead; but in most cases, since the average height of clouds is not more than 7 or 8 miles above sea-level, if indeed that—a base-line of 150 yards, or even less, will be found to work well. If, however, the clouds to be analysed lie near the horizon, the base-line will probably have to be greater. Let us take the *extreme* case of clouds which are 20 miles above sea-level, and viewed when they appear to be resting on the observer's horizon, which we will further suppose is the sea, while the height of the observer above sea-level is one-fifth of a mile. Such a cloud would be, if visible, 310 miles from the observer, as the crow flies, and the base-line must be at least a mile and a half, no doubt two miles would be better, but then such a case is not likely to occur, and I only quote it to give some idea of the superior limit to the base-line necessary for our purpose. Under ordinary

circumstances with clouds anywhere up to 3 miles in vertical height, and the height of the observer above sea-level less than 400 feet, a base-line of rather over a quarter of a mile will be sufficient to exhibit even the furthest clouds (which may be taken as about 50 miles off) in relief. The stereoscopic cloud photographs shown were taken as snap-shots from ships going 15 knots, with an interval of about a minute between the exposures of the two negatives which together constitute one of the pairs of photographs; and this implies a base-line of approximately 450 yards. Some of them exhibit the curve of the sky, *i. e.* the curve of the earth, very beautifully. Subsequent experience has shown the writer that in many cases the base-line was unnecessarily long, with the result that the resulting photographs are more difficult to combine subjectively in the stereoscope than need have been the case.

There are two considerations which affect this third great field for experiment:—

(1) The clouds are moving objects; hence if they are moving fast in a line at right angles to the direction in which the camera is pointed, the camera itself need not be moved, save perhaps to give it a slight rotation about its axis. The two views can be taken from the same spot at an interval of time long or short at the operator's will, and depending also on the rate of the clouds' motions, and also on the rate at which the clouds may be changing their forms;—for

(2) The clouds are continually changing, and hence theoretically two SIMULTANEOUS photographs should be taken at the proper distance apart. Such an arrangement is absolutely necessary in all cases where the height, distance, and velocity of the clouds are to be measured from the photographs; but for a picture giving a general view of the cloud layers, this is scarcely necessary, as the photographs exhibited sufficiently prove.

Lastly, not with clouds only, but with any distant moving object, a stereoscopic presentation of that object may be obtained by taking two successive photographs of it, with a suitable time-interval between the two. The principle involved is plain. A change in the position of an object,

whilst the observer, or rather his camera, remains at rest, is precisely equivalent to a change in the position of the camera in the opposite direction, whilst the object remains at rest. The results of the former case are, however, often very curious, for although the moving object is presented in relief by the photographs, yet naturally its surroundings are not so ; in fact these last may be different in the two views, in which case the stereoscopically presented object stands forth on a ground which changes slowly from that on one picture to that on the other, through retinal rivalry, caused by the gross dissimilarity of the two views. This retinal rivalry lies indeed at the root of all stereoscopic vision, for it is this retinal rivalry, caused by the dissimilarity of the two stereoscopic views, which prompts the eyes to change their angle of convergence, so as to bring corresponding points of images of the two pictures to "corresponding points" of the two retinas.

Here this paper must close : in the first part of it I have restated as simply as I could the fundamental truths on which stereoscopy rests. This led at once to the deduction of a working estimate of the average distance at which true stereoscopic sensation fails. In the last part I have pointed out three great fields of survey work :—Military, Geographical, and Meteorological, in which the great extension of the base-line advocated in this paper may, I hope and believe, prove a valuable, if not an indispensable implement in further research.

Upton Park, Bucks,
May 10th, 1907.

Discussion.

Mr. J. Tennant expressed his interest in the paper, and said he had shown pictures of clouds, taken by the same method as that employed by the author, in a paper published in 1897. He was not certain that the feeling of convergence was the only thing which determined stereoscopic vision. The estimate of 43 feet as the limit of stereoscopic vision seemed to him too small, as he had observed stereoscopic effects at distances up to 125 yards. In 1896 he devised a stereoscopic camera and used it to measure the heights and

depths of clouds. In some of his cloud experiments he had used a base-line $\frac{3}{4}$ of a mile long. The method was applicable to many subjects and was a most valuable one.

Mr. C.W. S. Crawley did not quite understand the author's stereoscopic limit which he put at 43 feet. The only definite limit to the angle θ was the smallest difference of angle of convergence, say θ_0, by which we could judge difference of distance. In ordinary use, many things prevented our employment of stereoscopic power to the full, but he thought 43 feet was much below the limit in any case. Von Helmholtz, from a rough test, gave us θ_0 as about 1 minute, which would mean that we could say by pure stereoscopic power that an object at 200 metres was nearer than one at infinity.

As a matter of fact, von Helmholtz's figure was much too large; for English eyes at any rate. The speaker had repeated the rough experiments in a way which, while it allowed the stereoscopic power fuller scope, did not allow any help from other sources. He had tested all sorts and ages of people and found the usual value of θ_0 about 10 secs.; giving a limiting distance of 1200 metres. Several were, however, much better than this and had θ_0 about 2 secs. of arc, or a limiting distance of 6000 metres. Personally, the speakers varied from 3 seconds when in good health to 8 or 10 secs. when not; and he found a stereoscopic test most useful in hepatometry. In answer to a question, the speaker said these values were fully borne out by others on the Forbes range-finder, where judgment of distance by any other means was absolutely impossible.

He quite agreed with the author that wide-base stereoscopic photography might be much more widely used in surveying and in cloud-work, and hoped that the paper would greatly promote such use. He thought the author would be interested in some of the work of Dr. Pulfrich of Jena, who had gone a long way in the former branch.

XLI. *On the Measurement of Mutual Inductance by the Aid of a Vibration Galvanometer.* *By* ALBERT CAMPBELL, *B.A.** (From the National Physical Laboratory.)

[Plate XIII.]

CONTENTS.

1. *Introductory.*

THE determination of a self inductance by comparing it with a condenser by means of Anderson's Method can be made with ease and accuracy, as the two adjustments are independent. The use of a vibration galvanometer in this method as carried out by Rosa and Grover (Bulletin of the Bureau of Standards, vol. i. p. 291, 1905) greatly increases the ease of manipulation and the sensitivity of reading. For several years past we have tested our standards of mutual inductance against the B.A. Standard Air Condensers (as well as Mica Standards) which have been standardized by Maxwell's Commutator Method. For this purpose I have used Carey Foster's Method †, in which (as in Anderson's Method) the adjustments are easy and the formula simple. The connexions are shown in fig. 1.

R is a non-inductive resistance (capable of carrying a fair amount of current); M is the mutual inductance to be measured, its secondary coil being in series with an adjustable resistance in the branch AyF ; K is a condenser, and G a ballistic galvanometer. When there is no throw on G on reversing the battery then

$$M=10^{-6}KRr,$$

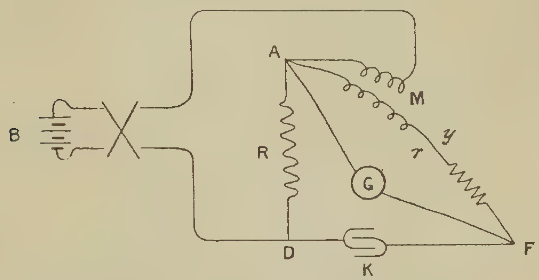
where M is in henries and K in microfarads, and r is the resistance of the branch AyF. To increase the sensitivity a

* Read May 24, 1907.

† Phil. Mag. [5] vol. xxiii. p. 121, Feb. 1887.

secohmmeter arrangement was sometimes used. Knowing the value of the vibration galvanometer in other cases, I attempted to apply it to this one, but found that the method had to be modified in order to make it applicable. I found

Fig. 1.

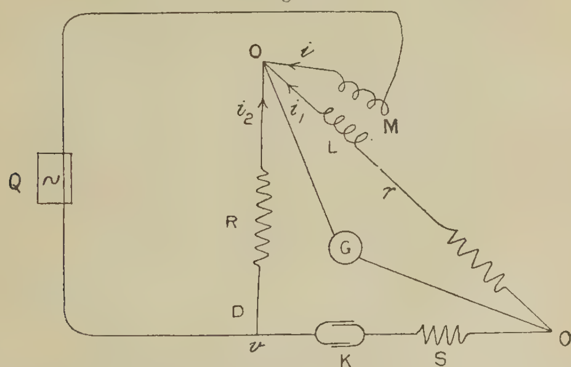


later that the modification I had introduced had been already suggested by Rowland *. As the use of the vibration galvanometer, however, is a great advantage (and novel in this method, so far as I know), I think it will be of interest to describe the complete method.

2. *Theory of Modified Carey Foster Method.*

In fig. 2 let Q be a source of alternating current and G a vibration galvanometer or its equivalent. The necessary

Fig. 2.



modification consists in adding a resistance S in series with the condenser K. Let the resistances of the other branches

* Phil. Mag. [5] vol. xlv. pp. 65–85 (1898).

be R and r, and let the instantaneous values of the currents be i, i_1, and i_2 as marked. Let the instantaneous potentials on the terminals of the galvanometer G be 0 and 0, so that there is no deflexion (*i. e.* the condition of balance is to hold at every instant). Let v be the instantaneous value of the potential at D, and q the charge in the condenser.

Then

$$i = -i_1 - i_2$$

and

$$0 = i_1 r + L\frac{di_1}{dt} + M\frac{di}{dt}$$

$$= i_1 r + L\frac{di_1}{dt} - M\frac{di_1}{dt} - M\frac{di_2}{dt}. \quad . \quad . \quad . \quad (1)$$

Also

$$v = i_2 R,$$

$$q = K(v - i_1 S),$$

$$i_1 = \dot{q} = K\left(\dot{v} - S\frac{di_1}{dt}\right);$$

$$\therefore \quad \dot{v} = \frac{i_1}{K} + S\frac{di_1}{dt}.$$

So

$$i_2 R = v = \text{const.} + \int \frac{i_1}{K} dt + SKi_1;$$

therefore

$$R\frac{di_2}{dt} = \frac{i_1}{K} + S\frac{di_1}{dt}.$$

Substituting in (1) we have

$$i_1\left(r - \frac{M}{RK}\right) + \left(L - M - \frac{MS}{R}\right)\frac{di_1}{dt} = 0.$$

Now, as will be shown below, we may assume all the currents to be sinusoidal, i. e., $i_1 = I_1 \sin pt$, and hence

$$r - M/RK = 0 \text{ and } L - M - \frac{MS}{R} = 0.$$

Thus for a balance we must have the following two conditions satisfied, viz.:

$$M = 10^{-6} KRr, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$L = M\frac{R+S}{R}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where K is in microfarads. When $S=0$, $L=M$; and this is therefore the minimum permissible value for L. When L is less than M in the pair of coils under test, a third inductive coil must be inserted in the branch OrO to bring the value of L up to at least M, otherwise no balance can be obtained. It will be noticed (*a*) that if R be kept constant the conditions (2) and (3) can be satisfied by *independent* adjustments of K or r and S; (*b*) that the adjustments are independent of the frequency. Accordingly a balance can always be obtained with great facility *.

3. *Vibration Galvanometers.*

Since Vibration Galvanometer methods are familiar to very few experimenters in this country, I think the following general description will be of interest.

By a *Vibration Galvanometer* is meant one in which the natural vibration frequency of the moving part can be adjusted to be the same as the frequency of the source of alternating or pulsating current used. The two main advantages in using a tuned galvanometer or other instrument are as follows :—(*a*) when the instrument is in tune with the source of current the vibratory motion of the moving part is enormously increased, due to resonance; thus a high sensitivity is obtained, usually about 100 times greater than that without tuning. (*b*) Since the sensitivity is so very much greater for the proper frequency than for others, when

* The above investigation can be carried out more readily by the use of the operators $Lp\sqrt{-1}$ and $1/Kp\sqrt{-1}$, but to some readers the method given will be clearer.

Equation (1) becomes

$$(r+Lp\sqrt{-1}-Mp\sqrt{-1})i_1=Mp\sqrt{-1}\,.\,i_2\,;$$

and the next equations give

$$Ri_2=\left(S-\frac{\sqrt{-1}}{Kp}\right)i_1.$$

Hence

$$R(r+Lp\sqrt{-1}-Mp\sqrt{-1})=Mp\sqrt{-1}\left(S-\frac{\sqrt{-1}}{Kp}\right).$$

Separating the real and imaginary parts we have

$$M=KRr$$

and $$L=M\frac{R+S}{R}$$ as before.

the wave form is other than a sine curve the instrument, if tuned to the fundamental frequency, responds to this, and is practically unaffected by the harmonics ; if the instrument is tuned to one of the harmonics instead, all but this component is practically ignored by it. For this reason the theory of each method in which a tuned instrument is used can be worked out on the assumption that the wave forms of voltage and current are all pure sine curves.

The deflexion is usually proportional to the amplitude.

This use of a tuned instrument in null methods is, I believe, due to Max Wien *, in one of whose papers (*Wied. Ann.* xliv. p. 689, 1891) will be found a very complete discussion of a number of his methods of measuring inductance and capacity.

In all his earlier experiments the tuned instrument was an "Optical Telephone," in which the motion of the diaphragm was magnified by the use of a small mirror with light spot and scale. The sensitivity of this was 3×10^{-7} amp. per mm. at 1 m.

In 1896 Rubens's Vibration Galvanometer appeared † ; it consists of a series of very small magnets or soft-iron needles fastened to the middle of a tightly strung torsion wire and in a field due to two strong permanent magnets, round whose pole-pieces are coils carrying the alternating current to be measured. The tuning is effected by altering the effective length of the clamped torsion wire and by adjustment of the magnets, and the sensitivity obtained is said to be four times greater than that of the optical telephone.

Some years later ‡ M. Wien brought out a more sensitive form, in which the small magnet system is between the poles of a small electromagnet magnetized entirely by the current to be measured.

4. *Moving Coil Vibration Galvanometer.*

After using an instrument of the Rubens type for some time, I designed another of moving coil type, which I have found more convenient.

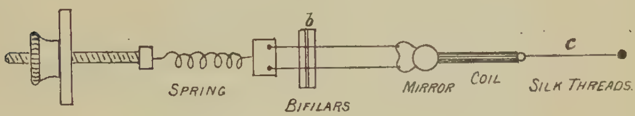
* Max Wien, *Wied. Ann.* xlii. p. 593 (1891); xliv. p. 681 (1891); xliv. p. 689 (1891); lvii. p. 249 (1896); lviii. p. 353 (1898). See also E. Orlich, *Elektrotechn. Zeitschr.* vol. xxvi. (1903).

† Rubens, *Wied. Ann.* lvi. p. 27 (1896).

‡ M. Wien, *Ann. der Physik*, iv. p. 425 (1901).

It consists of an electromagnet (or permanent magnet) with a rather narrow air gap in which is suspended a very light and small coil with bifilar control, which can be regulated by altering the tension by means of an adjustable spring or weight (as in some oscillographs).

Fig. 3.



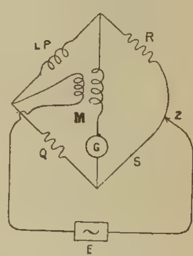
In fig. 3 is shown one arrangement of the suspended system which I have used (shown horizontally for convenience of printing).

Below the coil is a fastening of one or more silk threads *c*. The range of frequency obtainable depends on the moment of inertia of the moving part, the tension, width of bifilars, &c. In one specimen the ordinary range (from 50 to 100 ∼ per sec.) can be obtained by simply tightening or loosening the spring by the screw adjustment, while by placing an adjustable bridge *b* (fig. 3) under the bifilars the range can be extended to 700 or 800 ∼ per sec. The readiness with which the frequency can be adjusted appears to be one of the advantages in the bifilar type. As the frequency is raised the sensitivity decreases in the inverse ratio. With given magnetic field, in normal use (*i. e.* with resonance) the sensitivity only depends on the damping moment, which is both mechanical and electrical. For example, if the moment of inertia and the control torque be both increased in the same proportion without altering the damping, then both the frequency and the sensitivity remain unchanged. It is of importance, therefore, to keep the damping small. For many purposes at frequencies of 20 to 200 ∼ per second, sufficient sensitivity can be obtained even when using a fairly large mirror (1 cm. diameter), but for higher frequencies it is advisable to reduce this size considerably. The control torque is usually strong, the tension being of the order of 0·5 to 1 kgm.

5. *Practical Working of Galvanometer and Hughes-Rayleigh Method.*

The best type of current to use is a nearly pure sine-curve alternating current of very steady frequency (see Rosa and Grover above), but an interrupted current can be used with good accuracy. It is desirable to be able to set the frequency of the current by gradual and fine adjustment for the exact tuning, and for this purpose a wire interrupter like that of Wien is effective. It is merely a monochord solidly supported with fine adjustment of tension and maintained electrically with a mercury break as tuning-forks are. When the galvanometer is in resonance (which is known by the maximum elongation of the spot of light with a given current), it does not follow that it is responding to the fundamental frequency given by the wire *; it may be in resonance with one of the harmonics. In order to determine the actual frequency to which it is answering, a usual method is to test it by a condenser and a variable self-inductance brought to resonance. Another method which I have found very convenient for the same purpose is that of Hughes's Inductance Bridge † as developed by Lord Rayleigh ‡, in which a mutual inductance is compared against an independent self-inductance.

Fig. 4.



The connexions are shown in fig. 4, where M is a variable mutual inductance; L a self-inductance of resistance P; Q, R, and S non-inductive resistances; E a source of alternating or intermittent current of steady frequency, and G the vibration galvanometer.

$R+S$ is kept constant, Z being a slider by which S, which is entirely a slide-wire, can be gradually varied down to zero.

* It is a curious fact that a Rubens Galvanometer with given control sometimes has two points of resonance near one another, *e. g.* 40 and 43 ~ per second in one specimen. (See also Rosa and Grover, *loc. cit.*)

† Prof. E. D. Hughes, Jour. Inst. Elect. Eng. xv. p. 6, Jan. 1886.

‡ Lord Rayleigh, Phil. Mag. Dec. 1886.

As Lord Rayleigh has shown, the conditions for a balance are

$$QR - SP = p^2 ML \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$M(P+Q+R+S) = SL, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $p = 2\pi n$, n being the frequency.

Let Q as well as R+S have a fixed value. For good sensitivity the resistance P usually will have a temperature coefficient not negligible, as the whole or part of the arm may be of copper.

Two cases arise according as we consider P (1) known and constant, or (2) only approximately known and variable.

Case 1.—Let P be constant, and hence

$$P+Q+R+S = \text{const.} = a \text{ (say)} ;$$

also

$$R+S = b.$$

Let L also be constant and known, and let a balance be obtained by varying M and the position of Z.

Then we have

$$(2\pi n)^2 = p^2 = \frac{a}{L^2}\left(\frac{QR}{S} - P\right)$$

$$= \frac{a}{L^2}\left\{\frac{Q(b-S)}{S} - P\right\} . \quad . \quad . \quad (6)$$

Thus n is expressed in terms of the single variable S, and the slide-wire may be marked directly with the values of the frequency deduced from (6).

If wide range of frequencies is to be dealt with, various values may be given to L. Thus, if L be changed to some sub-multiple L/q, the scale readings for n will merely have to be multiplied by q throughout, provided that P is kept constant. Perhaps a better way is to keep L unchanged, and alter P+Q and R+S each in the ratio $1:q$, the alterations being made in P and R only, leaving Q and S as before ; from formula (6) it will be seen that, for the same scale reading, n will become qn. The value of the variable M does not require to be known. The following example, giving values which I have actually used, may be of practical interest. P=25 ohms, Q=5 ohms, R+S=4 ohms, L=0·1066 henry. The graduation of the slide-wire corresponds to Table I.

TABLE I.

n.	S.	Temperature-coefficient of n.
∼ per sec.	ohm.	°/₀ per degree C.
10	0·642	−0·43
15	0·608	−0·32
20	0·562	−0·21
30	0·471	−0·08
40	0·392	−0·01
50	0·320	+0·04
60	0·260	+0·08
70	0·213	+0·09
80	0·175	+0·10
90	0·147	+0·11
100	0·126	+0·11
110	0·107	+0·12
120	0·090	+0·12

Since $M=0·00313\,S$, its range of variation will be from about 2 millihenries at $n=0$ down to about 0·28 millihenry at $n=120$.

This case is usually sufficient to discriminate the actual frequency to which the galvanometer is responding, the exact frequency of the source being determined by comparison with a standard fork or stroboscopically.

Case 2.—If P be not exactly known, a temperature correction may be applied to the scale readings. Thus in the above example, in which P was entirely of copper, the temperature coefficients of n at various points of the scale are shown in the third column of Table I. It will be noticed that the correction may become large only below 30∼ per second.

If the variable M has been accurately calibrated, p may be obtained by the equation

$$p^2=\frac{(Q+S)(R+S)}{ML}-\left(\frac{S}{M}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For example, the interrupter was tuned to unison with a standard fork which gave 119·9∼ per sec.: with $L=0·1000$ henry the method gave $n=120·0$ and 119·8∼ per sec. in two observations.

Although I have described this method as used for determining the frequency to a first approximation, it can be used, with an accurately known frequency, to determine L

and M in terms of P, Q, R, S, and n; and the use of the vibration galvanometer avoids the difficulty of obtaining a pure sine-curve current.

6. *Results obtained by Modified Carey Foster Method.*

A series (A) of careful tests were made by this method on a standard mutual inductance of nominal value 0·05 henry. This standard consists of a pair of coils of silk-covered wire wound in two deep channels turned on a cylinder of marble, the whole being soaked in hot paraffin wax after winding. It is illustrated in Plate XIII.

It was originally adjusted by Carey Foster's method, using a ballistic galvanometer with an air condenser as standard. A series (B) of tests were made on it, using Carey Foster's Method with a secohmmeter commutator.

It was also tested (C) by a quite independent method, viz., Kirchhoff's Absolute Method with a ballistic galvanometer *.

In Table II. are shown some of the results of these sets of tests.

TABLE II.

Set.	Condenser.	K.	R.		Mean of set.	Greatest error from mean of set.
		mfd.				Parts in 10,000.
A	Mica	0·9994	100·00	0·05009	0·05009	2
	,, ...	0·8003	,,	0·05008		
	,, ...	0·5003	,,	0·05010		
B	Air	0·04165	20	0·05008	0·05010	6
	Mica	0·3000	50	0·05013		
	,, ...	0·5000	50	0·05013		
	,, ...	0·5000	20	0·05008		
C	....	....	......	0·05019	0·05014	10
				0·05013		
				0·05014		
				0·05011		
				0·05013		
				0·05015		
				0·05016		

* Maxwell's Elect. & Mag. vol. ii. § 759; Kirchhoff, Pogg. *Ann.* lxxvi. April 1849; and Glazebrook, Sargant, & Dodds, Phil. Trans. pp. 223-268 (1883).

In sets (A) and (B) the frequency was of the order of 40 and 100 ∼ per second respectively.

The agreement of sets (A) and (B) appears satisfactory; while the slightly higher result given by (C) is probably within the limits of possible error of the method as used *. The readings in (A) were sensitive to 2 or 3 in 10,000. The resistance R was specially wound to avoid capacity and inductance. An inductance of 0·1000 henry was added in the branch OrO. The total self-inductance of this branch by comparison with our standard inductances was 0 765 henry.

Hence, by (3),

$$M = L\left(\frac{R}{R+S}\right) = \frac{0{\cdot}765 \times 100}{1525{\cdot}5} = 0{\cdot}0501 \text{ henry},$$

which is good agreement with the results given by the formula

$$M = 10^{-6}\,\mathrm{KR}r.$$

Thus we infer that in the mica condenser used there is no appreciable apparent series resistance such as Rowland and other experimenters found in some cases.

I would remark that, after using method (A), I feel confident that it is still capable of much higher accuracy than that shown above.

In conclusion I would express my best thanks to Dr. Glazebrook for most valued advice and help.

Discussion.

Mr. A. Russell thanked the author for his most interesting and instructive demonstration. He considered that the bifilar method of suspending the vibrating coil was a great improvement on the single wire method described by Wien in the *Annalen der Physik* in 1901. In the 'Physical Review' of last December Mr. Wells states that an alteration of 0·3 per cent. in the length (about 12 cm.) of the

* The main probable errors in (C) seem to be due to three causes, namely: the variations (natural and other) in the earth's horizontal magnetic field; some uncertainty in the time measurements; and the magnetizing effect of the current on the galvanometer needles, not necessarily the same for a steady current and a sudden rush.

stretched wire of a vibration galvanometer, when tuned, diminished the amplitude of the oscillation to less than one-fifteenth of its former value. The resonance point is very sharply defined. Hence to work such an instrument at its greatest sensibility, it must be most accurately tuned and it must be supplied with a current of almost absolutely constant frequency. When used to measure currents, the shape of the wave must be known. Mr. Campbell, however, apparently uses his vibration galvanometer mainly in null methods, and so any kind of interrupted or alternating current can be used, the shape of the wave being almost immaterial. He noticed that Mr. Campbell obtained his balance very rapidly by adjusting both the tension of the suspending spring of his galvanometer and, by means of a simple interrupting device, the frequency of the supplied current during the course of the same experiment. The suggested method of measuring frequency was extremely ingenious. He asked whether Mr. Campbell had found it necessary to take into account the "effective internal resistance" of the condensers he used in his tests.

Dr. Eccles asked the author for information as to the differences in performance between vibration galvanometers like the one shown, which employed rotational vibrations, and string galvanometers such as that of Einthoven, where the vibrations are transverse. The latter is a tuned instrument, and, it has been stated, has been used in the measurement of inductance, with what method and degree of success Mr. Campbell might know.

Mr. Rollo Appleyard said the author's methods should be applicable to the case of an inductance shunted by a capacity. In that event it would be useful for measurements of the inductance of telephone cables. If it could be adapted to that purpose, it would be of great service in affording a rapid means of determining the "attenuation constant," when frequency was of importance.

Dr. R. S. Willows said the suspension used by the author was an improvement on those in general use, which were liable to be broken when changing the frequency.

Mr. Campbell, in reply to Mr. Russell, said that the second formula in the modified Carey Foster method involved the

resistance in series with the condenser. With a mica condenser it gave results in such close agreement with the first formula, that it precluded the possibility of any apparent resistance in the condenser of the order observed in certain condensers by Rowland and by Mr. Russell. The author was not aware of the use of the Einthoven string galvanometer as a vibration galvanometer, but if the damping could be made small enough there seemed no reason why it should not be so used. In reply to Mr. Appleyard, he thought that one of M. Wien's methods would be applicable.

XLII. *Note on the Rate of Decay of the Active Deposit from Radium.* *By* W. Wilson *and* W. Makower *.

In some experiments in which the ionization produced by the α rays from radium C was balanced against that produced by the more penetrating β and γ rays, it was found that after a short time these two ionizations were no longer exactly equal, however carefully they had at first been adjusted to equality. A similar effect has been noticed by Bronson †, and has been attributed to the slowly moving β rays emitted by radium B, recently discovered by Schmidt ‡. Since these rays are emitted by radium B, whereas the α and the more penetrating rays are emitted by radium C, it is to be expected that the rate of decay as measured by these two types of radiation will be different.

Although this explanation seemed probable, it was thought to be of interest to test this point somewhat more carefully, to make sure that this explanation is really the correct one. The following experiments show conclusively that this is the case.

As the whole effect to be measured was very small, it was necessary to make somewhat careful experiments to test the matter, and the method used was as follows.

Two ionization-vessels A and B (fig. 1) were connected to one pair of quadrants of an electrometer, the other pair of

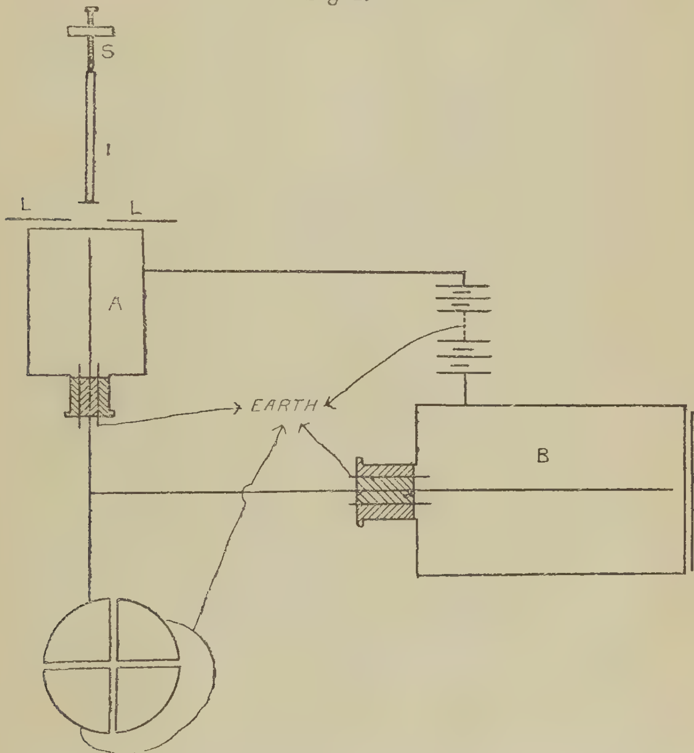
* Read May 24, 1907.

† Bronson, Phil. Mag. July 1906.

‡ Schmidt, *Phys. Zeit.* vi. p. 897 (1905).

which was permanently connected to earth. The vessel A was 6·4 cms. long and 5 cms. in diameter, and its end was closed by an aluminium-leaf which would allow the α rays to pass through. The vessel B was 15·4 cms. long and had a diameter of 8·4 cms., its end being closed by a copper plate thick enough to absorb all the α rays, but thin enough to allow much of the β radiation to pass through.

Fig. 1.



A wire which had been exposed to the radium emanation for a sufficient length of time to allow the deposit to assume a steady state was broken in two pieces, one of which was fixed outside the vessel B, the other being fastened to an iron bar I which, by means of the screw S, could be moved towards and away from the vessel A. The vessel A was connected to one terminal of a battery of two hundred small storage-cells, the middle point being connected to earth and the other

terminal to B. Two lead screens, L L, were made which could slide so as to leave an opening between them, and were placed close to the end of the vessel A. The distance between the plates could be varied from the whole width of the vessel to nothing. By altering the width of this opening and varying the distance of the wire from the ionization-vessel A, a balance could be obtained between the ionizations produced in A and B by the two wires respectively.

After removal from the emanation to which they had been exposed, the wires were left for a sufficient length of time to allow the radium A practically to disappear. The balance was then made, roughly at first by means of the lead screens and then more carefully by the screw S. In the earlier experiments the *difference* of the ionizations produced in the two vessels in one minute was measured at intervals of about five minutes, the value of the *whole* ionization in the vessel B also being determined from time to time. It was found that the needle did not move uniformly from its position of rest to its final position, but moved irregularly, as is often the case when the difference between two nearly equal large ionizations is being measured *. These irregularities could, however, be to some extent eliminated by measuring the difference of the ionizations produced in five minutes instead of one minute.

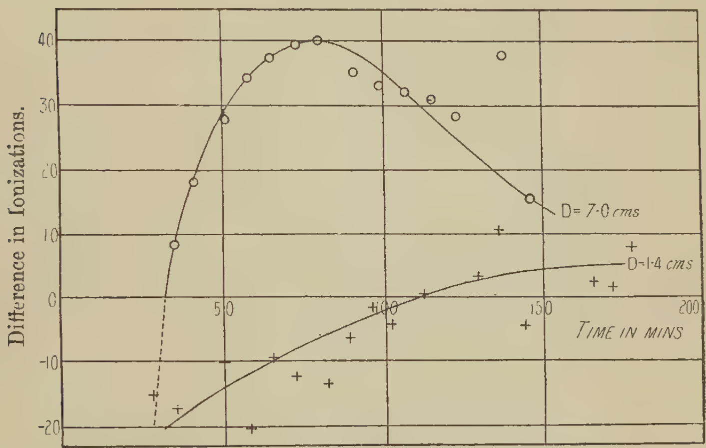
The results obtained in this manner for two distances D of the wire from the vessel A, 7 cms. and 1·4 cms. respectively, are shown in fig. 2, where the ordinates represent *differences* between the ionizations in A and B, and the abscissæ times reckoned from the moment at which the wire is removed from the emanation. The *whole* ionization in each vessel was 1600 scale-divisions, 48 minutes after removal from the emanation, so that the deviations of the points from the curves (fig. 2) in no case amounted to as much as one per cent. of this quantity. The activity decreased with time according to the usual laws †. It will be noticed that for small distances of the wire from the vessel A the difference in the observed rates of decay of the deposit on the two wires is small, but for large distances (7 cms.) it is considerable.

* Bronson, Phil. Mag. Jan. 1906.

† Miss Brooks, Phil. Mag. Sept. 1904.

Now it is known that the range of the α particle from radium C is just over 7 cms.* ; consequently when the wire is at this distance from the vessel A the α rays contribute but little to the ionization in this vessel, which is for the most

Fig. 2.

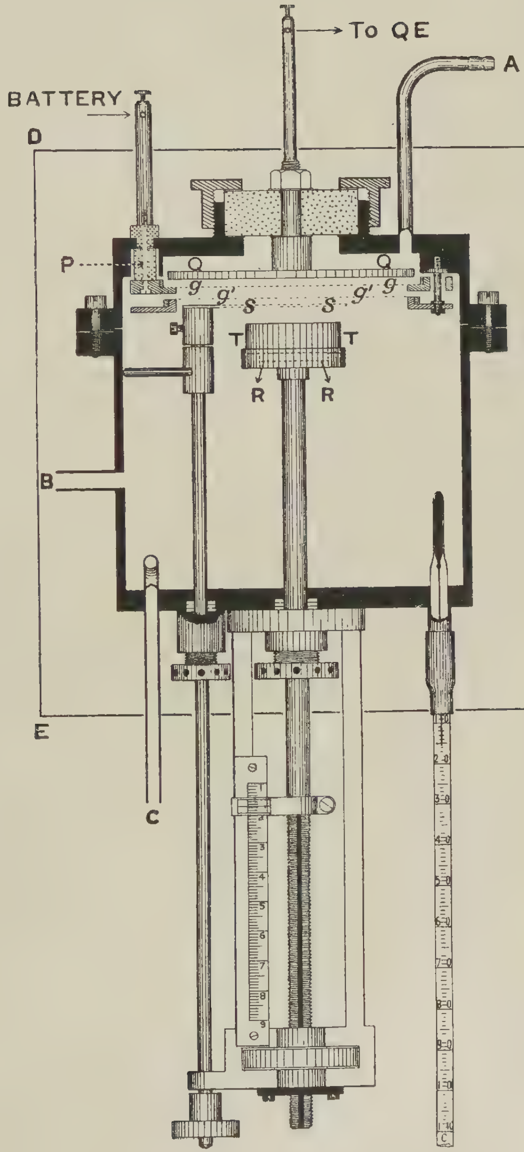


part due to the rays emitted by radium B. If the observed effect is due to the slowly moving β rays emitted by radium B, the difference in the rate of decay of the activity as measured in the two vessels A and B should therefore be greater in these circumstances than when measurements are made with the active wire near to A. Moreover, the ionization in the vessel A should decay more rapidly than that in the vessel B. This is, in fact, found to be the case, so that we may conclude that the difference in the rates of decay as measured in the two vessels is due to the slowly moving β rays emitted by radium B.

To further test this point, the rays were made to pass between the pole-pieces of a small electromagnet before entering the vessel A. On exciting the magnet, the slowly moving β rays were deflected and the balance was disturbed, the ionization in the vessel A at once falling below that in B. This fact confirms the conclusion given above.

Physical Laboratory,
The University, Manchester.

* Bragg & Kleeman, Phil. Mag. Sept. 1905.


To QE
A
BATTERY
D
G
P
Q
Q
g
g'
S
S
g'
g
T
T
R
R
B
E
F
C

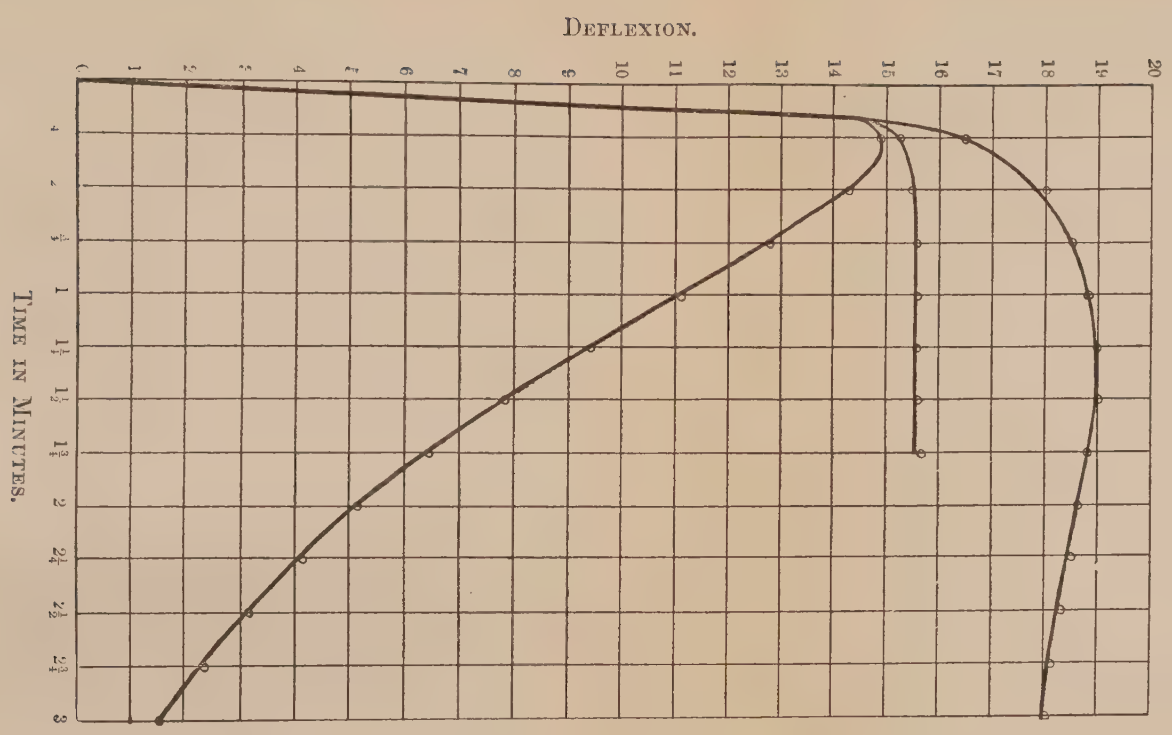


Fig. 4.

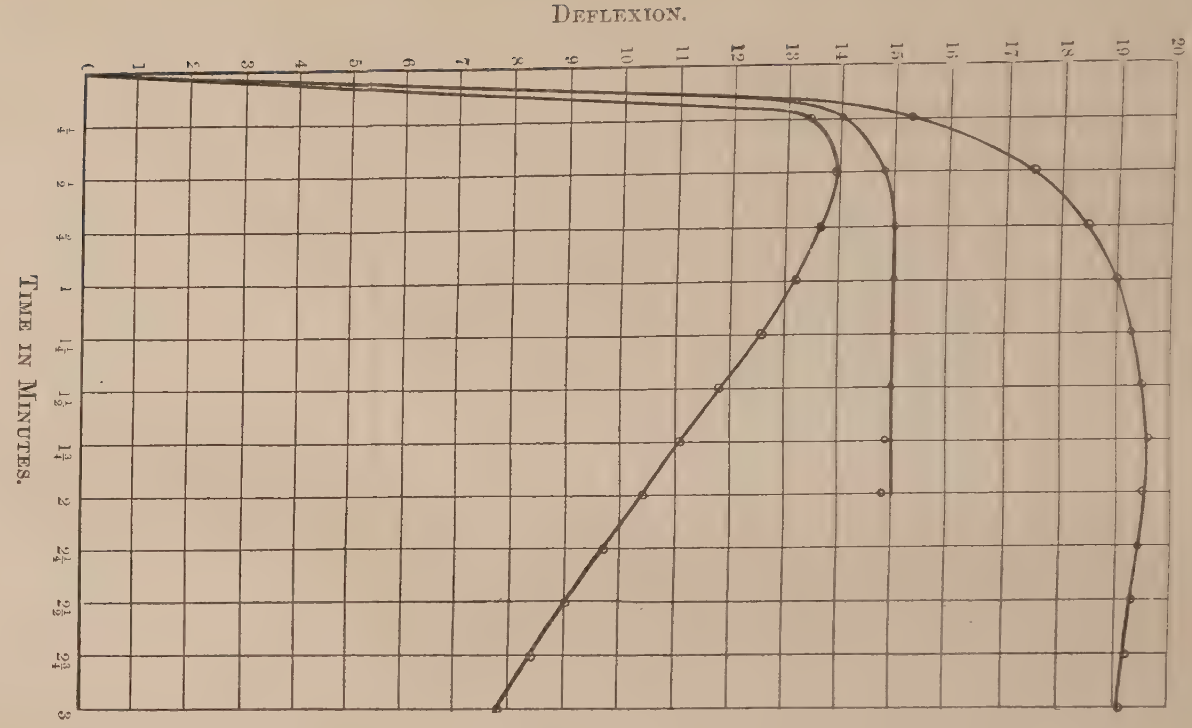


Fig. 3.

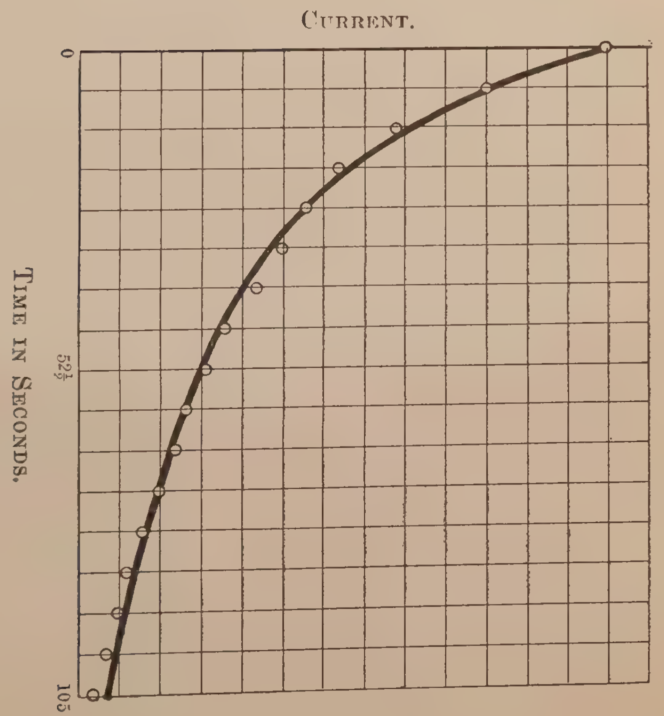


Fig. 7.

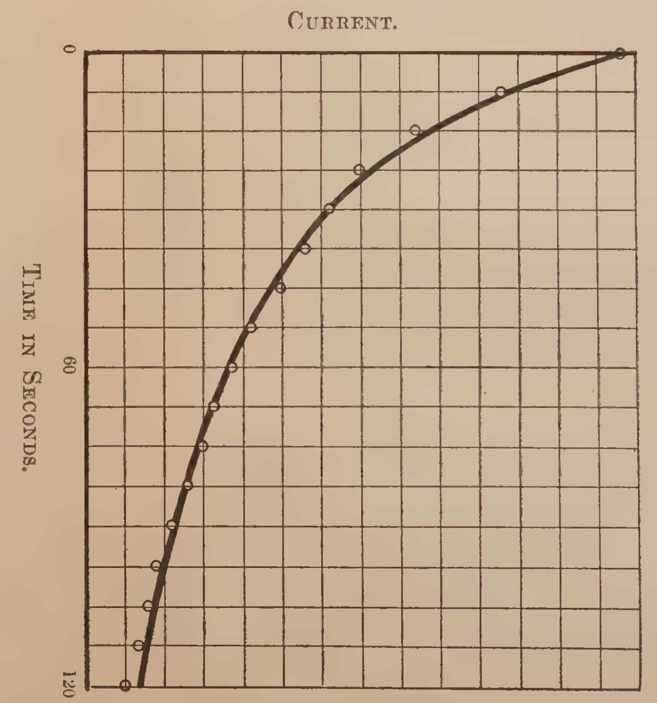


Fig. 6.

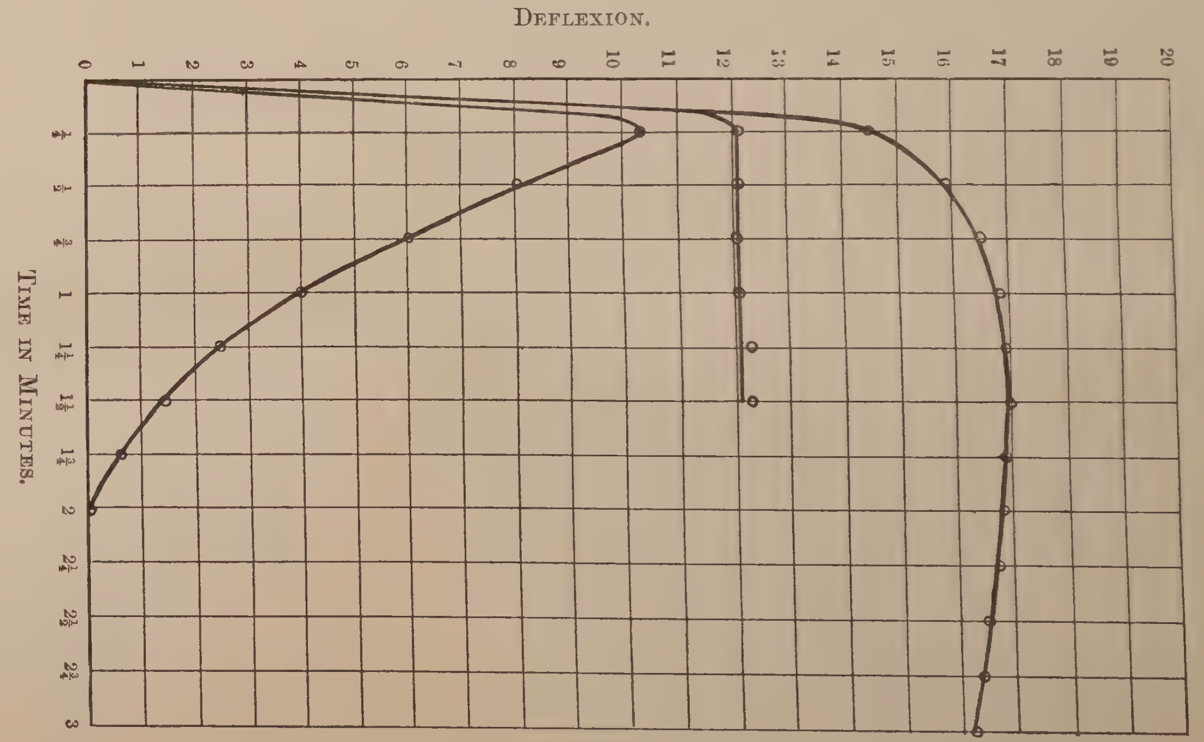


Fig. 5.

INDEX.

A.

B.

C.

G.

H.

I.

L.

M.

N.

O.

P.

Q.

R.

S.

T.

Printed by Taylor and Francis, Red Lion Court, Fleet Street.